AN EXTREMAL PROBLEM IN GRAPH THEORY

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The distance $d_G(u,v)$ between vertices u, v of a graph G is the least number of edges in any u-v path of G; $d_G(u,v) = \infty$ if u and v lie in distinct components of G. A graph $G = (V,E)$ is *distance-critical*, if for each $x \in V$ there are vertices u, v (depending on x) such that $d_G(u,v) < d_{G-x}(u,v)$. Let g(n) denote the largest integer such that $|E| \le \binom{n}{2} - g(n)$ for every distance-critical graph on n vertices. It follows from the results of this note that g(n) is of the order of magnitude of $n^{3/2}$; possibly, one has $g(n) \sim \sqrt{2}\, n^{3/2}$.

THEOREM 1. *A graph $G = (V,E)$ is distance-critical iff to each vertex x of G there corresponds a pair $\{u,v\} \subseteq V$ such that $uv \notin E$, and $(yu \in E$ and $yv \in E) \longleftrightarrow y = x$ for each $y \in V$.* □

More generally, we are interested in the graphs $G = (V,E)$ satisfying the following condition, where $[V]^r$ denotes the collection of the r-subsets of V:

(*) There is a mapping M from an n-element subset S of V into $[V]^r$ such that $[yv \in E$ for each $v \in M(x)] \longleftrightarrow y = x$.

Let $f_1(r,n)$ denote the largest integer such that $|[V]^2 - E| \ge f_1(r,n)$ for every graph satisfying (*).

THEOREM 2. *For each r there is a $c_r > 0$ such that $f_1(r,n) \ge [c_r + o(1)] \cdot n^{1+1/r}$.*

Proof: It is convenient to prove a slightly stronger statement: For each r there is N_r, $c'_r > 0$, such that if G satisfies (*) with $n \ge N_r$

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then $|\{uv\notin E:\ uv\cap S\neq\emptyset\}|\geq c'_r\,n^{1+1/r}$. The proof is by induction on r. The assertion clearly holds for $r=1$. Let $k\geq 2$ and suppose that the assertion holds for $r=k-1$. Let $G=(V,E)$ be a graph satisfying (*) with $r=k$. Let S_o be a maximal subset of S with the property that the k-sets $M(x)$, $x\in S_o$, are pairwise disjoint.

CASE I: $|S_o|\geq n^{1/k}$. By assumption, for each fixed $x\in S_o$ and each $y\in S-(M(x)\cup\{x\})$, at least one of the edges joining y and an element of $M(x)$ is missing. Since the sets $M(x)$, $x\in S_o$, are pairwise disjoint, at least $n^{1/k}\cdot[n-(k+1)n^{1/k}]\sim n^{1+1/k}$ edges are missing.

CASE II: $|S_o|\leq n^{1/k}$. Let $A=\cup(M(x);\ x\in S_o)$. Thus $|A|\leq kn^{1/k}$. By the maximality of S_o, $A\cap M(y)\neq\emptyset$ for each $y\in S$. Let α be a function $S\to A$ such that $\alpha(y)\in M(y)$ for each $y\in S$. Let A_o denote the set $\{a\in A: |\alpha^{-1}(A)|\geq N_{k-1}\}$. Let $S'=\cup(\alpha^{-1}(a): a\in A_o)$. Since $|S'|\geq n-N_{k-1}kn^{1/k}$, we have $|S'|=n+o(n)$. Consider a fixed $a\in A_o$. The mapping M_a: $x\mapsto M(x)-\{a\}$ defined on $\alpha^{-1}(a)$ satisfies the condition of (*) with $r=k-1$. By induction hypothesis, $|\{xy\notin E:\ xy\cap\alpha^{-1}(a)\neq\emptyset\}|\geq c'_{k-1}n^{k/(k-1)}$ for sufficiently large n. Summing over all $a\in A_o$ and dividing by 2 (because each edge is counted at most twice) we obtain by an elementary estimate:

$$|\{xy\notin E:\ xy\cap S\neq\emptyset\}|\geq\frac{1}{2}c'_{k-1}\cdot\left(\frac{n+o(n)}{kn^{1/k}}\right)^{k/(k-1)}\cdot kn^{1/k}\sim\frac{1}{2}c'_{k-1}k^{-1/(k-1)}\cdot n^{1+1/k}.$$

This completes the proof. □

Let $f_2(r,n)$ denote the largest integer such that $|E|\leq\binom{n}{2}-f_2(r,n)$ for every graph $G=(V,E)$ satisfying (*) with $S=V$ and $[M(x)]^2\cap E=\emptyset$ for each $x\in V$. Clearly, $f_1\leq f_2$ but we were not able to establish a better upper bound for f_1 then the following one for f_2:

THEOREM 3. *For* $r \geq 2$, $f_2(r,n) \leq [(r!)^{1/4}+o(1)]\ n^{1+1/4}$.

Proof: Let $r \geq 2$, n be given. Let $G_o = (V_o,E_o)$ be a regular graph of degree r containing no triangles, with $|V_o|$ nearly equal $(r!n)^{1/r}$. Let $M_o(x) = \{y: yx \in E_o\}$ for each $x \in V_o$. Let V_1 be a set, $V_1 \cap V_o = \emptyset$, such that there is a bijection

$$M_1:V_1 \to \{W \in [V_o]^r : [W]^2 \cap E_o = \emptyset \text{ and } W \neq M_o(x) \text{ for any } x \in V_o\}.$$

Let $G = (V,E)$ be a graph defined by $V = V_o \cup V_1$ and $E = E_o \cup [V_1]^2 \cup \{xy: x \in V_1, y \in M_1(x)\}$. Then G and $M = M_o \cup M_1$ satisfy the conditions stated above. Moreover, $|V| \sim n$ and $\binom{n}{2}-|E| \sim (r!)^{1/r}$. The details are left to the reader.

We conjecture that for $r \geq 2$,

$$\lim f_1(r,n)/n^{1+1/r} = \lim f_2(r,n)/n^{1+1/r} = (r!)^{1/r}.$$

However, the optimal constants c_r calculated from our proof of Theorem 2 form a sequence converging to 0. In particular, one has $c_2 = 1/\sqrt{2}$. Since $f_2(2,n) = g(n)$, we have:

COROLLARY. $[1/\sqrt{2}+o(1)]n^{3/2} \leq g(n) \leq [\sqrt{2}+o(1)]n^{3/2}$. □

We suspect that, in fact, $g(\frac{1}{2}k(k-5)+k) = \frac{1}{2}k^3-3k^2+\frac{7}{2}k$. An example of a distance-critical graph realizing this bound is obtained from the proof of Theorem 3 by taking $r = 2$ and G_o = cycle of length k.

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