AN EXTREMAL PROBLEM IN GENERALIZED RAMSEY THEORY

S.A. Burr, P. Erdös, R.J. Faudree, C.C. Rousseau and R.H. Schelp

Abstract

A connected graph G of order n is called *m-good* if $r(K_m,G) = (m-1)(n-1) + 1$. Let $f(m,n)$ be the largest integer q such that *every* connected graph of order n and size q is m-good and let $g(m,n)$ be the largest q for which *there exists* a connected graph G of order n and size q which is m-good. Asymptotic bounds are given for f and g.

1. Introduction.

One of the most notable yet simply proved results in generalized Ramsey theory is the theorem of Chvátal [5],

$$r(K_m,T) = (m-1)(n-1) + 1, \tag{1}$$

where K_m is the complete graph of order m and T denotes an arbitrary tree of order n. This result suggests many different avenues for research. In fact, $r(K_m,G) \geq (m-1)(n-1) + 1$ for every connected graph G of order n and it is natural to seek to determine those graphs for which equality holds. A connected graph G of order n will be called *m-good* if $r(K_m,G) = (m-1)(n-1) + 1$. Chvátal's theorem shows that every connected $(n, n-1)$ graph is m-good and so suggests the introduction of the following extremal functions. Let $f(m,n)$ denote the largest integer q such that *every* connected (n,q) graph is m-good and let $g(m,n)$ denote the largest integer q for which *there exists* a connected (n,q) graph which is m-good. Our most precise results are for the case $m = 3$ and the major portion of this paper will be devoted to this case. For this reason and in order to gain simplicity of notation, we shall let $f(3,n)$ and $g(3,n)$ be denoted $f(n)$ and $g(n)$ respectively.

Terminology and notation not specifically mentioned will be in accordance with that in the well-known text of Harary [9]. A graph with n vertices and q edges will be referred to as an (n,q) graph. Our conventions with respect to Ramsey theory run as follows. Let $V = \{v_1,v_2,\ldots,v_p\}$ denote a set of vertices. Then $[V]^2$ denotes the

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set of all unordered pairs of these vertices. By a *two-coloring* we mean a partition $[V]^2 = (R,B)$. Equivalently, we ascribe to each edge of the complete graph of order p a color, either red or blue. This two-coloring defines two edge-induced graphs of order p and we use $\langle R\rangle$ and $\langle B\rangle$ as symbols for these graphs. Let F and G be ordinary graphs without isolated vertices. The statement $K_p \to (F,G)$ means that if $|V| = p$ then for every possible two-coloring (R,B) of $[V]^2$, either $\langle R\rangle$ contains (a subgraph isomorphic to) F or $\langle B\rangle$ contains G. In detail, either there exists a one-to-one map $\sigma\colon V(F)\to V$ such that $\{u,v\} \in E(F)$ implies $\{\sigma(u),\sigma(v)\} \in R$ or there exists a one-to-one map $\tau\colon V(G) \to V$ such that $\{u,v\} \in E(G)$ implies $\{\tau(u),\tau(v)\} \in B$. The *Ramsey number* $r(F,G)$ is the smallest natural number p such that $K_p \to (F,G)$. Properties of $\langle R\rangle$ and $\langle B\rangle$ will be denoted in an obvious way. Thus, for example, if $v \in V$, then $N_R(v)$ and $N_B(v)$ denote the neighborhoods of v in $\langle R\rangle$ and $\langle B\rangle$ respectively. Let X be a subset of V. We shall use the symbols $X_R(v)$ and $X_B(v)$ to denote $N_R(v) \cap X$ and $N_B(v) \cap X$ respectively.

The arguments used in this paper are similar in nature to those used by certian of the authors in other investigations. It may be useful to the reader to refer to these studies, in particular to [2], [3], and [4].

2. *Low order values of the extremal functions.*

The following table gives $f(n)$ and $g(n)$ for $n \leq 6$.

TABLE I. Low order values of f and g

n	f(n)	g(n)
2	1	1
3	2	2
4	5	5
5	7	8
6	8	12

For $n \leq 4$, these values are trivially determined. The values of $f(5)$ and $g(5)$ are contained in the work of Clancy [6]. The $(5,8)$ graph $K_5 - P_3$ is 3-good and this graph provides us with the example which shows that $g(5) = 8$. It also shows that $f(5) \geq 7$ since every connected $(5,q)$ graph with $q \leq 7$ is a subgraph of $K_5 - P_3$ and, therefore, 3-good. The $(5,8)$ graph $K_5 - 2K_2$ is not 3-good and so $f(5) = 7$. For $n = 6$ there is a strikingly similar situation. The Ramsey numbers $r(K_3,G)$ for all connected graphs of order six have been determined by three of the authors [8] and we now draw upon those results. The $(6,12)$ graph $K_6 - P_4$ is 3-good, and this is the example which shows that $g(6) = 12$. It also shows that $f(6) \geq 8$ since every connected $(6,q)$ graph with $q \leq 8$ is a subgraph of $K_6 - P_4$. On the other hand, the $(6,9)$ graph $K_6 - 2K_3$ is not 3-good. Thus $f(6) = 8$.

3. *Asymptotic Bounds.*

The results of the last section, though certainly interesting, are probably in no way indicative of $f(n)$ and $g(n)$ in general. We thus turn to the main subject of this paper, namely general upper and lower bounds for these two extremal functions. Several preliminary results are needed.

LEMMA 1.1. *Let* G *be a graph of order* n *and let* $H = G - x_o$ *where* x_o *is a vertex of degree* d *in* G. *If* $p \geq (d+1)(n-1) + 1$ *and* $K_p \rightarrow (K_3,H)$, *then* $K_p \rightarrow (K_3,G)$.

Proof. Let $V(G) = \{x_o,x_1,\ldots,x_{n-1}\}$ and suppose that x_o is of degree d in G, its neighborhood being $\{x_1,x_2,\ldots,x_d\}$. With $H = G-x_o$ suppose that $K_p \rightarrow (K_3,H)$, and with $|V| = p$ let (R,B) be a two-coloring of $[V]^2$ such that $\langle R\rangle$ contains no K_3. Then there exists an embedding $\sigma\colon V(H) \rightarrow V$ of H into $\langle B\rangle$. Let Y denote the set of vertices in V which are distinct from $\sigma(x_1),\ldots,\sigma(x_{n-1})$ and note that $|Y| \geq d(n-1) + 1$. If any vertex of Y is adjacent in $\langle B\rangle$ to each of the vertices $\sigma(x_1),\ldots,\sigma(x_d)$, then $\langle B\rangle$ contains G. If not, then at least one of the vertices $\sigma(x_1),\ldots,\sigma(x_d)$ has degree at least n in $\langle R\rangle$. Since $\langle R\rangle$ contains no K_3, this gives K_n, and so G, in $\langle B\rangle$. □

Armed with this result, we may now prove a general, albeit crude, upper bound for $r(K_3,G)$ where G is an arbitrary (n,q) graph.

LEMMA 1.2. *If G is an (n,q) graph, then $r(K_3,G) \leq n + 2q$.*

Proof. The proof is by induction on n. Since G has no isolated vertices, we start with $n = 2$ and $G \cong K_2$ where the result is obviously true. With $n > 2$ let G be an (n,q) graph. Then G has a vertex x_o of degree $d \leq 2q/n$. Upon deleting this vertex, we obtain the $(n-1, q-d)$ graph H. Set $p = n + 2q$. Since $p > (n-1) + 2(q-d)$, it follows from the induction hypothesis that $K_p \rightarrow (K_3,H)$. In addition, $p > (d+1)(n-1) + 1$. Consequently, $K_p \rightarrow (K_3,G)$. □

In what follow, the term *suspended path* is often used. A suspended path of length ℓ in a graph G is a path in G $\{x_o,x_1,\ldots,x_\ell\}$ in which the vertices $x_1, x_2, \ldots, x_{\ell-1}$ are each of degree two in G.

LEMMA 1.3. *Let G be a graph of order n. (a) Suppose that there is a vertex of G which is of degree one and let H denote the graph obtained by deleting this vertex. If $K_{2n-1} \rightarrow (K_3,H)$ then $K_{2n-1} \rightarrow (K_3,G)$. (b) Suppose that G contains a supended path of length three, $\{u,v_1,v_2,w\}$, and let H denote the graph obtained from G by replacing this suspended path by one of length two, $\{u,v,w\}$. If $K_{2n-1} \rightarrow (K_3,H)$ then $K_{2n-1} \rightarrow (K_3,G)$.*

Proof. (a) This is a special case of Lemma 1.1. (b) With $|V| = 2n-1$ let (R,B) be an arbitrary two-coloring of $[V]^2$, and suppose that $\langle R\rangle$ contains no K_3 and $\langle B\rangle$ contains no copy of G. Let $\sigma\colon V(H)\rightarrow V$ be an embedding of H into $\langle B\rangle$. Let X denote the set of vertices of V which are exterior to this copy of H and note that $|X| = n$. For simplicity of notation, let $X_R\{\sigma(v)\}$ and $X_B\{\sigma(v)\}$ be represented by just X_R and X_B respectively. Since, by assumption, $\langle B\rangle$ contains no copy of G, every vertex in X_B is adjacent to $\sigma(u)$ and $\sigma(w)$ in $\langle R\rangle$. It now follows that since $\langle R\rangle$ contains no K_3, the only edges of $[X]^2$ which can be in R are of the type $\{x,y\}$

where $x \in X_R$ and $y \in X_B$. If no such edge exists, then X spans K_n in <B>. Rejecting this possibility, we note that if $x \in X_R$ is adjacent in <R> to some vertex $y \in X_B$, then x is adjacent in <B> to both $\sigma(u)$ and $\sigma(w)$. If there were two such vertices x_1, x_2, then the embedding of H could be extended to an embedding of G, $\tau: V(G) \to V$ by setting $\tau(v_1) = x_1$, $\tau(v_2) = x_2$ and $\tau = \sigma$ otherwise. Thus, by assumption, there is a *unique vertex* $x \in X_R$ which is adjacent in <R> to one or more vertices of X_B. Consider the graph spanned by X together with $\sigma(v)$ in <B>. If $|X_B| = 1$ then the graph spanned by X in <B> is complete except for one edge. If $|X_B| \geq 2$, then the graph spanned by X in <B> contains a K_{n-1} and $\sigma(v)$ is adjacent in <B> to at least two vertices of this K_{n-1}. In either case, <B> contains G, and so we have reached a contradiction. □

If neither hypothesis (a) nor (b) of the last lemma holds, then there are some important consequences.

LEMMA 1.4. *With $k \geq 0$ let H be a connected $(\ell, \ell+k)$ graph which has no vertex of degree one and no suspended path of length three. If $k = 0$, then $H \simeq C_3$ and, otherwise, $\ell \leq 5k$. This bound is sharp.*

Proof. If $k = 0$, then H is a unicyclic graph. Since there are no vertices of degree one, H is, in fact, a cycle. Since there is no suspended path of length three, $H \simeq C_3$. Now suppose that $k \geq 1$. Let h denote the number of vertices of degree at least three in H. Since $k \geq 1$, it follows that $h \geq 1$. For each vertex v of degree two in H, delete v and join the two vertices to which it is adjacent by an edge. We thus obtain a multigraph M with h vertices and $h + k$ edges. Since M has no vertices of degree two, $3h \leq 2(h+k)$ and so $h \leq 2k$. Now if H is regained by inserting the vertices of degree two, the fact that there is at most one such vertex for each edge of M implies that there are at most $3k$ such vertices. Consequently, $\ell \leq 5k$. The simple example $H \simeq K_{2,3}$ shows that the bound is sharp. □

We are now prepared to prove general upper and lower bounds for $f(n)$ and $g(n)$.

THEOREM 1. *(a) For all* $n \geq 4$, $f(n) \geq (17n + 1)/15$. *(b) Let* $\varepsilon > 0$ *be fixed. Then, if* n *is sufficiently large,* $f(n) < (27/4+\varepsilon)n(\log n)^2$.

Proof. (a) With $0 \leq k \leq (2n + 1)/15$, let G be a connected $(n,n+k)$ graph and suppose that $K_{2n-1} \to (K_3,G)$. By repeated application of Lemma 1.3, if necessary, we obtain a connected $(\ell,\ell+k)$ graph H such that (i) H has no vertex of degree one, (ii) H has no suspended path of length three, and (iii) $K_{2n-1} \to (K_3,H)$. Since $n \geq 4$, $H \neq C_3$. Consequently, by Lemma 1.4, $\ell \leq 5k$. On the other hand, $\ell \geq 4k$; otherwise, by Lemma 1.2, $r(K_3,H) \leq 14k - 3 < 2n-1$. Let t denote the number of vertices of degree two in H. Then $2(\ell+k) \geq 2t + 3(\ell-t)$. Since $\ell \geq 4k$, it follows that $t \geq 2k$. By deleting as many vertices of degree two as necessary, we find a graph F and a vertex x of degree two in F such that $K_{2n-1} \to (K_3,F)$ but $K_{2n-1} \to (K_3,F-x)$. Since F is of order $\leq 5k$ and $3(5k-1) + 1 \leq 2n-1$, this contradicts Lemma 1.1 and so the result is proved.

(b) Choose ℓ to be the least integer t such that $r(K_3,K_t) > 2n-1$. Let G be the graph consisting of a K_ℓ together with a path of length $n - \ell$ attached to one of its vertices. Then G is a connected (n,q) graph where $q = \binom{\ell}{2} + (n-\ell)$ and G is not 3-good. In [10], Spencer reconsiders an early application of the probabilistic method by one of the authors. Spencer shows that

$$r(K_3,K_t) > (1/27 - o(1))(t/\log t)^2. \tag{2}$$

Using this result, it follows that ℓ is such that $q < (27/4 + \varepsilon)n(\log n)$ when n is sufficiently large. □

THEOREM 2. *There exist positive constants* A *and* B *such that*

$$An^{3/2}(\log n)^{1/2} < g(n) < Bn^{5/3}(\log n)^{2/3}$$

for all sufficiently large values of n.

Proof. The proof of the lower bound relies on a simple example together with a recent result of Ajtai, Komlós, and Szemerédi [1], namely $r(K_3,K_s) < cs^2/\log s$ for all sufficiently large values of s.

Set $s = [\sqrt{n(\log n)/6c}]$ where c is the constant which appears in the Ajtai, Komlós, Szemerédi result. Let t be the smallest integer for which $n - 1 - ts < r(K_3,K_s)$. Then $K_{n-1} \to (K_3, tK_s)$ and so $K_{2n-1} \to (K_3,G)$, where $G = K_1 + H$ and H is the graph consisting of t disjoint copies of K_s together with $n - 1 - ts$ isolated vertices. Thus G is a connected graph of order n and size $q = t\binom{s}{2} + n - 1$ which is 3-good. Using the fact that $r(K_3,K_s) < cs^2/\log s$, our choices for s and t yield $q > An^{3/2}(\log n)^{1/2}$, where $A = 3^{-1}(6c)^{-1/2}$.

The proof of the upper bound is based on a probability theorem due to Lovász. This theorem was first used by Lovász and one of the authors in [7], and it has been employed by Spencer to obtain lower bounds for certain Ramsey numbers in [10]. A proof of the Lovász result and a clear presentation of the strategy of its application are given in the paper of Spencer and by referring to this paper the reader will have no difficulty in following the present argument. The needed result is contained in the proof of Theorem 2.1 of Spencer's paper [10]. It is expressed in purely arithmetic terms as follows. Let G be an arbitrary (n,q) graph and suppose that there exist positive numbers a, b, and P such that $P < 1$, $aP^3 < 1$, $b(1-P)^q < 1$,

$$\log a > 3NaP^3 + N^n b(1-P)^q \tag{3}$$

and

$$\log b > \tfrac{1}{2}Nn^2aP^3 + N^n b(1-P)^q. \tag{4}$$

Then $r(K_3,G) > N$. This is a typical application of the probability method. If (3) and (4) are satisfied and the edges of K_N are randomly two-colored with each edge being red with independent probability P, then there is a positive probability that <R> contains no K_3 and <B> contains no copy of G. In our application of this result, we introduce constants C_1 through C_4 by setting

$$a = C_1, \tag{5}$$

$$b = \exp(C_2 n \log n), \tag{6}$$

$$P = C_3 n^{-2/3}(\log n)^{1/3}, \tag{7}$$

$$q = [C_4 n^{5/3}(\log n)^{2/3}], \tag{8}$$

and we set $N = 2n$. It is easily verified that (3) and (4) are satisfied when n is sufficiently large, provided that C_1 through C_4 are chosen so that

$$C_1 > 1, \tag{9}$$

$$C_3C_4 > C_2 + 1, \tag{10}$$

and

$$C_2 > C_1C_3^{\ 3}. \tag{11}$$

These inequalities are satisfied by

$$C_1 = 1 + \varepsilon/2, \tag{12}$$

$$C_2 = (1 + \varepsilon)/2, \tag{13}$$

$$C_3 = 2^{-1/3}, \tag{14}$$

$$C_4 = 3\cdot 2^{-2/3} + \varepsilon, \tag{15}$$

where $\varepsilon > 0$. Thus, if $q = [(3\cdot 2^{-2/3} + \varepsilon)n^{5/3}(\log n)^{2/3}]$ and n is sufficiently large, every (n,q) graph G satisfies $r(K_3,G) > 2n$. □

4. *More General Results*

We now turn to the general problem of estimating $f(m,n)$ and $g(m,n)$. The arguments which will be used are basically the same as in the case of $m = 3$. For this reason, we shall give only those proofs which require some less than obvious modification of the corresponding argument for $m=3$.

LEMMA 3.1. *If* $m \geqq 3$ *and* G *is an* (n,q) *graph, then*

$$r(K_m,G) \leqq (n + 2q)^{\alpha},$$

where $\alpha = (m-1)/2$.

Proof. The proof is by induction on $m+n$. The result has been proved for all n where $m = 3$ (Lemma 1.2) and it is clearly true for all m where $n = 2$. With $n \geqq 3$ and $q \leqq \binom{n}{2}$ let G be an (n,q) graph. With $m \geqq 4$ set $p = [(n + 2q)^{\alpha}]$, where $\alpha = (m-1)/2$. With $|V| = p$ let (R,B) be an arbitrary two-coloring of $[V]^2$ and suppose that <R> contains no K_m and <B> contains no copy of G. Let x be a vertex of degree $d = \delta(G) \leqq [2q/n]$ in G and let $H = G-x$. By induction, $r(K_m,H) \leqq p$, so there must be an embedding $\sigma\colon V(H) \to V$ of H into <B> . Suppose that the neighborhood of x in G is $\{x_1,x_2,\ldots,x_d\}$ and consider the vertices $\sigma(x_1)$, $\sigma(x_2)$, ..., $\sigma(x_d)$.
Since none of the $p(n-1)$ vertices exterior to the copy of H can be adjacent to each of $\sigma(x_1),\sigma(x_2),\ldots,\sigma(x_d)$, in <B>, if $p - (n-1) \geqq (n-1) \geqq d(r-1) + 1$ then at least one of the vertices $\sigma(x_1)$

through $\sigma(x_d)$ must have degree at least r in <R> . Finally, if we set $r = r(K_{m-1},G)$ we reach a contradiction to our assumption that <R> contains no K_m and <B> contains no copy of G. The needed inequality here is $d(r-1) + n \leq p$. By induction, we have $r \leq (n + 2q)^{\alpha-1/2}$. Using this inequality together with $d \leq [2q/n]$, it is not difficult to establish the fact that $d(r-1) + n \leq p$ and so completes the proof. □

LEMMA 3.2. *Let G be a graph of order n and set $p = n-1+r(K_{m-1},G)$.*

(a) Suppose that there is a vertex of G which is a degree one and let H denote the graph obtained by deleting this vertex. If $K_p \to (K_m,H)$ then $K_p \to (K_m,G)$.

(b) Suppose that G contains a suspended path of length $m^2 - 3m + 4$ and let H denote the graph obtained from G by replacing this suspended path by one of length $m^2 - 3m + 3$. If $K_p \to (K_m,H)$ then $K_p \to (K_m,G)$.

Proof. With $|V| = p$ let (R,B) be a two-coloring of $[V]^2$ and suppose that <R> contains no K_m and <B> contains no copy of G. In view of the fact that $K_p \to (K_m,H)$ and $p = n - 1 + r(K_{m-1},G)$, we see that there is a copy of H in <B> and, disjoint from this, a K_{m-1} in <R>. Let $\sigma: V(H) \to V$ be an embedding of H into <B> .

(a) Let v denote the vertex of degree one and let u be the vertex to which it is adjacent in G. Then $\sigma(u)$ is adjacent in <R> to all the vertices exterior to the copy of H, in particular to all of the vertices of the K_{m-1} in <R>. This gives K_m in <R> and so a contradiction.

(b) With $t = m^2-3m+3$ let $\{x_1,x_2,\ldots,x_{t+1}\}$ be the suspended path in H which, if lengthened by one, produces G. For use in the following argument, define the successor operation π by $\pi(x_i) = x_{i+1}$, $i = 1,2,\ldots,t$. Since <R> contains no K_m, each vertex of $X = \{\sigma(x_1),\sigma(x_2),\ldots,\sigma(x_t)\}$ is adjacent to at least one vertex of the red K_{m-1} in <B> . Since $t = (m-2)(m-1) + 1$, this means that there is a vertex v, one of the vertices of the red K_{m-1}, such that $|X_B(v)| \geq m-1$. Consider the set Y consisting of v together with $\{\pi(x) \mid x \in X_B(v)\}$. Since, by assumption, the suspended path cannot be lengthened to produce a copy of G, it follows that Y spans a complete graph in <R>. Thus, we have found a K_m in $\langle R \rangle$ and so a contradiction. □

In addition to the preceeding lemmas, the theorem to follow relies on certain information concerning the classical Ramsey numbers. If $m \geq 3$ is fixed, there are constants c_1 and c_2 such that, for all sufficiently large values of n,

$$c_1 \left(\frac{n}{\log n}\right)^{\frac{m+1}{2}} < r(K_m, K_n) < c_2\, n^{m-1} \, \frac{\log \log n}{\log n}. \qquad (16)$$

Also, we shall assume a familiarity with Spencer's proof of the lower bound in (16) by means of the Lovász theorem [10]. Having given this orientation, we now give, without further discussion, the following theorem.

THEOREM 3. *With* $m \geq 3$ *fixed, set* $\alpha = 2/(m-1)$, $\beta = 4/(m+1)$, $\gamma = m/(m-1)$, $\delta = (m+2)/m$ *and* $\varepsilon = 1 - \binom{m}{2}^{-1}$. *Then, there are positive constants* A, B, C, D *such that, for all sufficiently large values of* n,

$$n + An^{\alpha} < f(m,n) < n + Bn^{\beta}(\log n)^{2}$$

and

$$Cn^{\gamma} < g(m,n) < Dn^{\delta}(\log n)^{\varepsilon}.$$

5. Question.

The bounds given in Theorem 2 and 3 leave much to be desired and, in this sense, there are many open questions left by this work. However, there is one particular question which should be mentioned. It is particularly annoying that we have not been able to answer this question. Does $f(n)/n \to \infty$ as $n \to \infty$?

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Department of Mathematics
Memphis State University
Memphis, Tennessee 38152
U.S.A.

P. Erdos
Mathematics Institute
Hungarian Academy of Sciences
Budapest, Hungary