Some old and new problems in various branches of combinatorics

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During my very long life I published very many papers which consist almost entirely of open problems in various branches of combinatorial mathematics (i.e. graph theory, combinatorial number theory, combinatorial geometry and combinatorial analysis). First of all I give a list (with I hope few (or no) omissions):

1. Problems and results in graph theory and combinatorial analysis, Proc. 5th British Comb. Conf. Aberdeen 1975, Cong. Numerantium XV, 169-192.

2. Problems and results on finite and infinite graphs, Recent Advances in Graph Theory, Proc. Symp. Prague 1974, Acad. Praha 1975, editor M. Fiedler, 183-190.

3. Extremal problems in graph theory, Theory of Graphs and Its Applications, Proc. Symp. Smolenice 1963, Acad. Press, New York 1964, M. Fiedler, editor, 29-36.

4. Problems and results on finite and infinite combinatorial analysis, Coll. Math. J Bolyai 10, Finite and Infinite Sets, Keszthely, Hungary 1973, 403-424.

5. Some unsolved problems in graph theory and combinatorial analysis, Comb. Math. and Its Applications, Oxford Conference 1969, Acad. Press, London 1977, 97-109.

6. Problems and results in chromatic graph theory, Proof Techniques in Graph Theory, Acad. Press, New York 1969, 27-35.

7. Extremal problems on graphs and hypergraphs, Hypergraph Seminar held at Columbus, Ohio 1972, Lecture Notes in Math. 411, Springer Verlag, 75-83.

8. Topics in combinatorial analysis, Proc. Second Conference in Combinatorics, Graph Theory and Computing (1971), 2-20.

9. Some new applications of probability methods to combinatorial analysis and graph theory, ibid. vol. 5 (1974), 39-54.

10. Some recent progress on extremal problems in graph theory, ibid. vol. 6 (1975), 3-14.

11. Some recent problems and results in graph theory, combinatorics and number theory, vol. 7 (1976), 3-14.

12. Problems and results in combinatorial analysis, vol. 8 (1977), 3-12.

13. Problems and results in combinatorial analysis and combinatorial number theory, vol. 9 (1978), 29-40.

14. Some extremal problems on families of graphs, Comb. Math. Proc. Int. Conf. Canberra (1977), Lecture Notes in Math. 686, 13-21.

15. P. Erdös and D.J. Kleitman, Extremal problems among subsets of a set, Discrete Math. 8 (1974), 289-294, see also Proc. Second Chapel Hill Conference 1970, 144-170.

16. Problems and results in graph theory and combinatorial analysis, Problèmes combinatoires et théorie des graph, Coll. Internationaux Centre Nat. Rech. Sci. 260, Orsay 1976, 127-129.

17. Problems and results in combinatorial analysis, Combinatorics, Proc. Symp. Pure Math XIX Amer. Math. Soc. 1971, 77-89.

18. Some old and new problems in combinatorial analysis, Proc. 2nd Intern. Conf. on Comb. Math. (New York 1978), to appear.

I refer to these papers by their number.

In the present paper I first of all give a progress report on some of my favourite problems and later I state a few recent problems and give some proofs of new results.

1. Hajnal and I stated in 1961 the following problem: Denote by $m_k(r)$ the smallest integer for which there is a k-chromatic r-uniform hypergraph of $m_k(r)$ edges. Determine or estimate $m_k(r)$ as accurately as possible. (In the older literature the edges of a two chromatic hypergraph were said to have property B. This concept was first used by Miller). We proved $m_3(r) \le \binom{2r-1}{r}$, more generally $m_k(r) \le \binom{(k-1)r-k+2}{r}$, also $m_3(2) = 3$, $m_3(3) = 7$. $m_3(4)$ is unknown. Later I proved

$$(1) \qquad \log 2.2^n < m_3(n) < c_1 n^2 2^n.$$

The lower bound in (1) was improved by W. Schmidt to $(1 - \frac{2}{n})2^n < m_3(n)$. I conjectured that $m_3(n)/2^n \to \infty$ and I further conjectured that to every r there is a c_r which tends to infinity with r so that if $\{A_1,\dots,A_t\}$, $|A_i| \ge r$, $1 \le i \le t$ is a three chromatic family of sets then

$$(2) \qquad \sum_{i=1}^{t} \frac{1}{2^{|A_i|}} > c_r.$$

Beck proved both these conjectures, he proved in fact that $m_3(n) > c\, n^{1/3}\, 2^n$. It would be interesting to get an asymptotic formula for $m_3(n)$ and for $m_k(n)$.

Following G. Dirac we call the family $\{A_1,\dots,A_t\}$ critical if it is three chromatic but if we omit any of the sets A_i the remaining family is two chromatic. Assume that $\{A_1,\dots,A_t\}$ is critical and $\max_{1\le i\le r} |A_i| \ge r$, then perhaps (2) remains true.

It is well known and easy to see that $m_2(k) = \binom{k}{2}$. In other words every k-chromatic (ordinary i.e. $r = 2$) graph has at least $\binom{k}{2}$ edges, equality only for the complete graph $K(k)$. The generalization for hypergraphs fails in view of $m_3(3) = 7$ and the smallest complete three chromatic hypergraph for $r = 3$

is $K_3(5)$ with 10 edges. I conjectured nearly twenty years ago that for $k > k_0(r)$ $m_r(k) = \binom{(k-1)r+1}{r}$, equality only for the complete r-graph $K_r((k-1)r+1)$. This conjecture is still open even for $r = 3$.

P. Erdös and A. Hajnal, On a property of families of sets, Acta Math. Acad. Sci. Hungar. 12 (1961), 87-123.

P. Erdös and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, Finite and Infinite Sets, Coll. Math. Soc. J Bolyai 10, Keszthely 1973, North Holland/Amer. Elsevier, 609-627.

J. Beck, On three-chromatic hypergraphs, Discrete Math. 29 (1978), 127-137.

2. Rényi and I conjectured that almost all graphs $G(n;[c\ n \log n])$ are Hamiltonian for sufficiently large c, and in fact we conjectured that this holds for every $c > \frac{1}{2}$. ($G(n;\ell)$ denotes a graph of n vertices and ℓ edges). Pósa proved our first conjecture and the second was recently proved by Komlós and Szemerédi, their proof will soon appear. At the end of 9, Spencer and I formulate the following conjecture: Let $G(n;t)$ be a random graph of n vertices and t edges with the added condition that we know that every vertex has valency ≥ 2. Is it then true that for $t = \varepsilon\, n\log n$ ($\varepsilon > 0$ arbitrary) almost all of these graphs are Hamiltonian. This was also proved by Komlós and Szemerédi. We further stated in 9 several other conjectures all (or most) of which were also proved by Komlós and Szemerédi.

I conjectured that there is an interesting function $f(c)$, $f(c) \to 1$ as $c \to \infty$, so that for $c > c_0$ almost all $G(n;c,n)$ have a path of length $> f(c)n$. Szemerédi disagreed, he believed that the expected length of the longest path is $o(n)$. Komlós and Szemerédi now proved that I was right, in fact they proved that my conjecture holds for every $c > 1$. $f(c) = 0$ for $c \leq \frac{1}{2}$ follows from results of Rényi and myself - the largest connected component in fact has size $o(n)$. The behaviour of $f(\ )$ for $\frac{1}{2} < c \leq 1$ is not yet cleared up. As I put it in a nutshell, Komlós and Szemerédi proved that I was right - I would have preferred it if I would have proved that Szemerédi was right!

Rényi and I proved that if $\ell_n = \frac{1}{2} n \log n + n f(n)$ where $f(n) \to \infty$ as $n \to \infty$ and n is even then almost all graphs $G(n;\ell_n)$ have a matching (or a linear factor). During my last visit to Jerusalem (1979II) Professor Shamir suprised me with a very pretty and perhaps difficult problem: Let $n = r\,m$, $|S| = n$. Consider the random r-uniform hypergraph of n vertices and ℓ_n edges. How large must ℓ_n be so that with probability tending to 1 our hypergraph should have an r-matching i.e. m vertex disjoint edges? For $r = 2$ Rényi and I completely solved this problem. For reasons which are hard to explain (maybe not so hard, the two greatest evils old age and stupidity are adequate explanations) I neglected to ask this beautiful and natural question. Many questions on random hypergraphs can be settled easily if one settled $r = 2$. Shamir's question seems to be different and we have no idea what to expect. It is not at all clear if $\ell_n = [n^{1+\varepsilon}]$ suffices for such a matching - note even for $r = 3$.

Joel Spencer and I recently proved the following conjecture of Burtin: Denote by $G^{(n)}$ the graph determined by the edges of the n-dimensional cube. $G^{(n)}$ has 2^n vertices and $n\,2^{n-1}$ edges. Choose each edge of $G^{(n)}$ with probability $\frac{1}{2}$. Then the resulting graph $G^{(n)}$ is connected with probability tending to $\frac{1}{e}$, (which is the probability that $G^{(n)}$ has an isolated vertex). Our paper with Spencer will soon appear.

Füredi recently studied the random subgraphs of the lattice graph of the plane (i.e. two lattice points are joined if they are neighbours), he obtained several interesting results which no doubt can be extended to higher dimensions.

Neither he nor Spencer and I could so far decide whether if one chooses edges in our graphs with increasing probability (i.e. one studies the "evolution of random subgraphs") then does the "giant component" suddenly appear. Rényi and I proved that this happens for the random subgraphs of $K(n)$ if the number of edges is $\frac{n}{2}(1+\varepsilon)$. This unexpected phenomenon was perhaps the most interesting result of Rényi and myself. One more problem on the evolution of random graphs which Rényi and I found very interesting but due to the untimely death of Rényi I never investigated: In our paper "On the evolution of random graphs" we studied the size and distribution function of the size of the largest component. Similarly one should study the second largest component. Of particular interest seems to be the maximum expected size of the second largest component of the evolving random graph.

P. Erdös and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hung. Acad. Sci. 5 (1960), 17-61 and On the existence of a factor of degree one of a connected random graph, Acta Math. Acad. Sci. Hungar. 17 (1966), 359-368.

L. Pósa, Hamiltonian cycles in random graphs, Discrete Math. 14 (1976), 359-364.

3. Denote by $f(n; G(k,\ell))$ be the smallest integer for which every $G(n;f(n;G(k,\ell)))$ contains G has a subgraph. These types of problems were started by P. Turán who determined $f(n;K(r))$ for every r. W. Brown, V.T. Sós, A. Rényi and I proved that

$$(1) \qquad f(n;c_4) = (\tfrac{1}{2} + o(1))n^{3/2}.$$

We conjectured that if p is a prime or a power of a prime, then

$$(2) \qquad f(p^2+p+1;c_4) = \tfrac{1}{2}(p^3+p) + p^2 + 1.$$

(2) was recently proved for infinitely many values of p by Füredi.

Reimann and E. Klein (Mrs. Szekeres) proved that there is a bipartite $G(n)$ which has no C_4 and has $(1+o(1))n^{3/2}/2\sqrt{2}$

edges. Reimann further observed that $\frac{1}{2\sqrt{2}}$ is best possible. The following problem has been unsettled for more than 10 years: Let $G(n)$ be a graph of n vertices which contains no C_3 and no C_4. Is it true that $G(n)$ can not have more than $(1+o(1))n^{3/2}/2\sqrt{2}$ edges? This problem is still open but Simonovits and I proved this, if we assume that $G(n)$ contains no C_4 and no C_5. Our proof will appear soon.

$$f(n;C_4) = \frac{1}{2}n^{3/2} + \frac{n}{4} + O(n^{1/2})$$

is conjectured in 10.

An old and nearly forgotten conjecture of mine states that if G is a bipartite graph of $[n^{2/3}]$ white and n black vertices and more than cn edges then it contains a C_6. It is easy to see that it contains a C_8. Clearly many generalizations and extensions are possible.

Simonovits and I published since 1958 resp. 1966 many papers on extremal problems on graphs, here I only stated a few very recent results. Nevertheless I want to call attention again to the old problem of Turán which dates back to 1940. Denote by $f(n;K^{(r)}(t))$ the smallest integer for which every uniform r-graph on n vertices and $f(n;K^{(r)}(t))$ hyperedges contains a $K^{(r)}(t)$. It is easy to see that

$$\lim_{n\to\infty} f(n;K^{(r)}(t))/\binom{n}{r} = c_{r,t} \tag{3}$$

exists. $c_{2,t} = 1 - \frac{1}{t-1}$ follows from Turán's theorem, but the value of $c_{r,t}$ is not known for a single $t > r > 2$. Túran conjectured that $f(3n;K^{(3)}(4)) = n^3 + 2n\binom{n}{2} + 1$ and $f(2n;K^{(3)}(4)) = n^2(n-1) + 1$. I offer 1,000 dollars for a proof of these conjectures and the determination of $c_{r,t}$ for all $t > r > 2$.

The second problem is one of my old problems: Let $G_i^{(r)}(n_i)$ be a sequence of r-graphs having n_i vertices $n_1 < n_2 < \dots$. The edge density of the sequence is the largest α for which there is a sequence $x_i \to \infty$ as $i \to \infty$ so that $G_i^{(r)}(n_i)$ has

P. Erdös, On some extremal problems on r-graphs, Discrete Math. 1 (1971), 1-6.

W.G. Brown, P. Erdös and M. Simonovits, On multigraph extremal problems, Coll. Internat. C.N.R.S. 260, Problèmes Combinatoire et Théorie des Graphes, 1972, 63-66, Extremal problems for directed graphs, J. Comb. Theory, 15 (1973), 77-93.

Last year the comprehensive book of B. Bollobás appeared which contains an immense material and very extensive literature on extremal problems in graph theory: Extremal Graph Theory, London Math. Soc. Monographs No. 11, Acad. Press 1978.

4. Let $|S| = n$, and consider families $F(n;k)$ of subsets $\{A_i\}$, of S, $|A_i| = k$. A family is called m-intersecting if every m of them have a non-empty intersection. Let $f(n;k,m)$ be the cardinal number of the largest family $F(n;k)$ such that every m-intersecting subfamily of it is necessarily $(m+1)$-interesting. Ko, Rado and I proved that for $n \geq 2k$ $f(n;k,1) = \binom{n-1}{k-1}$. I conjectured in 8 that

$$(1) \qquad f(n;k,2) = \binom{n-1}{k-1} \quad \text{for} \quad k \geq 3,\ n \geq \frac{3k}{2}.$$

Chvatal proved (1) for $k = 3$, more generally he proved

$$f(n;k,k-1) = \binom{n-1}{k-1} \quad \text{for} \quad k \geq 3,\ n \geq k+2$$

and conjectured

$$f(n;k,m) = \binom{n-1}{k-1} \quad \text{for} \quad 1 \leq m \leq k,\ n \geq \frac{m+1}{m}k.$$

$f(n;2,2) = [\frac{n^2}{4}]$ is the well known theorem of Turán (every $G(n;[\frac{n^2}{4}]+1)$ contains a triangle and $[\frac{n^2}{4}] + 1$ is best possible. A.Frankl just informs me that he proved (1) for $n > n_0(k)$ and proved that Chvatal's conjecture is asymptotically correct.

In 8 I state that Hajnal and I proved $n \to (c_1 \log n, \binom{4}{3})^3$ but $n \not\to [c_2 \log n, \binom{4}{3})^3$. I am afraid I was too optimistic, we only proved $n \to (c \log n/\log\log n, [{4 \atop 3}])^3$, the non arrow relation we really proved. Thus a gap of size $c \log\log n$ remains.

V. Chvatal, An extremal set intersection theorem, J. London Math. Soc. (2) 9 (1974), 355-359.

P. Erdös and A. Hajnal, On Ramsey-like theorems, problems and results, Proc. Conference Oxford 1972, Combinatorics, Inst. of Math. and Its Applications, 123-140.

5. Hajnal and I conjectured that for every k there is a graph G_k which has no $K(4)$ but if one colours its edges by k colours there always is a monochromatic triangle. This was proved for $k = 2$ by J. Folkman and for every k by Nesetril and Rödl. For infinite k our problem is open. Our simplest unsolved problem states: Let $c = \aleph_1$. Is there a graph G of power $\aleph_2$ which has no $K(4)$, but if one colours the edges of G by $\aleph_0$ colours, there always is a monochromatic triangle.

The graphs of Folkman, Nesetril and Rödl are enormous. This made me offer 100 dollars for the proof or disproof of the following problem: Is there a graph of at most 10^{10} vertices which has no $K(4)$, but for every colouring of its edges by two colours there always is a monochromatic triangle? I expect that such a graph exists.

I conjectured that for every k and r $(r > 3)$ there is a graph $G_{k,r}$ which not only does not contain a $K(r+1)$ but every two $K(r)$'s of which have at most two vertices in common and is such that for every colouring of the edges by k colours there always is a monochromatic $K(r)$. Nesetril and Rödl proved this conjecture too and also all the extensions for hypergraphs I could think of. In 5 I stated the following question of H. Dowker and myself: Is it true that every graph of girth greater than k can be directed in such a way that it contains no directed circuit and if one reverses the direction of any of its edges the resulting new digraph should also not contain a directed circuit?

Nesetril and Rödl answered this question affirmatively.

Hajnal and I conjectured that for every k and r there is an $f(k,r)$ so that if $\chi(G) > f(k,r)$ then G has a subgraph of girth greater than r and chromatic number greater than r. Rödl proved this for $r = 3$ and every finite k in a surprisingly simple way, but his estimation for $f(k,3)$ is probably far from being best possible.

In 1 I stated the following problem of M. Rosenfeld: Is it true that every finite graph G which has no triangle is a subgraph of the graph $G_{\sqrt{3}}$ whose vertices are the points of the unit sphere of Hilbert space and two points are joined if their distance is $> \sqrt{3}$. Alspach and Rosenfeld proved this if G is bipartite. Larman disproved the general case. In fact he showed that for every $\alpha > 2(\frac{2}{3})^{1/2}$ there is a triangle free G which can not be imbedded in G_α. He conjectures that this remains true for every $\alpha > 2^{1/2}$.

Very recently J. Spencer proved the following very attractive problem: A graph is said to have property R, if for every colouring of its edges so that two edges having a common vertex always have different colours, our graph has a rainbow circuit i.e. a circuit all whose edges have different colour.

Spencer's problem now states: Are there graphs of arbitrarily large girth having property R?

One would expect that this problem will yield to the probability method, but so far we had no success.

P. Erdös and A. Hajnal, On the chromatic number of graphs and set systems, Acta Math. Acad. Sci. Hungar. 16 (1966), 61-99.

J. Nesetril and V. Rödl, Partitions of finite relational and set systems, J. Comb. Theory (A) 22 (1977), 289-312. This deep paper has a very extensive list of references.

D.G. Larman, A triangle free graph which cannot be $\sqrt{3}$ imbedded in any Euclidean unit sphere, J. Comb. Theory Ser. A 24 (1978), 162-169.

A forthcoming book of R.L. Graham, B. Rothschild and J. Spencer on Ramsey theory will discuss the results of Nesetril and Rödl and of course many other topics not mentioned in this survey.

6. Now I state some new problems. First a few questions on block designs and finite geometries. I am not an expert in this field and I apologize in advance if some of "my" problems turn out to be well known.

I. Is it true that there is an absolute constant C so that every finite geometry contains a blocking set which meets every line in at most C points? More generally: Is it true that to every $\varepsilon > 0$ there is an absolute constant C_ε so that if $|S| = n$ and $A_i \subset S$, $1 \le i \le m(\varepsilon,n)$ is a system of subsets $\varepsilon n^{1/2} < |A_i| < n$ for which every pair of elements is contained in exactly one A_i, then there is a set $S_1 \subset S$ for which every i, $1 \le i \le m(\varepsilon,n)$

$$(1) \qquad 1 \le |A_i \cap S_1| < C_\varepsilon ?$$

By the way a well known theorem of de Bruijn and myself implies that $m(\varepsilon,n) \ge n$.

J. Freeman informs me that he and Bruen proved that there are infinitely many finite geometries for which there are blocking sets which satisfy (1).

II. Let $A_1,\dots,A_t$, $|A_i \cap A_j| \le 1$, $1 \le i < j \le t$ be an arbitrary family of sets. Is it then true that there is a finite geometry whose lines contain the $\{A_i\}$ as subsets? In other words there is a set S, $|S| = n^2 + n + 1$ and $B_1 \subset S$, $|B_i| = n + 1$, $1 \le i \le n^2 + n + 1$, $|B_i \cap B_j| = 1$, $1 \le i < j \le n_2^2 + n + 1$ and $A_i \subset B_i$ for $1 \le i \le t$. I have no idea how to attack this problem.

N.G. de Bruijn and P. Erdös, On a combinatorial problem, Nederl. Akad. Wetensch. Proc. 51 (1948), 1277-1279, see also Indigationes Math.

In 1 I conjectured: Let $1 < a_1 < \dots < a_k$ be a sequence of integers for which all the sums $a_i + a_j$ are distinct. Then there is a prime p and a perfect difference set $\bmod(p^2+p+1)$ which contains the a's. I expect that this conjecture will be more difficult.

Both conjectures can be slightly strengthened. First of all both conjectures could hold for all sufficiently large n for which a finite geometry exists, and in the second conjecture the perfect difference set $1 \le b_1 < \dots < b_{p+1} \le p^2 + p$ could satisfy $a_i = b_i$, $i = 1,2,\dots,k$. (I remind the reader that a perfect difference set satisfies $b_i - b_j$ represent all the non zero residues $\bmod(p^2+p+1)$ exactly once. For references on these questions consult 1, p. 189). I was just informed that "my" problem on finite geometries is not new but was stated more than 10 years ago by T. Evans.

III. Jean Larson and I proved that for $n \to \infty$ and $|S| = n$ there is a family of sets $A_i \subset S$, $|A_i| = (1+o(1))n^{1/2}$ $1 \le i \le n(1+o(1))$ and so that every pair of elements of S is contained exactly one A_i. The proof follows without much difficulty from the probability method and from $p_{k+1} - p_k < p_k^{1-c}$ (the p's are the sequence of primes).

Here are a few further older problems of mine on block designs. Is it true that there is an absolute constant c so that for every n there is a system $A_i \subset S$, $|S| = n$ so that every pair is contained in exactly one A_i and for every T the number of indices $|A_i| = T$ is less than $c\,n^{1/2}$? It is not hard to prove that if this is true then it is best possible i.e. there always is a c_1 so that for some T the number of indices $|A_i| = T$ is greater than $c_1 n^{1/2}$.

Let $x_1 \ge x_2 \ge \dots \ge x_n \ge 2$ be a sequence of integers for which there are sets A_i, $|A_i| = x_i$, $A_i \subset S$, $|S| = n$ and every pair is in exactly one A_i. I conjectured that the number of such sequences is greater than $n^{c_3 n^{1/2}}$. If correct it is easily seen to be best possible apart from the value of c_3.

On the other hand I conjectured that if the elements of S are points in the plane and the A_i are the lines joining the points then the number of the sequences $x_1 \geq x_2 \geq \ldots$ is less than $\exp c_4 n^{1/2}$. It is again easy to see that if true this is best possible apart from the value of c_4. I expect that this problem is much harder than the previous one.

7. I state now a few miscellaneous recent problems.

I. At the problem session of our meeting J. Dinitz stated the following problem of his which I find very interesting and challenging: Let $A_{i,j}$, $|A_{i,j}| = n$, $1 \leq i \leq n$, $1 \leq j \leq n$ be an arbitrary family of n^2 sets of size n. Prove that there always is an $x_{i,j} \in A_{i,j}$ which form an incomplete latin square (i.e. each element occurs at most once in every row and every column).

Observe that if the $A_{i,j}$ are all the same set the answer is trivially affirmative. Further if $|A_{i,j}| \geq n$ is replaced by $|A'_{i,j}| \geq 2n - 1$ the answer is also trivially affirmative. Also Gupta has some related perhaps more general problems.

II. Hanani and I proved that if $n = \binom{t}{2} + j < \binom{t+1}{2}$ and G_n is a graph of n edges then it contains the largest number of subgraphs $K(r)$ $(r \leq t)$ if G_n has $t + 1$ vertices $x_1, \ldots, x_{t+1}$ where $G(x_1, \ldots, x_t)$ is complete and x_{t+1} is joined to j of the x's. The proof is not difficult. During my last visit in the spring of 1979 Mr. Allon asked me if our theorem remains true for hypergraphs. I could not settle this interesting question which as far as I know is still open, but I asked: Let $G(r)$ be any graph of r vertices. Let G_n be the graph of n edges which contains the largest number of subgraphs isomorphic to our $G(r)$. Can one characterize G_n and in particular when is G_n our graph with Hanani. Allon has several interesting results on this problem which I hope he will publish soon. The simplest case which remained unsettled was if $G(r)$ is an odd cycle $(r > 3)$.

P. Erdős, On the number of complete subgraphs contained in certain graphs, Publ. Math. Inst. Acad. Sci. Hungar. 7 (1962), 461.

III. Let $G(n)$ be a graph of n vertices $m < n(1-\varepsilon)$. Assume that every set of m vertices of our $G(n)$ contains an edge - in other words the largest independent set in our (n) is less than m. $f(n;m)$ is the largest integer so that there always exist a subgraph of m vertices of our $G(n)$ which has a subgraph of m vertices of our $G(n)$ which has at least $f(n;m)$ edges i.e. $f(n;m)$ is the largest integer so that if every induced subgraph of m vertices contains an edge then there is a subgraph of m vertices and $f(n;m)$ edges. We have

(1) $$c_1 m < f(n;m) < c_2 m \log n.$$

The lower bound in (1) is almost immediate, the upper bound is given by the probability method. Is the upper bound best possible? If $m = c \log n$ the answer is "easily" seen to be affirmative (easily but not trivially). As far as I see the most interesting open question is whether the upper bound in (1) is best possible for $m = [n^{1/2}]$?

There are several possible modifications of this problem which might be of some interest. Let $G(n)$ be a graph where we either assume that $G(n)$ does not contain a $K(m)$ and the largest independent set is less than m, or we assume that $G(n)$ has $\frac{1}{2}\binom{n}{2}$ edges. Denote by $e_{max}(G_m(n))$ resp. $e_{min}(G_m(n))$ the largest respectively the smallest integer for which there is an induced subgraph of m vertices of $G(n)$ containing $e_{max}(G_m(n))$ respectively $e_{min}(G_m(n))$ edges. Put

$$A(n;m) = \min_{G(n)} (e_{max}(G_m(n)) - e_{min}(G_m(n))$$

where the minimum is taken over all admissible graphs. Determine or estimate $A(n;m)$ as accurately as possible and compare it to $f(n;m)$.

Many further generalizations and extensions seem promising e.g. for hypergraphs but here I do not pursue this subject any further.

IV. In 1 (p. 189) I stated the following conjecture. Let $G(n)$ be a graph of n vertices, assume that every subgraph of $[\frac{n}{2}]$ vertices contains more than $[\frac{n^2}{50}]$ edges than $G(n)$ contains a triangle. Unfortunately I got nowhere with this interesting conjecture. Further I asked: Denote by $f(\alpha,n)$ the smallest integer so that if every induced subgraph of $[\alpha n]$ vertices contains $f(\alpha,n)$ edges then $G(n)$ contains a triangle. Determine or estimate $f(\alpha,n)$ as well as possible. Perhaps the determination of

$$\lim_{n\to\infty} f(\alpha,n)/n^2 = g(\alpha)$$

is not hopeless. So far I had no success. By Turán's theorem $f(1,n) = [\frac{n^2}{4}] + 1$.

Perhaps the following new question may be of interest. Let $G(n)$ be a graph of $[\frac{n^2}{4}]$ edges which has no triangle. By Turán's theorem $\overline{G(n)}$ ($\overline{G}$ is the complementary graph of G) must then contain a $K([\frac{n+1}{2}])$. Assume now that $G(n)$ has no triangle and the largest clique of $\overline{G(n)}$ is $h(n)$. Determine or estimate $\max_G e(G(n))^* = F_{h(n)}(n)$. The problem makes sense only if $h(n)$ is large enough for such a graph to exist. I proved nearly 20 years ago that if $h(n) > c\, n^{1/2} \log n$ then such a graph exists. V.T. Sós and I proved that if $h(n) = [\frac{n}{k}]$ then (k fix $n \to \infty$)

$$(1) \qquad F_{[\frac{n}{k}]}(n) > \frac{c\, n^2}{k \log k}$$

On the other hand trivially $F_{[\frac{n}{k}]}(n) < \frac{n^2}{2k}$ (since every vertex has valency $< \frac{n}{k}$). It would seem likely that $F_{\frac{n}{k}}(n) = o(\frac{n^2}{k})$, and perhaps in fact (1) is best possible.

Another problem which is perhaps more closely related to my original problem states as follows: Let $f(n;T)$ be the largest integer for which there is a $G(n;f(n;T))$ which contains no

$^*e(G)$ denotes the number of edges of G.

triangle and for which every induced subgraph of $\frac{n}{2}$ vertices contains at least T edges. My conjecture stated in 1, if true, implies that the problem makes sense only for $T \le \frac{n^2}{50}$. For $T = \frac{n^2}{50}$ I think $f(n;T) = [\frac{n^2}{5}]$. Further generalizations are possible if instead of looking at induced subgraphs of $[\frac{n}{2}]$ vertices we insist that every induced subgraph of $[\alpha n]$ vertices has at least T edges, but we do not pursue this question at present.

The final remark. Consider the graphs $G(n;[c\,n^2])$ $o < c < \frac{1}{2}$, $n \to \infty$. It easily follows from the probability method that there is a $G(n;[c\,n^2])$ so that if $\ell_n/\log n \to \infty$ every induced subgraph of ℓ_n vertices has $(c+o(1))\ell_n^2$ edges. Thus in particular there is a $G(n;[cn^2])$ so that every induced subgraph of $[\frac{n}{2}]$ vertices has $(\frac{c}{4} + o(1))n^2$ edges. On the other hand I proved that if $n > n_0(r)$ then every such graph must contain a $K(r)$. I hope to return to this question in the near future (assuming that there is a future for me).

P. Erdös, Graph theory and probability II, Canad. J. Math. 13 (1961), 346-352. See also, P. Erdös and J. Spencer, Probabilistic methods in combinatorics, Acad. Press, New York, 1974.

V. Denote by $A(n;k)$ the least common multiple of the integers $n+1,\dots,n+k$. In my lecture at our meeting I stated the following conjecture: Let n_k be the smallest integer for which

$$A(n_k;k) > A(n_k+k;k).$$

Then $\lim_{k\to\infty} n_k/k = \infty$. During the meeting I found a simple proof of this conjecture which I now present. (In fact the proof is so simple that only the remark made in 2 explains that I did not ind it right away).

Let $rk \le n_k$ $(r+1)k$, r fixed k large. Denote by $P(u,v)$ the product of the primes in the interval (u,v). Clearly

$$(1) \qquad A(n_k,k) = P(1,k) \prod_{i=1}^{r} P(\frac{n_k}{i}, \frac{n_k+k}{i})\, P(k, \frac{n_k+k}{r+1})Q_1$$

where the factor Q_1 comes from the primes which divide $A_k(n_k,k)$ by an exponent greater than one. Similarly

$$(2) \qquad A(n_k+k,k) = P(1,k) \prod_{i=1}^{r+1} P(\frac{n_k+k}{i}, \frac{n_k+2k}{i})P(k, \frac{n_k+2k}{r+2})Q_2.$$

The prime factors of Q_1 and Q_2 are all less than $(n_k+2k)^{1/2}$. The contribution of each of them is less than $\exp(c(n_k+2k)^{1/2})$. Thus

$$(3) \qquad \max(Q_1,Q_2) < \exp(n_k+2k)^{1/2}(\log n_k+2k).$$

By the prime number theorem for fixed i and large k

$$(4) \qquad P(\frac{n_k}{i}, \frac{n_k+k}{i}) = \exp((1+o(1)\ \frac{k}{i}).$$

From (1), (2), (3) and (4) we easily obtain for fixed r and large k that

$$(5) \qquad \frac{A(n_k+k,k)}{A(n_k,k)} \neq > \exp((1-\varepsilon)\ \frac{k}{r+1})\ \exp(-(rk)^{1/2+\varepsilon})$$

(5) clearly gives $A(n_k+k,k) > A(n_k,k)$ for every fixed r and large k, which proves our assertion $\lim_{k\to\infty} n_k/k = \infty$. In fact using sharper forms of the prime number theorem this proof gives without difficulty that there is a $c > 0$ so that $n_k > k^{1+c}$. On the other hand I conjecture that

$$(6) \qquad \lim_{k\to\infty} n_k/k^2 = 0.$$

The proof of (6) probably will not be very difficult, but I have not done it as yet.

The following old conjecture of mine seems very much more difficult: Let $n + k < m$ then

$$(7) \qquad A(n,k) \neq A(m,k).$$

At the moment I do not see how to attack (7). I asked for solutions of

(8) $A(n;k) > A(m;\ell),\ \ell > k,\ m \geq n + k.$

The referee of one of my papers found two solutions. Selfridge showed that (8) has no solution for $k < 7$ but that there are 18 solutions for $k = 7$ and probably the number of solutions tends to infinity as k tends to infinity, but as far as I know it is not yet known that (8) has infinitely many solutions. Selfridge further observed that if (8) holds then $n < A(1;k-1)$. It would be interesting to estimate the largest solution of (8).

Two final questions: Put $h(n) = \max(\ell - k)$ where the maximum is extended over all solutions of (8), n fixed k,ℓ,m are variable. Estimate $h(n)$ from above and below as accurately as possible. The exact determination of $h(n)$ is of course hopeless. Probably $h(n) \to \infty$ as $n \to \infty$ but perhaps for every $\varepsilon > 0$ $h(n)/n^{\varepsilon} \to 0$.

Are there infinitely many values of n so that for every $k,\ 1 \leq k \leq n - 1$

(9) $A(n-k,k) < A(n,k)$?

e.g. $n = 10$ and $n = 12$ satisfy (9), but I expect that for large n the solutions of (9) will be rare and I do not see a proof that (9) has infinitely many solutions.

P. Erdös, Some unconventional problems in number theory, Math. Magazine 52 (1979), 67-70.