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# Bases and Nonbases of Square-Free Integers

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A basis is a set $A$ of nonnegative integers such that every sufficiently large integer $n$ can be represented in the form $n = a_i + a_j$ with $a_i, a_j \in A$. If $A$ is a basis, but no proper subset of $A$ is a basis, then $A$ is a minimal basis. A nonbasis is a set of nonnegative integers that is not a basis, and a nonbasis $A$ is maximal if every proper superset of $A$ is a basis. In this paper, minimal bases consisting of square-free numbers are constructed, and also bases of square-free numbers no subset of which is minimal. Maximal nonbases of square-free numbers do not exist. However, nonbases of square-free numbers that are maximal with respect to the set of square-free numbers are constructed, and also nonbases of square-free numbers that are not contained in any nonbasis of square-free numbers maximal with respect to the square-free numbers.

## 1. INTRODUCTION

Let $A = \{a_i\}$ be a set of nonnegative integers, and let $2A = \{a_i + a_j\}_{i,j=1}^{\infty}$ consist of all sums of two not necessarily distinct elements of $A$. The sum set $2A$ is an *asymptotic basis of order* 2, or, simply, a basis, if $2A$ consists of all but finitely many nonnegative integers. The basis $A$ is *minimal* if no proper subset of $A$ is a basis. Minimal bases were introduced by Stöhr [15], and have been studied by Erdös, Härtter, and Nathanson [2, 3, 5, 10, 11, 13]. If the set $A$ is not a basis, that is, if there are infinitely many numbers not of the form $a_i + a_j$ with $a_i, a_j \in A$, then $A$ is an *asymptotic nonbasis of order* 2, or, simply, a nonbasis. The nonbasis $A$ is *maximal* if every proper superset of $A$ is a basis. Maximal nonbases were introduced by Nathanson [13], and have been studied by Erdös, Hennefeld, Nathanson, and Turjányi [4 — 7, 12, 14, 17].



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Although minimal bases have been constructed, and also bases no subset of which is minimal, it is usually extremely difficult to decide whether a given "natural" basis for the integers does or does not contain a minimal basis. It is not known, for example, whether the basis consisting of the set of sums of two squares $\{m^2 + n^2\}_{m,n=0}^{\infty}$ contains a minimal basis. In this paper we consider the set $Q$ of squarefree numbers. Cohen [1], Estermann [8], Evelyn and Linfoot [9], and Subhankulov and Muhtarov [16] proved that $Q$ is an asymptotic basis of order 2, and have provided estimates for the number of representations of an integer as the sum of two square-free numbers.

We shall show that the set $Q$ contains a minimal basis. More generally, a basis $A$ is called *r-minimal* if, for any $a_1, a_2, ..., a_k \in A$, the set $A\backslash\{a_1, ..., a_k\}$ is a basis if $k < r$ but a nonbasis if $k \geqslant r$. The 1-minimal bases are precisely the minimal bases. We shall construct for every $r \geqslant 1$ an $r$-minimal asymptotic basis of order two consisting of square-free numbers. We shall also construct an $\aleph_0$-minimal basis of square-free numbers; that is, a basis $A$ such that $A\backslash A^*$ is a basis for every finite subset $A^* \subseteq A$, but $A\backslash A^*$ is a nonbasis for every infinite subset $A^* \subseteq A$. In particular, an $\aleph_0$-minimal basis is an asymptotic basis of order 2 that does not contain a minimal asymptotic basis.

There cannot exist a maximal asymptotic nonbasis of order 2 consisting entirely of square-free numbers. However, there do exist nonbases that are maximal with respect to $Q$. More generally, a nonbasis $A$ is called *s-maximal* with respect to $Q$ if, for any $q_1, q_2, ..., q_k \in Q\backslash A$, the set $A \cup \{q_1, ..., q_k\}$ remains a nonbasis for $k < s$, but becomes a basis for $k \geqslant s$. The 1-maximal nonbases with respect to $Q$ are precisely the nonbases that are maximal with respect to $Q$. For every $s \geqslant 1$ we shall construct a set of square-free numbers that is an $s$-maximal nonbasis with respect to $Q$, and also a nonbasis of square-free numbers that is not contained in any nonbasis of square-free numbers that is maximal with respect to $Q$.

*Notation.* $Q$ denotes the set of positive square-free integers, and $2 = p_1 < p_2 < p_3 < \cdots$ denote the prime numbers in ascending order. The cardinality of the set $S$ is $|S|$, and the relative complement of $T$ in $S$ is $S\backslash T$. By $[a, b]$ (resp. $[a, b)$) we denote the interval of integers $n$ such that $a \leqslant n \leqslant b$ (resp. $a \leqslant n < b$). For $n \geqslant 1$, let $S(n) = \{a \in [n/2, n] \cap Q \mid n - a \in Q\}$. Let $f(n)$ denote the number of representations of $n$ as a sum of two square-free numbers. Then $f(n) = 2\,|S(n)|$ if $n/2 \notin Q$ and $f(n) = 2\,|S(n)| - 1$ if $n/2 \in Q$, hence $|S(n)| \geqslant f(n)/2$ for $n \geqslant 1$. The integer part of the real number $x$ is denoted $[x]$.

## 2. Some Lemmas

Lemma 1. *Let $m_0, m_1, ..., m_t$ be pairwise relatively prime integers $\geqslant 1$, let $s$ be any integer, and let $R_i \subseteq [0, m_i - 1]$ for $i = 1, 2, ..., t$. Let*

$$Z = \{a \in [K+1, L] \mid a \equiv s \pmod{m_0} \text{ but } a \not\equiv r_i \pmod{m_i} \text{ for all } i = 1,\dots, t \text{ and } r_i \in R_i\}$$

*Then*

$$|Z| \geqslant \frac{L-K}{m_0} \prod_{i=1}^{t} \left(1 - \frac{|R_i|}{m_i}\right) - \prod_{i=1}^{t} (m_i - |R_i|).$$

*Proof.* Let $S_0 = \{s\}$ and let $S_i = [0, m_i - 1] \setminus R_i$ for $i = 1, 2,\dots, t$. Then $|S_i| = m_i - |R_i|$, and $a \not\equiv r_i \pmod{m_i}$ for all $r_i \in R_i$ if and only if $a \equiv s_i \pmod{m_i}$ for some $s_i \in S_i$. Let $m = m_0 m_1 \cdots m_t$. Since the moduli $m_i$ are pairwise relatively prime, the Chinese remainder theorem implies that there is a set $S \subseteq [0, m-1]$ with $|S| = |S_1| \cdots |S_t|$ such that $Z = \{a \in [K+1, L] \mid a \equiv s \pmod{m}$ for some $s \in S\}$. The interval $[K+1, L]$ contains $[(L-K)/m]$ complete sets of residues modulo $m$, each of which contains $|S|$ elements of $Z$. Therefore,

$$\begin{aligned} |Z| &\geqslant \left[\frac{L-K}{m}\right] |S| > \left(\frac{L-K}{m} - 1\right) |S| \\ &= \frac{L-K}{m_0} \prod_{i=1}^{t} \frac{|S_i|}{m_i} - \prod_{i=1}^{t} |S_i| \\ &= \frac{L-K}{m_0} \prod_{i=1}^{t} \left(1 - \frac{|R_i|}{m_i}\right) - \prod_{i=1}^{t} (m_i - |R_i|). \end{aligned}$$

LEMMA 2. *Let $f(n)$ denote the number of representations of $n$ as a sum of two square-free numbers. Then*

$$f(n) > n\left(\prod_{i=1}^{\infty} \left(1 - \frac{2}{p_i^2}\right) - \epsilon\right)$$

*for every $\epsilon > 0$ and all $n > n_0(\epsilon)$. In particular, the square-free numbers form an asymptotic basis of order 2.*

*Proof.* Let $\epsilon > 0$. Choose $p_t$ so large that $\sum_{i=t+1}^{\infty} 1/p_i^2 < \epsilon/4$. If $a \in [1, n]$ and $a \notin Q$ or $n - a \notin Q$, then $a \equiv 0 \pmod{p_i^2}$ or $a \equiv n \pmod{p_i^2}$ for some prime $p_i \leqslant n^{1/2}$. The number of $a \in [1, n]$ such that $a \equiv 0$ or $n \pmod{p_i^2}$ for some $p_i > p_t$ is at most

$$\sum_{p_t < p_i \leqslant n^{1/2}} 2\left(\left[\frac{n}{p_i^2}\right] + 1\right) < \frac{\epsilon}{2} n + 2n^{1/2}. \tag{1}$$

Let $Z = \{a \in [1, n] \mid a \not\equiv 0 \text{ or } n \pmod{p_i^2}$ for all $i = 1, 2,\dots, t\}$. Let $m_0 = s = 1$, let $m_i = p_i^2$ for $i = 1, 2,..., t$, and let $R_i$ consist of the least nonnegative residues of 0 and $n$ modulo $p_i^2$. Then $|R_i| = 1$ or 2. By Lemma 1,

$$|Z| \geqslant n \prod_{i=1}^{t} \left(1 - \frac{|R_i|}{p_i^2}\right) - \prod_{i=1}^{t} (p_i^2 - |R_i|)$$

$$> n \prod_{i=1}^{\infty} \left(1 - \frac{2}{p_i^2}\right) - \prod_{i=1}^{t} (p_i^2 - 1). \tag{2}$$

The number of $a \in [1, n]$ such that both $a \in Q$ and $n - a \in Q$ is precisely $f(n)$; estimates (1) and (2) imply that

$$f(n) > n \prod_{i=1}^{\infty} \left(1 - \frac{2}{p_i^2}\right) - \prod_{i=1}^{t} (p_i^2 - 1) - \frac{\epsilon}{2} n - 2n^{1/2}$$

$$> n \left(\prod_{i=1}^{\infty} \left(1 - \frac{2}{p_i^2}\right) - \epsilon\right)$$

for all $n > n_0(\epsilon)$. Since $\prod_{i=1}^{\infty} (1 - 2/p_i^2) > 0$, it follows that $f(n) > 0$ for $n$ sufficiently large, and so $Q$ is an asymptotic basis of order 2.

LEMMA 3. *Let $q_1, q_2,..., q_s$ be square-free numbers, and let $R_i$ consist of the least nonnegative residues of $q_1, q_2,..., q_s$ modulo $p_i^2$. Then the number of $w \leqslant n$ such that $w - q_1, w - q_2,..., w - q_s$ are simultaneously square-free is greater than*

$$n \left(\prod_{i=1}^{\infty} \left(1 - \frac{|R_i|}{p_i^2}\right) - \epsilon\right)$$

*for every $\epsilon > 0$ and all $n > n(\epsilon)$. In particular, the numbers $w - q_1,..., w - q_s$ are simultaneously square-free for arbitrarily large $w$.*

*Proof.* Since the $q_j$ are square-free, $0 \notin R_i$ and so $|R_i| \leqslant \min\{s, p_i^2 - 1\}$. Therefore, $\prod_{i=1}^{\infty} (1 - |R_i|/p_i^2) > 0$.

Let $q = \max\{q_1, q_2,..., q_s\}$. If $w \in [q + 1, n]$ and $w - q_j \notin Q$ for some $j = 1,..., s$, then $w \equiv q_j \pmod{p_i^2}$ for some prime $p_i < n^{1/2}$. For $\epsilon > 0$, choose $p_t$ so large that $\sum_{i=t+1}^{\infty} 1/p_i^2 < \epsilon/2s$. The number of $w \in [q + 1, n]$ such that $w \equiv q_j \pmod{p_i^2}$ for some $j = 1,..., s$ and $i > t$ is at most

$$\sum_{p_t < p_i < n^{1/2}} s \left(\left[\frac{n - q}{p_i^2}\right] + 1\right) < \frac{\epsilon}{2} n + sn^{1/2}. \tag{3}$$

Let $Z = \{w \in [q+1, n] \mid w \not\equiv q_j \pmod{p_i^2}$ for all $j = 1,..., s$ and $i = 1,..., t\}$. By Lemma 1,

$$|Z| \geqslant (n-q) \prod_{i=1}^{t} \left(1 - \frac{|R_i|}{p_i^2}\right) - \prod_{i=1}^{t} (p_i^2 - |R_i|). \tag{4}$$

Lemma 3 follows from estimates (3) and (4).

LEMMA 4. *Let $n \geqslant 1$, and let $p_j^2$ and $p_k^2$ be the two smallest primes such that neither $p_j^2$ nor $p_k^2$ divides $n$. Then $p_j p_k < c_1 \log^2 n$ for all $n > n_1$.*

*Proof.* Suppose $p_j < p_k$. Then $\prod_{i=1}^{k} p_i^2$ divides $np_j^2p_k^2$, and so $(\prod_{i=1}^{k} p_i)^2 \leqslant np_j^2p_k^2 < np_k^4$. By Chebyshev's theorem, $\theta(x) = \sum_{p \leqslant x} \log p > cx$ for some $c > 0$ and all $x \geqslant 2$. Exponentiating and squaring this inequality with $x = p_k$, we obtain

$$e^{2cp_k} < \left(\prod_{i=1}^{k} p_i\right)^2 < np_k^4.$$

But $p_k^4 < e^{cp_k}$ for all $k > t$, and so $e^{cp_k} < n$ for $k > t$. If $n > e^{cp_t}$, then $e^{cp_k} < n$ for all $k$. Therefore, $p_k < (1/c) \log n$ for all $n > n_1 = [e^{cp_t}]$. For $c_1 = 1/c^2$, we have

$$p_j p_k < p_k^2 < c_1 \log^2 n.$$

LEMMA 5. *Let $S(w) = \{a \in [w/2, w] \cap Q \mid w - a \in Q\}$. Let $A_u(w)$ consist of all square-free numbers $q > u$ except $q \in S(w)$, i.e. $A_u(w) = Q \backslash (S(w) \cup [1, u])$. Then $w \notin 2A_u(w)$. If $w > w^*$ and if $w > 8u + 4$, then $n \in 2A_u(w)$ for all $n \geqslant w/2$, $n \neq w$.*

*Proof.* If $w = q + q'$ with $q, q' \in Q$ and $q' \leqslant q$, then $w/2 \leqslant q \leqslant w$ and $q' = w - q \in Q$, hence $q \in S(w)$. Therefore, $q \notin A_u(w)$ and so $w \notin 2A_u(w)$.

By Lemma 2, there exists $0 < c_0 < c_2 = \prod_{i=1}^{\infty} (1 - 2/p_i^2)$ such that $f(n) > c_0 n$ for all $n > n_0$. If $w > c_0 n_0/2$ and $n > 2w/c_0$, then $f(n) > c_0 n > 2w$. But $A_u(w)$ is missing at most $w$ square-free integers, and so $n \in 2A_u(w)$ for all $n > 2w/c_0$.

Suppose that $w/2 \leqslant n \leqslant 2w/c_0$, and $n \neq w$. Then $w \leqslant |w(n-w)| < 2w^2/c_0$. Let $p_j, p_k$ be the two smallest primes such that neither $p_j^2$ nor $p_k^2$ divides $w(n-w)$. If $w > n_1$, then Lemma 4 implies that

$$p_j^2 p_k^2 < c_1^2 \log^4 |w(n-w)| < c_1^2 \log^4 (2w^2/c_0) \tag{5}$$

Let $Z^* = \{a \in [u+1, n-u-1] \mid a \equiv w \pmod{p_j^2}$ and $a \equiv n - w \pmod{p_k^2}\}$. Since $(p_j^2, p_k^2) = 1$, $Z^* = \{a \in [u+1, n-u-1] \mid a \equiv s \pmod{p_j^2 p_k^2}\}$

for some $s$. If $a \in Z^*$, then $w - a \equiv 0 \pmod{p_j^2}$ and so $a \notin S(w)$. Similarly, if $a \in Z^*$, then $w - (n - a) \equiv 0 \pmod{p_k^2}$ and so $n - a \notin S(w)$. Therefore, if $a \in Z^*$ and $a, n - a \in Q$, then $a, n - a \in A_u(w)$ and so $n \in 2A_u(w)$. Let $a \in Z^*$. If $a \notin Q$ or $n - a \notin Q$, then $a \equiv 0 \pmod{p_i^2}$ or $a \equiv n \pmod{p_i^2}$ for some prime $p_i \leqslant n^{1/2}$. By the choice of the primes $p_j$ and $p_k$, if $a \in Z^*$ then $a \equiv w \not\equiv 0, n \pmod{p_j^2}$ and $a \equiv n - w \not\equiv 0, n \pmod{p_k^2}$.

Let $0 < \epsilon < c_0 c_2/8$. Choose $p_t$ so large that $\sum_{i=t+1}^{\infty} 1/p_i^2 < \epsilon/2$. If $i \neq j, k$, then $(p_i^2, p_j^2 p_k^2) = 1$ and so $Z^*$ contains at most $2([(n - 2u - 1)/p_i^2 p_j^2 p_k^2] + 1)$ numbers $a$ such that $a \equiv 0, n \pmod{p_i^2}$. Therefore, the number of $a \in Z^*$ such that $a \equiv 0, n \pmod{p_i^2}$ for some prime $p_i > p_t$ is at most

$$\sum_{p_t < p_i \leqslant n^{1/2}} 2\left(\left[\frac{n - 2u - 1}{p_i^2 p_j^2 p_k^2}\right] + 1\right) < \frac{n}{p_j^2 p_k^2}\,\epsilon + 2n^{1/2}$$

$$< \frac{2\epsilon w}{c_0 p_j^2 p_k^2} + 2\left(\frac{2w}{c_0}\right)^{1/2} \tag{6}$$

Let $Z = \{a \in Z^* \mid a \not\equiv 0, n \pmod{p_i^2} \text{ for } i \leqslant t, i \neq j, k\}$. It follows from Lemma 1 and from $w > 8u + 4$ that

$$\begin{aligned} |Z| &\geqslant \frac{n - 2u - 1}{p_j^2 p_k^2} \prod_{\substack{i=1 \\ i \neq j,k}}^{t} \left(1 - \frac{2}{p_i^2}\right) - \prod_{\substack{i=1 \\ i \neq j,k}}^{t} (p_i^2 - 1) \\ &> c_2 \frac{w/2 - 2u - 1}{p_j^2 p_k^2} - c_3 \\ &> \frac{c_2 w}{4 p_j^2 p_k^2} - c_3 \end{aligned} \tag{7}$$

Combining estimates (5), (6), and (7), we conclude that the number of $a \in Z^*$ with $a, n - a \in Q$ is at least

$$\begin{aligned} &\frac{c_2 w}{4 p_j^2 p_k^2} - c_3 - \frac{2\epsilon w}{c_0 p_j^2 p_k^2} - 2\left(\frac{2w}{c_0}\right)^{1/2} \\ &> \frac{2}{c_0}\left(\frac{c_0 c_2}{8} - \epsilon\right)\frac{w}{p_j^2 p_k^2} - 2\left(\frac{2w}{c_0}\right)^{1/2} - c_3 \\ &> 1 \end{aligned}$$

for $w > n_2$. Therefore, if $w > w^* = \max\{c_0 n_0/2, n_1, n_2\}$ and $w > 8u + 4$, then $n \in 2A_u(w)$ for all $n \geqslant w/2, n \neq w$.

LEMMA 6. *Let $W = \{w_k\}_{k=0}^{\infty}$ be a sequence of integers such that $w_0 > w^*$ and $w_k > 8w_{k-1} + 4$ for all $k \geqslant 1$. Let $S(w_k) = \{a \in [w_k/2, w_k] \cap Q \mid w_k - a \in Q\}$. Let $A(W) = Q \setminus \bigcup_{k=1}^{\infty} S(w_k)$. Then $w_k \notin 2A(W)$ for $k \geqslant 1$, but $n \in 2A(W)$*

*for all* $n \geqslant w_1/2, n \notin W$. *If* $Q^* \subseteq [1, w_t] \cap Q$, *then* $w_k \notin 2(A(W) \cup Q^*)$ *for all* $k > t$, *but* $n \in 2(A(W)\backslash Q^*)$ *for all* $n \geqslant w_{t+1}/2, n \notin W$.

*Proof.* If $A \subseteq Q$ and $A \cap S(w_k) = \varnothing$, then $w_k \notin 2A$. Since $(A(W) \cup Q^*) \cap S(w_k) = \varnothing$ for all $k > t$, it follows that $w_k \notin 2(A(W) \cup Q^*)$ for all $k > t$. In particular, $w_k \notin 2A(W)$ for $k \geqslant 1$.

By Lemma 5, the sum set $2A_{w_{k-1}}(w_k)$ contains all $n \geqslant w_k/2, n \neq w_k$. If $k > t$ and $n \in [w_k/2, w_{k+1}/2), n \neq w_k$, then $n = q + q'$ where

$$\begin{aligned} q, q' &\in A_{w_{k-1}}(w_k) \cap [w_{k-1} + 1, w_{k+1}/2) \\ &= Q \cap [w_{k-1} + 1, w_{k+1}/2)\backslash S(w_k) \subseteq A(W)\backslash Q^* \end{aligned}$$

and so $n \in 2(A(W)\backslash Q^*)$ for all $n \geqslant w_{t+1}/2, n \notin W$. In particular, $n \in 2A(W)$ for all $n \geqslant w_1/2, n \notin W$.

## 3. Minimal Asymptotic Bases

Theorem 1. *There exists an* $\aleph_0$*-minimal asymptotic basis of order* 2 *consisting of square-free integers.*

*Proof.* We shall construct an increasing sequence of finite sets $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ whose union $A = \bigcup_{k=0}^{\infty} A_k$ is an $\aleph_0$-minimal basis of square-free numbers. Let $w_0 > w^*$, and let $A_0 = Q \cap [1, w_0]$. For any $q_1 \in A_0$, choose $w_1 > 8w_0 + 4$ such that $w_1 - q_1 \in Q$. Then $w_1 - q_1 \in [w_1/2, w_1]$, hence $w_1 - q_1 \in S(w_1)$. Let

$$A_1 = A_0 \cup \{[w_0 + 1, w_1] \cap Q\backslash S(w_1)\} \cup \{w_1 - q_1\}.$$

If $q_{k-1}, w_{k-1}$, and $A_{k-1}$ have been determined, let $q_k \in A_{k-1}$ and choose $w_k > 8w_{k-1} + 4$ such that $w_k - q_k \in Q$. Then $w_k - q_k \in S(w_k)$. Let

$$A_k = A_{k-1} \cup \{[w_{k-1} + 1, w_k] \cap Q\backslash S(w_k)\} \cup \{w_k - q_k\}.$$

This determines $q_k, w_k$, and $A_k$ for all $k$. Set $A = \bigcup_{k=0}^{\infty} A_k$.

The sequence $W = \{w_k\}_{k=0}^{\infty}$ satisfies $w_0 > w^*$ and $w_k > 8w_{k-1} + 4$ for all $k \geqslant 1$. Moreover, $A = A(W) \cup \{w_k - q_k\}_{k=1}^{\infty}$. The numbers $q_k \in A_k$ were chosen arbitrarily. Here is the crucial part of the construction: Choose the numbers $q_k$ so that, if $a \in A$, then $a = q_k$ for precisely one $k$. Then $A$ will be an $\aleph_0$-minimal basis.

Let $Q^*$ be a finite subset of $A$, say, $Q^* \subseteq [1, w_t]$. Since $A(W) \subseteq A$, Lemma 6 implies that $2(A\backslash Q^*)$ contains all sufficiently large $n \notin W$. If $w_k \in W$ and $k \geq 1$, then $w_k = (w_k - q_k) + q_k$ is the *unique* representation of $w_k$ as a sum of two elements of $A$. If $k > t$, then $w_k - q_k \in A\backslash Q^*$. Since each $a \in A$ is chosen only once as a number $q_k$, and since $Q^*$ is finite, it follows that, for $k$ sufficiently large, $q_k \in A\backslash Q^*$ and $w_k \in 2(A\backslash Q^*)$. Therefore, $A\backslash Q^*$ is an asymptotic basis for every finite subset $Q^*$ of $A$. But if $Q^*$ is an infinite subset of $A$, then $q_k \in Q^*$ for infinitely many $k$, hence $w_k \notin 2(A\backslash Q^*)$ for infinitely many $k$, hence $A\backslash Q^*$ is an asymptotic nonbasis.

COROLLARY. *There exists an asymptotic basis of order 2 consisting of square-free numbers that does not contain a minimal asymptotic basis of order 2.*

THEOREM 2. *For every $r = 1, 2, 3,\ldots$, there exists an $r$-minimal basis of order 2 consisting of square-free numbers.*

*Proof.* We shall construct an increasing sequence of finite sets of square-free integers $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$ such that $A = \bigcup_{k=0}^{\infty} A_k$ is an $r$-minimal basis. Choose $w_0 > w^*$ sufficiently large that $A_0 = Q \cap [1, w_0]$ contains at least $r$ numbers. Choose distinct integers $q_1^{(1)}, q_1^{(2)},\ldots, q_1^{(r)} \in A_0$. By Lemma 3, there exists $w_1 > 8w_0 + 4$ such that $w_1 - q_1^{(j)} \in Q$ for $j = 1,\ldots, r$. Let

$$A_1 = A_0 \cup \{[w_0 + 1, w_1] \cap Q\backslash S(w_1)\} \cup \{w_1 - q_1^{(j)}\}_{j=1}^{r}.$$

Suppose that the numbers $q_{k-1}^{(j)}$, $w_{k-1}$ and the set $A_{k-1}$ have been determined. Choose distinct integers $q_k^{(1)},\ldots, q_k^{(r)} \in A_{k-1}$. By Lemma 1, there exists $w_k > 8w_{k-1} + 4$ such that $w_k - q_k^{(j)} \in Q$ for $j = 1,\ldots, r$. Let

$$A_k = A_{k-1} \cup \{[w_{k-1} + 1, w_k] \cap Q\backslash S(w_k)\} \cup \{w_k - q_k^{(j)}\}_{j=1}^{r}.$$

This determines sets $A_k$ for all $k$. Let $A = \bigcup_{k=0}^{\infty} A_k$.

The sequence $W = \{w_k\}_{k=0}^{\infty}$ satisfies $w_0 > w^*$ and $w_k > 8w_{k-1} + 4$ for all $k \geq 1$, and $A = A(W) \cup (\bigcup_{k=1}^{\infty} \{w_k - q_k^{(j)}\}_{j=1}^{r})$. Moreover, $A \cap S(w_k) = \{w_k - q_k^{(j)}\}_{j=1}^{r}$. The numbers $q_k^{(1)},\ldots, q_k^{(r)} \in A_{k-1}$ can be chosen arbitrarily for each $k$. Choose them in such a way that *every* set $a_1,\ldots, a_r$ is chosen infinitely often for $q_k^{(1)},\ldots, q_k^{(r)}$; that is, if $Q^* \subseteq A$ and $|Q^*| = r$, then $Q^* = \{q_k^{(j)}\}_{j=1}^{r}$ for infinitely many $k$. Then $A$ will be an $r$-minimal basis.

Let $Q^*$ be a finite subset of $A$. Since $A(W) \subseteq A$, Lemma 6 implies that $2(A\backslash Q^*)$ contains all sufficiently large $n \notin W$. If $w_k \in W$, $k \geq 1$, then $w_k$ has exactly $r$ representations $w_k = (w_k - q_k^{(j)}) + q_k^{(j)}$ as a sum of two elements of $A$. If $|Q^*| < r$, then not all of these representations are destroyed, and so $w_k \in 2(A\backslash Q^*)$ for all $k \geq 1$. But if $|Q^*| = r$, then $Q^* = \{q_k^{(j)}\}_{j=1}^{r}$ for infinitely many $k$, hence $w_k \notin 2(A\backslash Q^*)$ for infinitely many $k$. Therefore, $A\backslash Q^*$ is a basis if and only if $|Q^*| < r$, and so $A$ is an $r$-minimal basis.

## 4. Maximal Asymptotic Nonbases

Theorem 3. *There does not exist a maximal asymptotic nonbasis of order 2 consisting of square-free numbers.*

*Proof.* Let $A \subseteq Q$ be a nonbasis. Let $n_1 < n_2 < n_3 < \cdots$ be the infinite sequence of numbers not belonging to $2A$. If $A$ is maximal, then $n_i - b \in A$ for every $b \notin A$ and $i$ sufficiently large. Choose $b$ so that $b \equiv 0 \pmod{2^2}$, $b \equiv 1 \pmod{3^2}$, $b \equiv 2 \pmod{5^2}$, and $b \equiv 3 \pmod{7^2}$. Then, $b, b-1, b-2, b-3$ are non-square-free numbers, hence $b - j \notin A$ for $j = 0, 1, 2, 3$, and so $n_i - b + j \in A$ for $j = 0, 1, 2, 3$ and all $i$ sufficiently large. But this is impossible, since there do not exist four consecutive square-free integers.

Theorem 4. *For every $s \geqslant 1$ there exists an asymptotic nonbasis of order 2 consisting of square-free numbers that is s-maximal with respect to the square-free numbers.*

*Proof.* Let $w_0 > w^*$, and choose $w_1 > 8w_0 + 4$ so that $|S(w_1)| \geqslant s$. Let $A_1 = [1, w_1] \cap Q \backslash S(w_1)$. Then $B_1 = [1, w_1] \cap Q \backslash A_1 = S(w_1)$. Choose $Q_2 \subseteq B_1$ with $|Q_2| = s - 1$. By Lemma 3, there exists $w_2 > 8w_1 + 4$ such that $w_2 - b \in Q$ for all $b \in B_1$. Let

$$A_2 = A_1 \cup \{[w_1 + 1, w_2] \cap Q \backslash S(w_2)\} \cup \{w_2 - b\}_{b \in B_1 \backslash Q_2}.$$

Suppose that $w_{k-1}$ and $A_{k-1}$ have been determined. Let $B_{k-1} = [1, w_{k-1}] \cap Q \backslash A_{k-1}$. Choose $Q_k \subseteq B_{k-1}$ with $|Q_k| = s - 1$. By Lemma 3, there exists $w_k > 8w_{k-1} + 4$ such that $w_k - b \in Q$ for all $b \in B_{k-1}$. Let

$$A_k = A_{k-1} \cup \{[w_{k-1} + 1, w_k] \cap Q \backslash S(w_k)\} \cup \{w_k - b\}_{b \in B_{k-1} \backslash Q_k}.$$

This determines sets $A_k$ for all $k$. Let $A = \bigcup_{k=1}^{\infty} A_k$.

The sequence $W = \{w_k\}_{k=0}^{\infty}$ satisfies $w_0 > w^*$ and $w_k > 8w_{k-1} + 4$ for all $k \geqslant 1$, and the set $A$ has the form

$$A = A(W) \cup \left( \bigcup_{k=2}^{\infty} \{w_k - b\}_{b \in B_{k-1} \backslash Q_k} \right).$$

Then $S(w_k) \cap A = \{w_k - b\}_{b \in B_{k-1} \backslash Q_k}$, but $b \notin A$ whenever $w_k - b \in A$, hence $w_k \notin 2A$ for all $k \geqslant 2$. Therefore, $A$ is an asymptotic nonbasis of order 2.

The $(s-1)$-element sets $Q_k \subseteq B_{k-1}$ can be chosen arbitrarily. Choose them in such a way that every $(s-1)$-element subset of $B = Q \backslash A$ is chosen as a $Q_k$ *infinitely often*. Then $A$ will be $s$-maximal with respect to $Q$.

Since $A(W) \subseteq A$, the sum set $2A$ contains all $n \geqslant w_1/2$, $n \notin W$. Let $Q^*$ be a finite subset of $B = Q \backslash A$, say, $Q^* \subseteq [1, q_t]$. Then $Q^* \subseteq B_{k-1}$ for every $k > t$. Suppose $|Q^*| \geqslant s$. If $k > t$, then $A$ contains all but $s - 1$ elements of the form $w_k - b$ with $b \in B_{k-1}$, and so $w_k - b \in A$ for some $b \in Q^*$, hence $w_k \in 2(A \cup Q^*)$. But if $| Q^* | = s - 1$, then $Q^* = Q_k$ for infinitely many $k$, and for each such $k$ the numbers in $\{w_k - b\}_{b \in Q^*}$ do not belong to $A$, hence $w_k \notin 2(A \cup Q^*)$. Therefore, if $Q^* \subseteq Q \backslash A$, then $A \cup Q^*$ is a nonbasis if and only if $| Q^* | < s$, and so $A$ is a nonbasis that is $s$-maximal with respect to $Q$.

THEOREM 5. *There exists an asymptotic nonbasis of order 2 consisting of square-free numbers that is not contained in any nonbasis of square-free numbers that is maximal with respect to the square-free numbers.*

*Proof.* By Lemma 2, there exists $c_0 > 0$ and $n_0$ such that $| S(w) | \geqslant f(w)/2 > c_0 w/2$ for all $w > n_0$. Let $w_0 > \max\{w^*, n_0\}$, and let $w_1 > \max\{8w_0 + 4, 4w_0/c_0\}$. Let $A_1 = [1, w_1] \cap Q \backslash S(w_1)$. Since $| S(w_1) | \geqslant f(w_1)/2 > c_0 w_1/2 > 2w_0$, there exists $q_1 \in S(w_1) \cap [(w_1 + 1)/2, w_1 - w_0 - 1]$. Then $w_1 - q_1 \in A_1$.

Suppose numbers $w_i$ and $q_i$ and sets $A_i \subseteq [1, w_i] \cap Q$ have been determined for all $i < k$. Let $B_{k-1} = [1, w_{k-1}] \cap Q \backslash A_{k-1}$. By Lemma 3, there exists $w_k > \max\{8w_{k-1} + 4, 4w_{k-1}/c_0\}$ such that $w_k - b \in Q$ for all $b \in B_{k-1}$. Let

$$A_k = A_{k-1} \cup \{[w_{k-1} + 1, w_k] \cap Q \backslash S(w_k)\} \cup \{w_k - b \mid b \in B_{k-1} \backslash \{q_i\}_{i=1}^{k-1}\}.$$

Since $f(w_k)/2 > c_0 w_k/2 > 2w_{k-1}$, there exists $q_k \in [(w_k + 1)/2, w_k - w_{k-1} - 1] \cap S(w_k)$. Then $w_k - q_k \in A_k$. Since $w_k - b \in [w_k - w_{k-1}, w_k]$ for all $b \in B_{k-1}$, it follows that $w_k - q_i \neq q_k$ for all $i < k$. This determines $w_k$, $q_k$, and $A_k$ for all $k$. Let $A = \bigcup_{k=1}^{\infty} A_k$.

The sequence $W = \{w_k\}_{k=0}^{\infty}$ satisfies $w_0 > w^*$ and $w_k > 8w_{k-1} + 4$ for all $k \geqslant 1$. Moreover,

$$A = A(W) \cup \bigcup_{k=2}^{\infty} \{w_k - b \mid b \in B_{k-1} \backslash \{q_i\}_{i=1}^{k-1}\}.$$

Then $n \in 2A$ for all $n \geqslant w_1/2$, $n \notin W$, but $w_k \notin 2A$ for all $k \geqslant 1$. Let $B = Q \backslash A$. If $b \in B$, say, $b \in [1, w_t] \cap Q \backslash A = B_t$, and if $b \neq q_i$ for any $i \leqslant t$, then $w_k - b \in A$ for all $k > t$, and so $w_k = (w_k - b) + b \in 2(A \cup \{b\})$ for all $k > t$. Thus, $A \cup \{b\}$ is a basis. It follows that if $Q^* \subseteq B$ and $A \cup Q^*$ is a nonbasis, then $Q^* \subseteq \{q_i\}_{i=1}^{\infty}$.

Suppose $Q^* \subseteq \{q_i\}_{i=1}^{\infty}$ and $w_k \in 2(A \cup Q^*)$. Since $w_k - q_i \notin Q^*$ for $i = 1, 2, \ldots, k - 1$, but $w_k - q_k \in A$, it follows that the only possible representation of $w_k$ as a sum of two elements of $A \cup Q^*$ is $w_k = (w_k - q_k) + q_k$. Therefore, $w_k \in 2(A \cup Q^*)$ if and only if $q_k \in Q^*$. Consequently, $A \cup Q^*$ is a nonbasis if and only if $Q^*$ does not contain infinitely many elements of $\{q_i\}_{i=1}^{\infty}$. Since there is no such maximal set $Q^*$, the nonbasis $A$ is not contained in a nonbasis of square-free numbers that is maximal with respect to the square-free numbers.

## 5. Problems

An asymptotic basis $A$ of nonnegative integers is an *infinitely oscillating basis* if it oscillates from basis to nonbasis to basis to nonbasis... as random elements are successively removed from, then added to, the set $A$. Equivalently, $A$ is an infinitely oscillating basis if, for all finite sets $A^*$, $B^*$ of nonnegative integers such that $A^* \subsetneq A$ and $B^* \cap A = \varnothing$, the set $(A \backslash A^*) \cup B^*$ is a basis if $|B^*| \geqslant |A^*|$ and a nonbasis if $|B^*| < |A^*|$. Similarly, an asymptotic nonbasis $A$ is an *infinitely oscillating nonbasis* if $A \cup \{b\}$ is an infinitely oscillating basis for every nonnegative integer $b \notin A$. Erdös and Nathanson [6] proved that there exist infinitely oscillating bases and nonbases. Moreover, they constructed a partition of the nonnegative integers into two sets $A$ and $B$ such that $A$ is an infinitely oscillating basis and $B$ is an infinitely oscillating nonbasis. Does there exist an infinitely oscillating basis of square-free numbers? That is, does there exist $A \subseteq Q$ such that, for all finite sets $A^* \subsetneq A$ and $B^* \subsetneq Q \backslash A$, the set $(A \backslash A^*) \cup B^*$ is a basis if and only if $|B^*| \geqslant |A^*|$? Is there a partition of the square-free numbers into two sets $A$ and $B = Q \backslash A$ such that $A$ is an infinitely oscillating basis and $B$ is an infinitely oscillating nonbasis?

A set $A$ of nonnegative integers is an asymptotic basis of order $h$ if every sufficiently large integer is the sum of $h$ terms of $A$; otherwise, $A$ is an asymptotic nonbasis of order $h$. Do there exist minimal bases and maximal nonbases of square-free numbers of orders $h > 2$?

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