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THE SIZE RAMSEY NUMBER

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Abstract

Let $\mathcal{C}$ denote the class of all graphs G which satisfy $G \to (G_1, G_2)$. As a way of measuring minimality for members of $\mathcal{C}$, we define the *size Ramsey number* $\hat{r}(G_1, G_2)$ by

$$\hat{r}(G_1, G_2) = \min_{G \in \mathcal{C}} |E(G)|.$$

We then investigate various questions concerned with the asymptotic behaviour of $\hat{r}$.

1. Introduction

Ramsey's theorem has inspired many striking and difficult problems. In this paper, we intend to add to this list of problems and to solve a few of the problems so introduced.

In its application to graphs, Ramsey's theorem is concerned with the assertion $G \to (G_1, G_2)$, the meaning of which is that in every partition (E_1, E_2) of $E(G)$, either $\langle E_1 \rangle \supseteq G_1$ or $\langle E_2 \rangle \supseteq G_2$. Given G_1 and G_2, we may define the class of graphs $\mathcal{C} = \mathcal{C}(G_1, G_2)$ by

$$\mathcal{C} = \{G \mid G \to (G_1, G_2)\}.$$

Ramsey's theorem establishes that $\mathcal{C}$ is non-empty by proving that for every m and n there is a minimum integer $r = r(K_m, K_n)$ such that $K_r \to (K_m, K_n)$. It follows that for every pair of graphs G_1, G_2, there is a *Ramsey number* $r(G_1, G_2)$, which is the smallest integer r such that $K_r \to (G_1, G_2)$. The Ramsey number can be viewed as a measure of minimality for members of the class $\mathcal{C} = \mathcal{C}(G_1, G_2)$ in that, as an equivalent definition, we may take

$$r(G_1, G_2) = \min_{G \in \mathcal{C}} |V(G)|.$$

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We now wish to introduce the idea of measuring minimality with respect to *size* rather than *order*. Accordingly, we define the *size Ramsey number* $\hat{r}(G_1, G_2)$ by

$$\hat{r}(G_1, G_2) = \min_{G \in C} |E(G)|.$$

For the purpose of comparing r and $\hat{r}$, we define $\hat{R}(G_1, G_2)$ by

$$\hat{R}(G_1, G_2) = \binom{r(G_1, G_2)}{2}.$$

From their definitions, we know that $\hat{r}(G_1, G_2) \leq \hat{R}(G_1, G_2)$. In the diagonal case $G_1 = G_2 = G$, the symbols $H \to (G)$, $r(G)$ and $\hat{r}(G)$ will have their obvious meanings. In this paper, most of our concern will be with the diagonal case.

There are two preliminary questions concerning the size Ramsey number which should be answered before posing others. The second of these two questions is best expressed in terms of the following definition, the purpose of which is to give a precise meaning to the idea that $\hat{r}(G)$ may be "significantly" less than $\hat{R}(G)$.

DEFINITION. *Let $\{G_n\}$ be an infinite sequence of graphs. Then $\{G_n\}$ is called an o-sequence if $\hat{r}(G_n) = o(\hat{R}(G_n))$ $(n \to \infty)$.*

The two questions are:

(i) Do there exist graphs G_1, G_2 such that $\hat{r}(G_1, G_2) = \hat{R}(G_1, G_2)$?

(ii) Do there exist o-sequences?

Question (i) is answered by the following theorem. This result is due to CHVÁTAL (personal communication).

THEOREM 1. *For all values of m and n, $\hat{r}(K_m, K_n) = \hat{R}(K_m, K_n)$. Moreover, if G is a connected graph of size $\leq \hat{R}$ such that $G \to (K_m, K_n)$, then $G \cong K_r$.*

PROOF. Let us first make an observation which provides the basic idea for the proof. Let (E_1, E_2) be a two-colouring of a graph G and suppose that u and v are two non-adjacent vertices of G. Consider the induced colourings of the two graphs $G - u$ and $G - v$. If, in both cases, $\langle E_1 \rangle \not\supseteq K_m$ and $\langle E_2 \rangle \not\supseteq K_n$, then the same is true in G. The reason is very simple. Any assumed monochromatic complete graph in the two-colouring of G cannot contain both u and v, since these two vertices are not adjacent. Hence, any such monochromatic complete graph would appear in the induced two-colourings of either $G - u$ or $G - v$. Clearly, this observation is special to complete graphs.

Let $G = G(V, E)$ be a connected graph of order p and size q $(\leq \hat{R})$ and suppose that $G \not\cong K_r$. We wish to prove that $G \nrightarrow (K_m, K_n)$. This is certainly true if $p < r$. We now take $p \geq r$ and make the induction hypothesis that the result holds for every graph of order $< p$ and size $\leq \hat{R}$ other than K_r. Since, $p \geq r, q \leq \hat{R}$ and $G \not\cong K_r$, we know that G is not complete. Let u and v be two non-adjacent vertices of G and set $G_u = G - v$, $G_v = G - u$, $W = V - \{u, v\}$ and $H = G - \{u, v\}$. If there exist two-colourings of G_u and G_v, agreeing on H, such that $\langle E_1\rangle \not\supseteq K_m$ and $\langle E_2\rangle \not\supseteq K_n$, then, by the observation made above, $G \nrightarrow (K_m, K_n)$. To establish the existence of such two-colourings, we employ the following device. Let $X = N(u) \cup N(v)$ where $N(u)$ and $N(v)$ denote the neighbourhoods of u and v respectively in G. Let H_u be the graph obtained from G_u by adding all edges of the form ux where $x \in X - N(u)$. Similarly define H_v and note that H_u and H_v are isomorphic graphs of order $p - 1$ and size $\leq q$. Moreover, for the reasons which follow, we may assume that $H_u, H_v \not\cong K_r$. It is clear that $H_u, H_v \cong K_r$ if and only if $H \cong K_{r-1}$ and $(N(u), N(v))$ is a nontrivial partition of W. If this were to be the case, then there is, for example, a vertex $w \in W$ such that u and w are not adjacent and $N(u) \cap N(w) \neq \emptyset$. Thus, we may simply consider $G - u$ and $G - w$ in the first place. Hence, it is clear that H_u and H_v satisfy the induction hypothesis. Moreover, since $H_u \cong H_v$, the existence of the desired two-colourings which agree on H is manifest. Clearly, the deletion of edges so that H_u returns to G_u and H_v returns to G_v spoils nothing so the desired two-colouring has been constructed. ▌

Remark. As we shall soon demonstrate, the determination of $\hat{r}(G_1, G_2)$ when G_1 and G_2 are not both complete can pose a basically new problem. Thus, Theorem 1 shows that the study of the size Ramsey number belongs to generalized, as opposed to classical, Ramsey theory. In the classical case, no new problems are created by the introduction of $\hat{r}$.

The answer to Question (ii) will follow from the following simple result, which gives $\hat{r}$ for stars.

Theorem 2. *For all values of m and n, $\hat{r}(K_{1,m}, K_{1,n}) = m + n - 1$.*

Proof. It is clear that in any two-colouring of $K_{1,m+n-1}$ either $\langle E_1\rangle \supseteq K_{1,m}$ or $\langle E_2\rangle \supseteq K_{1,n}$. Hence $\hat{r}(K_{1,m}, K_{1,n}) \leq m + n - 1$. In what follows, we suppose, without loss of generality, that $n \geq m$. Let G be a graph of size $q \leq m + n - 2$. It is clear that G has at most one vertex of degree $\geq n$. If there is no vertex of degree $\geq n$, then we may safely set $E_2 = E(G)$. If there is one vertex, v, of degree $\geq n$, then $\deg(v) \leq m + n - 2$ and every other vertex has degree less than m. In this case, we may colour $G - v$ arbitrarily and then colour the edges incident with v in such a way that $\langle E_1\rangle \not\supseteq K_{1,m}$ and $\langle E_2\rangle \not\supseteq K_{1,n}$. ▌

Corollary. *The sequence $\{K_{1,n}\}$ is an o-sequence.*

PROOF. It is known that

$$r(K_{1,n}) = \begin{cases} 2n-1 & n \text{ even} \\ 2n & n \text{ odd.} \end{cases}$$

Hence

$$\frac{\hat{r}(K_{1,n})}{\hat{R}(K_{1,n})} = \begin{cases} \dfrac{1}{n-1} & n \text{ even} \\ \dfrac{1}{n} & n \text{ odd} \end{cases}$$

and $\lim_{n\to\infty} \hat{r}(K_{1,n})/\hat{R}(K_{1,n}) = 0.$ ▌

In the scheme of this paper, Theorem 2 and its corollary have more than their face value. Taking a clue from the corollary, we shall study sequences of graphs which are obtained from a fixed graph by adding, in a prescribed way, progressive larger stars. We shall then ask whether or not such a sequence is an o-sequence.

2. Notation

In general, our graph-theoretic notation will follow [1] or [9]. However, some comments are in order concerning some of the more specialized notation which we shall use.

It is common to use $[X]^m$ to denote the collection of all m-element subsets of the set X. In a similar vein, we shall let $[X, Y]$ denote the collection of all pairs $\{x, y\}$ with $x \in X$ and $y \in Y$. To denote a complete graph on the vertex set X we shall write $[X]^2$. Also, $[X, Y]$ will signify the complete bipartite graph with parts X and Y.

Let v be a vertex in a graph G. We shall use $N(v)$ to denote the neighbourhood of v and $\deg(v) = |N(v)|$ will signify the degree of v. If $A = N(v)$ is a neighbourhood, then $\bar{A}$ will denote the closed neighbourhood, including v. Let X denote a set of vertices in G. Then, we shall use $X(v)$ to denote $X \cap N(v)$. With the underlying graph $G(V, E)$ understood and $X, Y \subseteq V$, $E(X)$ will denote $E \cap [X]^2$ and $E(X, Y)$ will denote $E \cap [X, Y]$.

Throughout this paper, we shall be concerned with partitions (E_1, E_2) of $E(G)$. Such a partition will be referred to as a *two-colouring* of G. To indicate $N(v)$, $\deg(v)$, $X(v)$ etc. in $\langle E_1\rangle$ and $\langle E_2\rangle$ we shall use a subscript. Thus, for example, $N_1(v)$ is the neighbourhood of v in the graph $\langle E_1\rangle$.

The graphs considered in this paper will be obtained by means of three "star" operations. The join of two graphs, symbolized by "+", is familiar.

Thus, given a graph G, the graph $G+\bar{K}_n$ is obtained from G by introducing n new vertices and by joining each vertex of G to each of these n additional vertices. If G is of order m, then $G+\bar{K}_n$ is of order $m+n$. In a similar way, we define $G\oplus\bar{K}_n$. This is the graph obtained from G by introducing, *for each vertex* v of G, n additional vertices and by joining v to these n vertices.

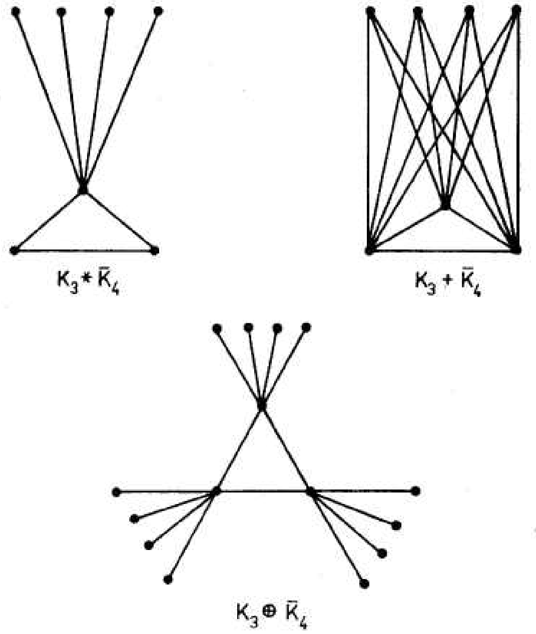

Fig. 1. Star operations

Thus, if G is of order m, then $G\oplus\bar{K}_n$ is of order $m(n+1)$. Finally, we let v be a particular vertex of G and define $G * \bar{K}_n(v)$ to be the graph obtained by adding n vertices and joining just v to the n additional vertices. If G is of order m, then $G * \bar{K}_n(v)$ is of order $m+n$. In case the choice of v is immaterial, we shall write $G * \bar{K}_n$. These three star operations are illustrated in Fig. 1.

In probabilistic arguments, we shall, in general, follow the notation of [4]. Thus, $B(n, p)$ denotes the binomial distribution characteristic of the sum of n independent random variables, each of which takes the value 1 with probability p and 0 with probability $1-p$. If A is an event, $P(A)$ will denote the probability of A and $\bar{A}$ will denote the complementary event. If X is a random variable, $\bar{X}$ will denote the expected value of X. As in [4], $G_{n,p}$ will denote a random graph on n labelled vertices where, for each pair $\{u, v\}$, $uv\in E(G)$ with independent probability p.

3. Statement of the problem

In order to motivate the subsequent development of this paper, we now pose two basic problems.

PROBLEM A. *Characterize those graphs G for which $\{G * \bar{K}_n\}$ is an o-sequence. Do the same for $\{G + \bar{K}_n\}$ and $\{G \oplus \bar{K}_n\}$.*

PROBLEM B. *Determine the asymptotic behaviour of $\hat{r}(K_m * \bar{K}_n)$ with m fixed and $n \to \infty$. Do the same for $\hat{r}(K_{m,n})$, $\hat{r}(K_m + K_n)$, and $\hat{r}(K_m \oplus \bar{K}_n)$.*

The fact that $\{K_{1,n}\}$ is an o-sequence suggests that there are other o-sequences to be obtained by means of the star operations, $* \bar{K}_n$, $+ \bar{K}_n$, and $\oplus \bar{K}_n$. This suggestion leads, in turn, to the formulation of Problem A. As we shall see, the characterization called for in Problem A turns out to be strikingly simple. We do not completely solve Problem B, but we are able to give useful upper and lower bounds in all cases.

4. Comments on the methods of solution

The methods which we shall use in considering the two problems just stated will involve only the simplest kind of combinatorial and probabilistic arguments. Because of their recurring use, three methods deserve to be brought to the attention of the reader.

The first method, which we shall call the *nested-neighbourhood method*, will be used to obtain upper bounds for $r(K_m * \bar{K}_n)$, $r(K_m + \bar{K}_n)$, and $r(K_m \oplus \bar{K}_n)$. By steadfastly assuming that we have a two-colouring (E_1, E_2) of K_p in which neither $\langle E_1 \rangle$ nor $\langle E_2 \rangle$ contains the graph in question, we shall find, in either $\langle E_1 \rangle$ or $\langle E_2 \rangle$, a sequence of vertices $x_1, x_2, \ldots, x_m$ and a corresponding sequence of neighbourhoods $A_1, A_2, \ldots, A_m$ such that $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_m$. Moreover, if p is sufficiently large, the neighbourhoods will be of sufficient cardinality that the nested-neighbourhood sequence will contain the graph in question. In this way we obtain the desired contradiction.

The second "method" is really a general purpose counterexample. Let the graph G and the natural number n be specified. Correspondingly, we define a *high-low colouring of G with respect to n*. First of all, n divides the vertices of G into "high" and "low" as defined by

$$H = \{v \mid \deg(v) \geq n\}$$

and

$$L = \{v \mid \deg(v) < n\},$$

respectively. Then, the high-low colouring of G with respect to n is defined by setting $E_1 = E(H, L)$ and $E_2 = E(H) \cup E(L)$. In particular, if $|E| \leq n^2/2$, the high-low colouring of $G(V, E)$ has several useful properties, which we now summarize.

FACT A. *Suppose that $|E| \leq n^2/2$ and let (E_1, E_2) be the high-low colouring of $G(V, E)$ with respect to n. Then*

(i) *$\langle E_1 \rangle$ contains no non-bipartite subgraph,*
(ii) *$\langle E_1 \rangle$ contains no two vertices of degree $\geq n$ which are adjacent, and*
(iii) *$\langle E_2 \rangle$ contains no vertex of degree $\geq n$.*

Part (i) follows since $\langle E_1 \rangle$ is bipartite. Part (ii) follows since any two such vertices must lie in H and they are joined by an edge in E_2. Finally, to establish part (iii) we note that the vertex and its neighbours must lie in H, but since $|E| \leq n^2/2$, we know that $|H| \leq n$.

The final "method" is simply an appeal to a known fact about random variables which have a binomial distribution.

FACT B. *Let X have the binomial distribution $B(n, p)$. Let $\varepsilon > 0$ and k be fixed. Then*

$$\lim_{n \to \infty} n^k P(|X - \bar{X}| > n\varepsilon) = 0.$$

This fact follows immediately from Bernstein's inequality applied in the binomial case ([10], p. 200).

5. Size Ramsey numbers for $G * \bar{K}_n$

The asymptotic behaviour of $\hat{r}(G * \bar{K}_n)$ is strongly influenced by whether or not G is bipartite. In fact, the results of this section will show that $\{G * \bar{K}_n\}$ is an o-sequence if and only if G is bipartite.

Let us first consider the case where G is bipartite. Our theorem in this case relies on two known results. The first is a result of GUY and ZNÁM which arises in their treatment of a problem of ZARANKIEWICZ [8].

LEMMA (Guy—Znám). *Suppose that $G \subseteq K_{M,N}$. If $|E(G)| \geq Nu$, where*

$$N\binom{u}{i} > (j-1)\binom{M}{i},$$

then $G \supseteq K_{i,j}$.

This result follows very simply from the pigeonhole principle and Jensen's inequality. The second result is, perhaps, less well-known and, for this reason, we shall give a short proof. The result is quoted by GUY in [7] and there attributed to CHVÁTAL and NIVEN.

LEMMA (Chvátal—Niven). *Let $f(m, k)$ denote the smallest integer n such that in every two-colouring of $[A, B] \simeq K_{2m,n}$ there is a monochromatic $K_{m,k}$. Then*

$$f(m, k) = (k-1)\binom{2m}{m} + 1.$$

PROOF. Consider the example in which for every $X \in [A]^m$, there are precisely $k-1$ vertices v ($\in B$) such that $A_1(v) = X$. This example shows that $f(m, k) > (k-1)\binom{2m}{m}$. Now we wish to prove that if (E_1, E_2) is an arbitrary two-colouring of $[A, B]$ where $|A| = 2m$ and $|B| = (k-1)\binom{2m}{m} + 1$, then either $\langle E_1 \rangle$ or $\langle E_2 \rangle$ contains a $K_{m,k}$. It is convenient to weaken our hypothesis by taking $|A| = 2m - 1$. Let the elements of $[A]^m$ be identified as $X_1, X_2, \ldots, X_N$, where $N = \binom{2m-1}{m}$. For each $v(\in B)$, let i (1 or 2) be determined by

$$|A_i(v)| = \max(|A_1(v)|, |A_2(v)|)$$

and let j be any index such that $X_j \subseteq A_i(v)$. Then assign the label (i, j) to vertex v. Since there are only $2\binom{2m-1}{m} = \binom{2m}{m}$ such labels and $|B| = (k-1)\binom{2m}{m} + 1$, it follows that k vertices of B must have the same label and so there is a monochromatic $K_{m,k}$. ∎

The following theorem gives upper and lower bounds for $\hat{r}(G * \overline{K}_n)$ when G is bipartite.

THEOREM 3. *Let G be a bipartite graph with parts A and B having cardinalities m and k respectively and suppose that $v \in A$. Then*

$$\hat{r}(G * \overline{K}_n(v)) \geq mn/2,$$

and, if n is sufficiently large,

$$\hat{r}(G * \overline{K}_n(v)) < 4m(2(n+k) - 1).$$

PROOF. For simplicity of notation, we shall let $G * \overline{K}_n(v)$ be denoted as simply G^*. The lower bound is based on the high-low colouring method. Given a graph of size $< mn/2$, in forming H and L we know that $|H| < m$. It follows that in the corresponding high-low colouring, neither $\langle E_1 \rangle$ nor $\langle E_2 \rangle$ contains G^*.

The upper bound is established by successively applying the two lemmas. Let (E_1, E_2) be an arbitrary two-colouring of $[A, B]$, where $|A| = 4m$ and

$|B| = N = 2(n+k) - 1$. If n is sufficiently large, then $N \geq (k-1)\binom{4m}{2m} + 1$. Hence, applying the Chvátal—Niven lemma, there is a monochromatic $[X, Y] \simeq K_{2m,k}$. We shall assume that $[X, Y] \subseteq E_1$. If, for any vertex $v\ (\in X)$, $|B_1(v)| \geq n+k$, then $\langle E_1 \rangle \supseteq G^*$. If not, then $|B_2(v)| \geq N - (n+k-1) = n+k$ for every $v\ (\in X)$ and so $|E_2(X, B)| \geq 2m(n+k) > mN$. Applying the Guy—Znám lemma with $u = m$, $i = m$ and $j = k$, we find that $\langle E_2 \rangle \supseteq [W, Z] \simeq K_{m,k}$. Since every vertex $v\ (\in W)$ satisfies $|B_2(v)| \geq n+k$, we find, in this case, that $\langle E_2 \rangle \supseteq G^*$. ▌

A lower bound for $\hat{r}(G * \overline{K}_n)$ when G is non-bipartite is already implicit in our observations concerning the high-low colouring method.

THEOREM 4. *If G is non-bipartite, then $\hat{r}(G * \overline{K}_n) > n^2/2$.*

PROOF. This follows from parts (i) and (iii) of Fact A. ▌

An upper bound for $\hat{r}(G * \overline{K}_n)$ when G is non-bipartite follows from an upper bound for $r(K_m * \overline{K}_n)$ $(m \geq 3)$. The following theorem gives $r(K_m * \overline{K}_n)$ precisely, provided n is sufficiently large. The proof of this theorem uses the technique of the proof of Theorem 2 in [2].

THEOREM 5. *Let $m \geq 3$ be fixed. If n is sufficiently large, then*

$$r(K_m * \overline{K}_n) = (m-1)(m+n-1)+1.$$

In particular, $r(K_3 * \overline{K}_n) = 2n+5$ for all $n \geq 1$ and $r(K_4 * \overline{K}_n) = 3n+10$ for all $n \geq 3$.

PROOF. Let $p = (m-1)(m+n-1)$ and consider the two-colouring of K_p defined by setting $\langle E_1 \rangle = (m-1)K_{m+n-1}$. In this two-colouring, $\langle E_1 \rangle$ contains no vertex of degree $\geq m+n-1$ and $\langle E_2 \rangle$ contains no K_m. Hence, $r(K_m * \overline{K}_n) > (m-1)(m+n-1)$.

The remainder of the proof utilizes a nested neighbourhood argument. Let (E_1, E_2) be an arbitrary two-colouring of K_p where

$$p = (m-1)(m+n-1)+1.$$

If n is sufficiently large, then $p \geq r(K_m)$, in which case either $\langle E_1 \rangle$ or $\langle E_2 \rangle$ contains a K_m. We shall assume that $\langle E_1 \rangle \supseteq [X]^2 \simeq K_m$. If, for any $v\ (\in X)$, $\deg_1(v) \geq m+n-1$, then $\langle E_1 \rangle \supseteq K_m * \overline{K}_n$. If not, we may select $v\ (\in X)$, set $A = N_2(v)$ and be sure that $|A| \geq (m-1)(m+n-1) - (m+n-2) = (m-2)(m+n-1)+1$. Assume that n is large enough that $|A| \geq r(K_m, K_{m-1})$ and consider the induced two-colouring of $\langle A \rangle$. Note that if $\langle A \rangle_2 \supseteq K_{m-1}$, then $\langle E_2 \rangle \supseteq K_m * \overline{K}_n$. Hence, we assume that $\langle A \rangle_1 \supseteq K_m$ and note that we may now simply repeat the argument from the point where we had assumed that $\langle E_1\rangle \supseteq K_m$. By repetition of the basic argument, we see that $\langle E_2\rangle$ contains a sequence of nested neighbourhoods $A_1 \supseteq A_2 \supseteq \ldots \supseteq A_{m-2}$, and a simple induction shows that $|A_k| \geq \geq (m-k-1)(m+n-1)+1$, for $k = 1, \ldots, m-2$. Let us set $B = A_{m-2}$ and remind ourselves that $|B| \geq m+n$. Finally, we consider the induced two-colouring of $\langle B\rangle$. If $\langle B\rangle_2 \supseteq K_2$, then $\langle E_2\rangle \supseteq K_m * \overline{K}_n$. If not, then $\langle E_1\rangle \supseteq K_{m+n}$ and, hence, $\langle E_1\rangle \supseteq K_m * \overline{K}_n$.

For the success of this proof, we see that it suffices to set n large enough so that $(m-k-1)(m+n-1)+1 \geq r(K_m, K_{m-k})$ for $k = 0, 1, \ldots, m-2$. A quick check using known Ramsey numbers then shows that $r(K_3 + \overline{K}_n) = = 2n+5$ for all $n \geq 1$ and $r(K_4 * \overline{K}_n) = 3n+10$ for all $n \geq 3$. ▌

We are now able to give the solution of Problem A for the sequence $\{G * \overline{K}_n\}$.

Corollary. *The sequence* $\{G * \overline{K}_n\}$ *is an o-sequence if and only if G is bipartite.*

Proof. If G is bipartite, then, by Theorem 3, $\hat{r}(G * \overline{K}_n) = O(n)$ $(n \to \infty)$, whereas, trivially, $\hat{R}(G * \overline{K}_n) > \binom{n}{2}$. Hence, if G is bipartite, then $\{G * \overline{K}_n\}$ is an o-sequence. If G is a non-bipartite graph of order m and if n is sufficiently large then, by Theorems 4 and 5,

$$n^2/2 < \hat{r}(G * \overline{K}_n) \leq \hat{r}(K_m * \overline{K}_n) \leq \hat{R}(K_m * \overline{K}_n) < m^2n^2/2.$$

Consequently, if G is non-bipartite, then $\{G * \overline{K}_n\}$ is not an o-sequence. ▌

6. Size Ramsey numbers for $G + \overline{K}_n$

In order that $\{G + \overline{K}_n\}$ be an o-sequence, G must be severely restricted. We shall show that $\{G + \overline{K}_n\}$ is an o-sequence if and only if G is an empty graph.

First, we take up the case where G is empty by considering the size Ramsey number of $\overline{K}_m + \overline{K}_n = K_{m,n}$.

Theorem 6. *Let $m \geq 2$ be fixed and suppose that n is sufficiently large. Then*

$$e^{-1}m2^{m-1}n < \hat{r}(K_{m,n}) \leq \frac{28}{9}m^2 2^{m-1}n.$$

Proof. Consider first the upper bound. Let (E_1, E_2) be an arbitrary two-colouring of $[A, B] \simeq K_{M,N}$. We may assume that $|E_1| \geq MN/2$. Hence, setting $u = M/2$ in the Guy—Znám lemma, $\langle E_1 \rangle \supseteq K_{m,n}$ if

$$N\binom{M/2}{m} > (n-1)\binom{M}{m}.$$

This will certainly be the case if we set $N = \left[\binom{M}{m} n \Big/ \binom{M/2}{m}\right]$. It follows that for all M ($\geq 2m$),

$$\hat{r}(K_{m,n}) \leq \frac{M\binom{M}{m}}{\binom{M/2}{m}} n.$$

In particular, if we set $M = m^2/2$, then the expression on the right hand side is asymptotic to $em^2 2^{m-1}n$ as $m \to \infty$. A more detailed study shows that if an upper bound of the form $cm^2 2^{m-1}n$ is to hold in general, then c must be taken somewhat greater than e, namely $\frac{28}{9}$, in order to take care of the worst case, which is $m = 3$.

The proof of the lower bound employs the probabilistic method. Suppose that $G(V, E)$ is a graph in which every two-colouring produces a monochromatic $K_{m,n}$. Let

$$A = \{v \mid \deg(v) \geq n\}$$

and

$$B = \{v \mid \deg(v) \geq m\}.$$

Then $|A| \leq 2|E|/n$ and $|B| \leq 2|E|/m$. If $G \supseteq [C, D] \cong K_{m,n}$ then, clearly, $C \subseteq A$ and $D \subseteq B$. Hence, setting $M = |A|$ and $N = |B|$, it must be true that every two-colouring of $K_{M,N}$ produces a monochromatic $K_{m,n}$. However, in a random two-colouring of $K_{M,N}$ in which $P(uv \in E_1) = P(uv \in E_2) = 1/2$, the probability that there is a monochromatic $K_{m,n}$ is not more than $2\binom{M}{m}\binom{N}{n}\Big/2^{mn}$. Moreover,

$$2\binom{M}{m}\binom{N}{n}\Big/2^{mn} \leq 2(M^m/m!)(N^n/n!)/2^{mn} < 2(M^m/m!)(eN/2^m n)^n.$$

Now suppose that $|E| < e^{-1}m2^{m-1}n$. Then $M < m2^m/e$ and $N < n2^m/e$ so that $(eN/2^m n) < 1$. It follows that if n is sufficiently large, then the probability that there is a monochromatic $K_{m,n}$ is less than 1. Hence, our assumption that $G \to (K_{m,n})$, where $|E| < e^{-1}m2^{m-1}n$ has led to a contradiction. ▌

If G is not empty, then a lower bound for $\hat{r}(G+\bar{K}_n)$ is contained in Fact A.

THEOREM 7. *If G is not empty, then $\hat{r}(G+\bar{K}_n) > n^2/2$.*

PROOF. This follows from parts (ii) and (iii) of Fact A. ∎

Our next objective is to obtain upper and lower bounds for $\hat{r}(K_m+\bar{K}_n)$. A lower bound of $n^2/2$ has been obtained already, but this can be strengthened. In order to obtain the stronger result, we shall refine the high-low colouring method. The refinement is based, in part, on the following result.

LEMMA. *Let B and C be positive constants and suppose that $G(Y, E)$ is a graph of order N ($\leq Bn$) such that for every v ($\in Y$), $\deg(v) \leq K = [Cn]$. Let δ be a fixed positive number and let M be a fixed natural number. If n is sufficiently large, there exists a partition $(Y_1, Y_2, \ldots, Y_M)$ of Y such that*

$$|Y_i(v_j)| \leq (1+\delta)Cn/M, \quad i = 1, \ldots, M,\ j = 1, \ldots, N.$$

PROOF. The proof uses the probabilistic method. Let $(Y_1, Y_2, \ldots, Y_M)$ be a random partition of the N vertices into M parts, where, for each i and j, $\mathsf{P}(v_j \in Y_i) = 1/M$. Let A denote the following event: for all i and j, $|Y_i(v_j)| \leq \leq (1+\delta)Cn/M$. It suffices to prove that $\mathsf{P}(A) > 0$.

For $j = 1, \ldots, N$, let $d_j = \deg(v_j)$ and note that $X_{ij} = |Y_i(v_j)|$ is a random variable having the binomial distribution $B(d_j, 1/M)$. Let X denote a random variable having the distribution $B(K, 1/M)$. Then

$$\mathsf{P}(\bar{A}) \leq \sum_{i=1}^{M}\sum_{j=1}^{N} \mathsf{P}(X_{ij} > (1+\delta)Cn/M) \leq MN\,\mathsf{P}(X > (1+\delta)Cn/M) \leq$$
$$\leq MN\,\mathsf{P}(|X - \bar{X}| > \mathrm{K}\delta).$$

Since K tends to infinity linearly with n whereas $N \leq Bn$, Fact B implies that $\mathsf{P}(\bar{A}) \to 0$ as $n \to \infty$ and, hence, $\mathsf{P}(A) > 0$ for all sufficiently large n. ∎

The desired lower bound for $\hat{r}(K_m+\bar{K}_n)$ is implied by the following result.

THEOREM 8. *Let ε be a fixed real number satisfying $0 < \varepsilon < 1$ and let $m \geq 3$ be a fixed natural number. If n is sufficiently large, then*

$$\hat{r}(K_m, K_{1,n}) > \max\begin{cases} n^2/2 \\ (1-\varepsilon)\left[\dfrac{(m-2)^2}{4}\right] n^2/2. \end{cases}$$

PROOF. The lower bound of $n^2/2$ follows from parts (i) and (iii) of Fact A. To improve the result when $m \geq 5$, we use a refinement of the high-low colouring method. Let $G(V, E)$ be a graph of size $|E| \leq (1-\varepsilon)LMn^2/2$, where $L+M = m-2$. Let (X, Y, Z) be the partition of V defined by setting

$$X = \{v \mid \deg(v) < n\},$$

$$Y = \{v \mid n \leq \deg(v) < M(1-\varepsilon)n\},$$

and

$$Z = \{v \mid \deg(v) \geq M(1-\varepsilon)n\}.$$

Further, in a way to be described presently, we shall partition Y into $(Y_1, Y_2, \ldots, Y_M)$ and Z into $(Z_1, Z_2, \ldots, Z_L)$. Thus, we partition V into $L+M+1 = m-1$ parts altogether. Now we two-colour $G(V, E)$ in the following way. For every $uv \in E$ set $uv \in E_1$ if u and v are in different parts and set $uv \in E_2$ if u and v are in the same part. It is clear that $\langle E_1 \rangle \not\supseteq K_m$ since for any set of m vertices there must be two vertices which are in the same part and therefore joined in E_2. To see that $\langle E_2 \rangle \not\supseteq K_{1,n}$ let us consider the three possible cases. If $v \in X$, then $\deg(v) < n$ in G and so $\deg(v) < n$ in $\langle E_2 \rangle$. Now suppose that $v \in Z$. Since each vertex in Z has degree at least $M(1-\varepsilon)n$ in G, we know that $|Z|\, M(1-\varepsilon)n/2 \leq |E| \leq LM(1-\varepsilon)n^2/2$ and hence, $|Z| \leq Ln$. Certainly, then, we can form the partition $(Z_1, \ldots, Z_L)$ of Z in such a way that $|Z_k| \leq n$ for $k = 1, \ldots, L$. Having done this, we are assured that if $v \in Z$ then $\deg(v) < n$ in $\langle E_2 \rangle$. Finally, let us suppose that $v \in Y$. Since each vertex in Y has degree at least n in G, we know that $|Y| n/2 \leq$ $\leq |E| \leq LM(1-\varepsilon)n^2/2$ and hence, $|Y| \leq LM(1-\varepsilon)n$. Now set $\delta < \varepsilon/(1-\varepsilon)$ and apply the previous lemma to the induced subgraph $\langle Y \rangle$. We thus obtain the desired partition $(Y_1, \ldots, Y_M)$ of Y such that for every $v \in Y$, $\deg(v) <$ $< n$ in $\langle E_2 \rangle$.

As a final step, we maximize LM subject to the condition $L+M =$ $= m-2$, obtaining the result $\left[\frac{(m-2)^2}{4}\right]$ when L and M are as nearly equal as possible. ▌

The upper bound for $\hat{r}(K_m + \bar{K}_n)$ stems from our knowledge of the ordinary Ramsey number $r(K_m + \bar{K}_n)$.

THEOREM 9. *For all values of m and n,*

$$r(K_m + \bar{K}_n) \leq 2^{2m-1}(n+1) - 1,$$

and, if m and $\varepsilon > 0$ are fixed and n is sufficiently large, then

$$r(K_m + \bar{K}_n) > [(2^m - \varepsilon)n].$$

PROOF. The upper bound is obtained by a nested neighbourhood argument. Let (E_1, E_2) be an arbitrary two-colouring of K_p. Arbitrarily select a vertex v and let X be the larger of the two neighbourhoods $N_1(v)$ and $N_2(v)$. Note that $|X| \geq (p-1)/2$. Consider the induced two-colouring of $\langle X \rangle$ and repeat the process starting with the selection of an arbitrary vertex. In this way obtain a sequence of neighbourhoods $X_1, X_2, \ldots, X_{2m-1}$. By induction, $|X_k| \geq (p+1-2^k)/2^k$ for $k = 1, 2, \ldots, 2m-1$. Of the $2m-1$ times the process is performed, the majority colour must be the same m times. Hence, in either $\langle E_1 \rangle$ or $\langle E_2 \rangle$, we find a nested neighbourhood sequence $A_1, A_2, \ldots, A_m$. Finally, since $|A_m| \geq |X_{2m-1}| \geq (p+1-2^{2m-1})/2^{2m-1}$, it follows that if $p \geq 2^{2m-1}(n+1) - 1$ then $|A_m| \geq n$ and so there is a monochromatic $K_m + \bar{K}_n$.

The lower bound is obtained by the probabilistic method. Consider the random graph $G = G_{N,\frac{1}{2}}$ where $N = [(2^m - \varepsilon)n]$. Let A denote the following event: $G \supseteq K_m + \bar{K}_n$. To describe this event, let $[V]^m = \{S_1, S_2, \ldots, S_k\}$ where $k = \binom{N}{m}$ and, for $j = 1, \ldots, k$, define the random variable X_j to be the number of vertices which are adjacent to all the vertices of S_j. Then

$$A = \bigcup_{j=1}^{k} \{G \mid [S_j]^2 \subseteq E(G) \text{ and } X_j \geq n\}.$$

Hence

$$\mathsf{P}(A) \leq \frac{\binom{N}{m} \mathsf{P}(X \geq n)}{2^{\binom{m}{2}}},$$

where X represents a typical X_j. If $\mathsf{P}(A) < 1/2$ we can be sure that $N < r(K_m + \bar{K}_n)$. Note that X has the binomial distribution $B(N-m, 1/2^m)$. Hence, $\bar{X} = (N-m)/2^m$ and $X \geq n$ implies that $|X - \bar{X}| > N\delta$, where $\delta = \varepsilon/4^m$. It follows that

$$\mathsf{P}(A) \leq \frac{N^m}{m!\, 2^{\binom{m}{2}}} \mathsf{P}(|X - \bar{X}| > N\delta).$$

Applying Fact B, we see that $\mathsf{P}(A) < 1/2$ for all sufficiently large n. ∎

REMARK. It is natural to raise the question of whether or not one of the two bounds in Theorem 9 is, in general, asymptotically correct. For $m = 1$ there is no question, since both bounds are asymptotic to $2n$. For $m = 2$ the issue has been settled in favour of the lower bound. ROUSSEAU and SHEEHAN [11] have proved that, for all n, $r(K_2 + \bar{K}_n) \leq 4n + 2$ and that if $4n + 1 = p^\alpha$ (a prime power) then $r(K_2 + \bar{K}_n) = 4n + 2$.

The results of this section contain the solution of Problem A for the sequence $\{G + \bar{K}_n\}$.

COROLLARY. *The sequence $\{G + \bar{K}_n\}$ is an o-sequence if and only if G is empty.*

PROOF. By Theorem 6, $\hat{r}(\overline{K}_m+\overline{K}_n)=O(n)$ $(n\to\infty)$, whereas, trivially, $\hat{R}(\overline{K}_m+\overline{K}_n)>\binom{n}{2}$. Hence, if G is empty, then $\{G+\overline{K}_n\}$ is an o-sequence. If G is a non-empty graph of order m then, by Theorems 7 and 9,

$$n^2/2<\hat{r}(G+\overline{K}_n)\leq\hat{r}(K_m+\overline{K}_n)\leq\hat{R}(K_m+\overline{K}_n)<2^{4m-3}n^2.$$

Consequently, if G is non-empty, then $\{G+\overline{K}_n\}$ is not an o-sequence. ▌

7. Size Ramsey numbers for $G\oplus\overline{K}_n$

The story here is the same as in the preceding episode: $\{G\oplus K_n\}$ is an o-sequence if and only if G is empty.

We may dispose of the case where G is empty by a simple observation. We note that $\hat{r}(\overline{K}_m\oplus\overline{K}_n)\leq(2m-1)(2n-1)$, since, in any two-colouring of $(2m-1)K_{1,2n-1}$ there is a monochromatic $mK_{1,n}=\overline{K}_m\oplus\overline{K}_n$.

To complete the story when G is non-empty, we need only one new result.

THEOREM 10. *Let m be fixed. If n is sufficiently large, then*

$$r(K_m\oplus\overline{K}_n)\leq(2m^2-m+1)(n+1)-1.$$

PROOF. The result will be obtained by using the nested neighbourhood argument. First of all, we make the following general observation. If (E_1,E_2) is a two-colouring of K_p where $p\geq r(K_m)$ then there is either a monochromatic $K_m\oplus\overline{K}_n$ or a monochromatic $K_{1,q}$ where $q=p-m(n+1)$. The reason is very simple. Since $p\geq r(K_m)$, we may suppose that $\langle E_1\rangle\supseteq K_m$. If each of these m vertices has degree at least $m(n+1)-1$ in $\langle E_1\rangle$, then $\langle E_1\rangle\supseteq\supseteq K_m\oplus\overline{K}_n$. If not, then there is certainly a vertex of degree at least $p--m(n+1)$ in $\langle E_2\rangle$.

Let (E_1,E_2) be a two-colouring of $[V]^2\simeq K_p$. Set $X_0=V$ and, for $i=1,2,\ldots,2m-1$, apply the observation made above to the induced two-colouring of $\langle X_{i-1}\rangle$. Assuming that $|X_{i-1}|\geq r(K_m)$ and that there is no monochromatic $K_m\oplus\overline{K}_n$, we obtain X_i as the neighbourhood in the resulting monochromatic star.

Clearly, $|X_k|\geq p-km(n+1)$ for $k=1,2,\ldots,2m-1$. Of the $2m-1$ monochromatic stars, m must be of the same colour. Hence, in either $\langle E_1\rangle$ or $\langle E_2\rangle$, there is a nested neighbourhood sequence $A_1,A_2,\ldots,A_m$. Let us re-index the A's so that $A_m\supseteq\bar{A}_{m-1}\supseteq\ldots\supseteq\bar{A}_1$. Now note that if $|A_k|\geq k(n+1)-1$ for $k=1,2,\ldots,m$, then there is a monochromatic $K_m\oplus\overline{K}_n$. But $|A_k|\geq|X_{2m-1}|\geq p-m(2m-1)(n+1)$. Hence, if $p==(2m^2-m+1)(n+1)-1$, then there must be a monochromatic $K_m\oplus\overline{K}_n$.

For the success of this proof, we see that it suffices to set n large enough so that $|X_{2m-2}| \geq r(K_m)$. Hence, if n is large enough that $(m+1)(n+1) - 1 \geq r(K_m)$ the result holds. ▌

Now we can give the solution of Problem A for the sequence $G \oplus \overline{K}_n$.

COROLLARY. *The sequence $\{G \oplus \overline{K}_n\}$ is an o-sequence if and only if G is empty.*

PROOF. We have observed that $\hat{r}(\overline{K}_m \oplus \overline{K}_n) = O(n)$ $(n \to \infty)$. Then, because of the trivial fact that $\hat{R}(\overline{K}_m \oplus \overline{K}_n) > \binom{n}{2}$, we see that $\{\overline{K}_m \oplus \overline{K}_n\}$ is an o-sequence. If G is non-empty, then $\hat{r}(G \oplus \overline{K}_n) > n^2/2$ by parts (ii) and (iii) of Fact A. Hence, using the result of Theorem 10, if G is a non-empty graph of order m, then

$$n^2/2 < \hat{r}(G \oplus \overline{K}_n) \leq \hat{r}(K_m \oplus \overline{K}_n) \leq \hat{R}(K_m \oplus \overline{K}_n) < 2m^4n^2.$$

Consequently, if G is non-empty, then $\{G \oplus \overline{K}_n\}$ is not an o-sequence. ▌

8. Open questions

Except for complete graphs and stars, we have given no general, exact results for $\hat{r}$. Some asymptotic results have been obtained, but there are many open questions. Theorem 5 and Theorem 8 imply that for n sufficiently large there exist constants a_1 and a_2 such that

$$a_1m^2n^2 \leq \hat{r}(K_m * \overline{K}_n) \leq a_2m^2n^2.$$

Although for this case the size Ramsey number is known up to a constant, the same cannot be said for $\hat{r}(K_{m,n})$, $\hat{r}(K_m + \overline{K}_n)$, and $\hat{r}(K_m \oplus \overline{K}_n)$. The known bounds are as follows:

$$b_1m2^{m-1}n \leq \hat{r}(K_{m,n}) \leq b_2m^2 2^{m-1}n \quad \text{for } n \text{ sufficiently large},$$

$$c_1m^2n^2 \leq \hat{r}(K_m + \overline{K}_n) \leq c_2 4^{2m}n^2,$$

$$d_1m^3n^2 \leq \hat{r}(K_m \oplus \overline{K}_n) \leq d_2m^4n^2,$$

where b_1, b_2, c_1, c_2, d_1 and d_2 are appropriate constants. These bounds are either explicitly given or implied by the results of Theorems 6, 8, 9 and 10. It would be nice to determine each of these size Ramsey numbers up to a constant.

In Section 6 it is shown, for m fixed and n sufficiently large, that $\{K_{m,n}\}$ is an o-sequence. The arguments used there for the lower bound are not valid when m is allowed to grow large with n. It is thus an open question as to whether $\{K_{n,n}\}$ is an o-sequence. Bounds for $r(K_{n,n})$ are

$$a_1n2^{n/2} \leq r(K_{n,n}) \leq a_2n2^n$$

and proved in [3]. By a straightforward probabilistic argument one can show that $\hat{r}(K_{n,n}) \geq b_1 n^2 2^{n/2}$. Hence, using the upper bound given in Theorem 6, one obtains

$$b_1 n^2 2^{n/2} \leq \hat{r}(K_{n,n}) \leq b_2 n^3 2^{n-1}.$$

Determination of the exact size Ramsey number for even a simple graph like a path, P_n, on n vertices seems quite difficult. It is well known (see [6]) that $r(P_n) = n + [n/2] - 1$. In [5] it is shown that $K_{n,n} \to P_n$. Thus $\hat{r}(P_n) \leq \leq n^2 < \hat{R}(P_n)$. It would be interesting to know if $\lim_{n\to\infty} \frac{\hat{r}(P_n)}{n}$ exists, and if so determine its value. An easier but still apparently difficult question is to determine if $\{P_n\}$ is an o-sequence.

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