SOME NUMBER THEORETIC PROBLEMS ON BINOMIAL COEFFICIENTS

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We state a few simple but probably very difficult problems about binomial coefficients. The problems came up at the International Combinatorial Conference in Canberra, August 1977.

Let $1 \le i < j \le \frac{1}{2}n$. First observe that

$$\left(\binom{n}{i},\ \binom{n}{j}\right) > 1. \qquad (1)$$

To prove *(1)* put

$$\binom{n}{j} = \binom{n}{i}\binom{n-i}{j-i} / \binom{j}{i}\ . \qquad (2)$$

From *(2)* we evidently have

$$\left(\binom{n}{i},\ \binom{n}{j}\right) \ge \binom{n}{i} / \binom{j}{i} \ge 2^{i}. \qquad (3)$$

Since $\binom{2p}{p}$ is not divisible by p, we have $(2p, \binom{2p}{p}) = 2$ hence equality in *(3)* for $i = 1$, $n = 2p$, $j = p$. For $i > 1$ there is always strict inequality in *(3)* and it seems likely that there is an $h(n)$ tending to infinity with n so that

$$\left(\binom{n}{i},\ \binom{n}{j}\right) \ge h(n) \qquad \text{for} \qquad 2 \le i < j \le \tfrac{1}{2}n.$$

Denote by $P(m,n)$ the greatest prime factor of (m,n), and in particular by $P(m,n)$ the greatest prime factor of m. A well known theorem of Sylvester and Schur states that $P\left(\binom{n}{i},\ \binom{n}{i}\right) > i$. This result generalizes the theorem of Tschebicheff according to which there always is a prime between n and $2n$.

Conjecture 1. For every $1 \le i < j \le \frac{1}{2}n$ we have

$$P\left(\binom{n}{i},\ \binom{n}{j}\right) \ge i\ . \qquad (4)$$

(4) if true is probably very deep. We also conjecture that

$$P\left(\binom{n}{i},\ \binom{n}{j}\right) > i \qquad (5)$$

holds except in a few special cases. First of all we discuss some of the counter-examples to *(5)*.

(5) fails for $i = 2$ and some special values of n, all of which seem to be powers of 2. Assume $n = 2^{r}$ to be of the form $pq + 1$ where p,q are primes, e.g.

$2^4 = 1 + 3 \times 5$, or $2^{11} = 1 + 23 \times 89$. Then $\binom{n}{2} = 2^{r-1}pq$ so that

$$P\left[\binom{n}{2}, \binom{n}{j}\right] = 2$$

provided that j is such that $\binom{n}{j}$ is not divisible by pq. The best chance for this to happen is if j is such that $j \equiv 0 (mod\ p)$, $j \equiv 1 (mod\ q)$. For instance in the previous two cases, when $n = 16$, $p = 3$, $q = 5$, $j = 6$ we have $\left[\binom{16}{2}, \binom{16}{6}\right] = 8$

and when $n = 2048$, $p = 23$, $q = 89$, $j = 713$ we have $\left[\binom{2048}{2} \binom{2048}{713}\right] = 2^{10}$. The same method can of course be used if $2^r - 1 = u - v$, $(u,v) = 1$ where u,v are not necessarily primes, but the chance of failure is much greater.

There are a few scattered counterexamples for $i = 3$ like $\left[\binom{10}{3}, \binom{10}{5}\right] = 2^2 \times 3$; the numbers $3^r + 1$ have the best chances. For $i \geq 4$ there should only be a few incidental counterexamples to (5). We only know one such counterexample, namely $\left[\binom{28}{5}, \binom{28}{14}\right] = 2^3.3^3.5$.

In trying to prove conjecture 1 the following further problem occurred. Put

$$f(n) = \min_{1<j\leq\frac{1}{2}n} \left[n, \binom{n}{j}\right].$$

From (2) it follows that

$$f(n) \geq p(n) \qquad (6)$$

where $p(n)$ is the smallest prime factor of n. If $n = p^k$ is a prime power then equality sign holds in (6), $f(p^k) = p$. For composite n we have the further inequality

$$f(n) \leq \frac{n}{P(n)} \qquad (7)$$

where $P(n)$ is the greatest prime power which divides n. To see this put $j = p^{\alpha}$ where $p^{\alpha} | n$, $p^{\alpha+1} \nmid n$, then

$$\left[n, \binom{n}{j}\right] = \frac{n}{p^{\alpha}}$$

as seen immediately from (2). If $n = pq$, $p < q$ then there is equality both in (6) and in (7). This again is seen from (2):

$$\binom{pq}{q} = p\binom{pq-1}{q-1}$$

hence $\binom{pq}{q}$ is divisible by p but not by q. This shows that $f(pq) = p = \frac{n}{q}$, and equality in (6) and (7) follows. Another case with equality in (7) is $f(30) = 6$; the high value of $f(3)$ is caused by the presence of 25 and 27 near 30. It would be of interest to characterize the composite n with $f(n) = n/P(n)$. Numbers of the form $\prod_{p \le k} p$ with $k > 6$ do not seem to have this property; for instance

$$f(210) = \left(210, \binom{210}{30}\right) = 14 < \frac{210}{P(210)} = 30.$$

There are also cases other than pq with equality in (6). For instance we have $f(3pq) = 3$, namely $\binom{3pq}{pq} \not\equiv 0 \pmod{pq}$, for infinitely many pairs of primes p,q. We leave the proof to the reader.

One more remark. Let n be composite. It is immediate that for infinitely many n, $f(n) \ge \sqrt{n}$, say when $n = p^2$. There are some n for which the strict inequality

$$f(n) > \sqrt{n} \qquad (8)$$

holds, e.g. $f(30) = 6$, $f(70) = 0$, $f(154) = 14$ (all of the form $2pq$). It seems likely that there are infinitely many n for which the inequality (8) is true, although we cannot prove this at present.

From (7) it follows that

$$f(n) < (1 + o(1))\frac{n}{\log n} . \qquad (9)$$

To prove (9) observe that the prime number theorem easily implies $P(n) \ge (1 + o(1))\log n$. Thus (7) implies (9). Perhaps it is true that for every $\alpha > 0$ there is an $n_0(\alpha)$ so that for every composite $n > n_0(\alpha)$, $f(n) < n/(\log n)^{\alpha}$.

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