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ON THE SCHNIRELMANN DENSITY OF k-FREE INTEGERS

By

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1. *Introduction*

Let k be an integer $\geqslant 2$ and $Q_k(x)$ the number of k-free integers not exceeding x. Write

$$Q_r(x) = \frac{x}{\zeta(k)} + E(x). \tag{1.1}$$

Then, as is well known, Gagenbauer [7] showed in 1885 that

$$E(x) = o(x^{1/k}).$$

This is improved to

$$E(x) \quad o(x^{1/k} \exp [-b(\log x \log x)^{1/2}]) \tag{1.2}$$

by Evelyn and Linfoot [6] in 1931. They also showed that $E(x) \neq o(x^{1/2k})$. In 1911, Axer [1] showed that, on the basis of the Riemann hypothesis,

$$E(x) = o(x^{1/(2k+1)+\epsilon}) \tag{1.3}$$

while if we assume that $E(x) = o(x^{1/(2k)+\epsilon})$, this implies the Riemann hypothesis. Very recently, in 1976, Montgomery and Vaughan announced the result that for $k=2$, on the Riemann Hypothesis, $E(x) = o(x^{(21/64)+\epsilon})$. This vastly improves Axer's result for $k=2$. Their result is not yet published.

Let D_k and d_k denote, respectively, the asymptotic and Schnirelmann densities of the k-free integers, i.e.

$$D_k = \lim_{x \to \infty} \frac{Q_k(x)}{x}; \tag{1.4}$$

$$d_k = \inf_{n > 0} \frac{Q_k(n)}{n}. \tag{1.5}$$

It is clear that $D_k = \frac{1}{\zeta(k)}$, but no such simple formula is known for d_k. In 1964, Kenneth Rogers [11] showed that

$$d_2 = \frac{53}{88} < \frac{6}{\pi^2} = D_2.$$

Actually

$$\frac{Q_2(176)}{176} = d_2 = \frac{53}{88} = \cdot 60227273.$$

In 1966, Stark [12] showed by analytic methods that for all $k \geqslant 2$,

$$d_k < D_k. \tag{1.6}$$

It may be noted that this implies that d_k is attained for some value of n. It is trivial that

$$d_2 \leqslant d_3 \leqslant \ldots \leqslant d_k \ldots \leqslant 1.$$

This follows from the observation that

$$Q_2 \subset Q_3 \subset \ldots \subset Q_k \subset \ldots$$

where Q_k = the set of k-free integers. As Duncan observed [4], it is easily proved that the asymptotic and Schnirelmann densities interlace:

$$d_k < D_k < d_{k+1} < D_{k+1}. \tag{1.7}$$

Also a simple estimate gives

$$d_k > 1 - \sum_{p \text{ prime}} p^{-k}. \tag{1.8}$$

This is due to Duncan [5], who also proved that d_{k+1} is closer to D_{k+1} than to D_k, and in fact

$$\frac{D_{k+1} - d_{k+1}}{D_{k+1} - D_k} < \frac{1}{2^k}. \tag{1.9}$$

In 1969, Orr [9] gave a non-analytic proof for the result (1.6). A simpler and entirely elementary proof (unpublished) of this is due to Hardy.

Orr also showed that if n_k denotes any of the values of n where $\frac{Q_k(n)}{n}$ attains the value d_k, then for $k \geqslant 5$

$$5^k \leqslant n_k < 6^k. \tag{1.10}$$

He also calculated d_k for $k = 3, 4, 5, 6$:

$$d_3 = \frac{157}{189},\ d_4 = \frac{145}{157},\ d_5 = \frac{3055}{3168},\ d_6 = \frac{6165}{6272},$$

$$n_3 = 378,\ n_4 = 2512,\ n_5 = 3168,\ 6336;\ n_6 = 31360.$$

These calculations are extended by G. E. Hardy for all $k \leqslant 75$. A table of values of n_k and the corresponding Q_k for $k \leqslant 75$ is given at the end of the paper. These values are obtained on computer using Theorem (1.13) below.

In a recent paper Diananda and Subbarao [3] obtained the following results, among others:

(1.11) $$d_k > 1-2^{-k}-3^{-k}-5^{-k},$$

thus improving (1.8);

(1.12) For $k \geqslant 5$, the largest n_k for any given k satisfies $\frac{1}{2}6^k \leqslant n_k < 6^k$.

(1.13) THEOREM. *For $k \geqslant 5$, there is an n_k so that*

(1) $3^k | n_k$ or $5^k | n_k$, or

(2) $2^k | n_k$ *and between* $n_k - 2^k$ *and* n_k *there is a multiple of* 3^k *or* 5^k.

(1.14) For $k \geqslant 5$, $d_k \geqslant 1-2^{-k}-3^{-k}-5^{-k}+\left[\dfrac{3^{-k}+2.5^{-k}}{6^k-3^k+1}\right]$.

In this note we are mainly concerned with the following questions that arise in this connection:

(1.15) Is d_k closer to $(1-2^{-k}-3^{-k}-5^{-k})$ or to D_k? What is the estimate for the difference between d_k and the closer one?

(1.16) For a given k, can we obtain precise information about the (maximal) value of n_k and preferably, can we get a formula for the exact value of this maximal value for infinitely many k if not for all k?

2. *An estimate for* $d_k \quad (1-2^{-k}-3^{-k}-5^{-k})$

In answer to the first question, we show that d_k is in fact closer to $1-2^{-k}-3^{-k}-5^{-k}$ than to D_k. In fact we prove a little more and answer a querry raised by Pomerance [10] by showing that:

(2.1) THEOREM. $$\frac{d_k-(1-2^{-k}-3^{-k}-5^{-k})}{D_k-d_k}=o\left(\frac{2}{3}\right)^k \to 0 \text{ as } k \to \infty.$$

Proof. Let $S=S_k$ denote the set of integers n_0 in the interval $(\frac{1}{2}6^k, 6^k)$ such that each n_0 is thc first multiple of 2^k following the first multiple of 3^k that follows some multiple of 5^k in the interval.

Let a=distance between the multiple of 5^k chosen and the immediately following multiple of 3^k:

b=distance between the multiple of 3^k and the immediately following multiple of 2^k.

Then $0 \leqslant a \leqslant 3^k$; $0 \leqslant b \leqslant 2^k$. Now

$$d_k \leqslant \frac{Q(n_0)}{n_0} = 1-2^{-k}-3^{-k}-5^{-k}+\frac{\left(\frac{n_0}{5^k}-\left[\frac{n_0}{5^k}\right]\right)+\frac{n_0}{3^k}-\left[\frac{n_0}{3^k}\right]\Big)}{n_0}$$

$$=1-2^{-k}-3^{-k}-5^{-k}+\left\{\frac{a+b}{5^k}+\frac{c}{3^k}\right\}\Big/ n_0$$

$$\leqslant 1-2^{-k}-3^{-k}-5^{-k}+\left(\frac{3^k+2^k}{5^k}+\frac{2^k}{3^k}\right)\Big/(\tfrac{1}{2}6^k)$$

$$=1-2^{-k}-3^{-k}-5^{-k}+O(1/9^k),$$

recalling that $n_0 \geqslant \frac{1}{2}6^k$. But

$$D_k-d_k=(5(k))^{-1}-d_k=\prod_P(1-P^{-k})-d_k$$

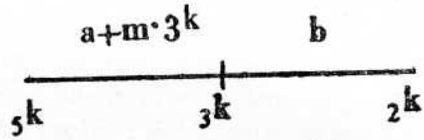
$$=1-2^{-k}-3^{-k}-5^{-k}+O(6^{-k})-d_k$$

$$=O(6^{-k}).$$

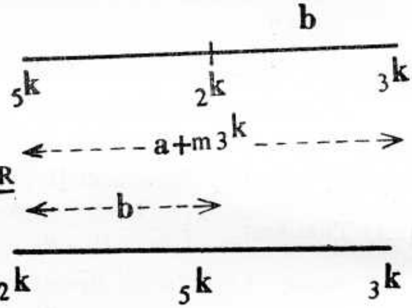
Hence the theorem follows.

3. *The value of* n_k

Theorem (1.13) shows that n_k is either a multiple of 3^k or 5^k or is a multiple of 2^k immediately following a mulitiple of 3^k or 5^k in the interval $(\frac{1}{2}6^k, 6^k)$. Thus we have the following possibilities:

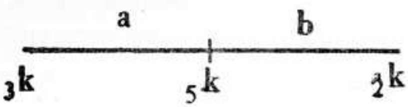


(3.1) $n_k = a$ multiple of 2^k following a multiple of 3^k that follows a multiple of 5^k.

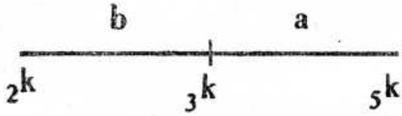


(3.2) $n_k = a$ multiple of 3^k. This may follow a multiple of 2^k or 5^k.

(3.3) $n_k = a$ multiple of 2^k following a multiple of 5^k.



(3.4) n_k is a multiple of 5^k. This may follow a multiple of 2^k or 3^k.



REMARK. Thus there are at most $2^k+\left(\frac{6}{5}\right)^k$ possible values for n_k, of which $\frac{1}{2}\left(\frac{6}{5}\right)^k$ arise from the situation (3.1), 2^k from (3.2), $\frac{1}{2}\left(\frac{6}{5}\right)^k$ from (3.3) and (3.4).

Using the values of d_k for $k \leqslant 75$ that we have, we found that n_k is as in (3.1)

with $m=0$ for $k=5$ and all k in the range $13 \leqslant k \leqslant 75$.

We therefore make the following conjecture:

(3.5) CONJECTURE. For all sufficiently large k, we have $d_k = Q_k(n_0)/n_0$ for some n_0 in the set S_k (in the interval $(\frac{1}{2}6^k, 6^k)$.

Alternatively,

(3.6) CONJECTURE. The Schnirelmann density d_k of the set of k-free integers is attained (for all sufficiently large k) at an integer n_0 which is the first multiple of 2^k following the first multiple of 3^k that follows some multiple of 5^k in $(\frac{1}{2}6^k, 6^k)$.

4. *Some remarks and problems*

We mention here some related problems that seem interesting.

(4.1) Let $p_1 < p_2 < \ldots$ be the sequence of consecutive primes greater than a fixed number $c > 2$. Let $S(p_1^2, p_2^2, \ldots)$ denote the set of integers no one of which is divisible by any p_i^2. Then its asymptotic density is clearly $\prod_{p_i > c} (1 - 1/p_i^2)$.

Is its Schnirelmann density less than its asymptotic density? What happens if $p_1 < p_2 < \ldots$ in any sequence of primes (not necessarily consecutive).

Are there arbitrarily large values of n such that the number of integers $\leqslant n$ which are not multiples of any p_i^2 is

$$< n \prod_{i=1}^{\infty} (1 - 1/p_i^2),$$

and also infinitely many integers m such that the number of integrs $\leqslant m$ which are not multiples of any p_i^2 is

$$> m \prod_{i=1}^{\infty} (1 - 1/p_i^2)?$$

Similar questions arise for k-free integers.

(4.2) If $\{p_1, p_2, ..., p_r\}$ be any finite set of primes, with the notation of (4.1), consider the set $S(p_1, p_2, ..., p_r)$. It is easy to show that for this set, the asymptotic and Schnirelmann densities are unequal.

(4.3) A similar question can be asked for an infinite set of primes. Pomerance [10] showed that there exists an infinite sequence of primes $p_1 < p_2 < ...$ $(\Sigma 1/p_i < \infty)$ so that that the Schnirelmann density of the set $S(p_1, p_2, ...)$ of integers n, no one of which is divisible by p_i for every i, is the same as the asymptotic density $\prod_i (1-1/p_i)$. Here is the outline of his proof.

Le $p_1 < p_2 < ... < p_k$ be any finite set of primes for which $S(p_1, ..., p_k)$ has Schnirelmann density $\alpha < \prod_{i=1}^{k} (1-1/p_i)$.

Consider the set of all primes $(p_1, ..., p_k, q_1, q_2, ...)$ for which $S(p_1, ..., p_k, q_1, ...)$ has asymptotic density $\geqslant \alpha$ and Schnirelmann density α. (Note that if the q's are very large, they will not change the Schnirelmann density.) Let

$$p_1 < p_2 < ... < p_k < p_{k+1} < p_{k+2} < ...$$

be a maximal set with this property (maximal means that no prime can be replaced by any smaller one). It is easy to see that $S(p_1, ..., p_k, p_{k+1} ...)$ has both Schnirelmann density and asymptotic density α.

Let $1 < a_1 < a_2 < ...$, $(a_i, a_j) = 1$, $\Sigma\, 1/a_i < \infty$ be any sequence of integers for which asymptotic density of $S(a_1, ...)$ is α. Denote by $S(\alpha)$ the lower bound of the Schnirelmann densities of these sequences. It is not hard to see that $S(\alpha) > 0$ for $\alpha > 0$. Determine or estimate $S(\alpha)$. If the condition $(a_i, a_j) = 1$ is dropped, it easily follows from a theorem of Besicovitch that $(\alpha) = 0$ for every $\alpha < 1$.

(4.5) Let n be any integer. The asymptotic density of the integer relatively prime to it is of course $\varphi(n)/n$. Denote its Schnirelmann density by $\alpha(n)$. We have $\alpha(n) < \varphi(n)/n$. $\alpha(n)$ has a mean value, but it may not be too easy to get a nice formula for it; also it has an asympotic distribution function, but $\alpha(n)$ is of course not multiplicative. It is easy to see that

$$\min_{1 \leqslant n \leqslant x} \alpha_n = 1/(p_{k+1} - 1) \text{ where } \prod_{i=1}^{k} p_i \leqslant x < \prod_{i=1}^{k+1} p_i.$$

Table of values of n_k

k	$Q_n(n_k)$	n_k
2	106	176
3	314	378
4	2 320	2 512
5	6 110	6 336
6	30 825	31 360
7	234 331	236 288
8	1 169 758	1 174 528
9	7 798 488	7 814 151
10	48 785 015	48 833 536
11	292 856 489	293 001 216
12	1 709 225 206	1 709 645 824
13	12 206 236 915	12 207 734 784
14	67 139 207 400	67 143 319 552
15	366 201 607 242	366 212 808 704
16	2 593 955 782 238	2 593 995 423 744
17	15 258 697 717 317	15 258 814 251 008
18	83 923 093 402 988	83 923 413 762 048
19	476 836 615 418 082	476 837 525 323 776
20	3 337 857 168 384 426	3 337 860 352 573 440
21	11 920 926 635 354 486	11 920 932 320 837 632
22	100 135 807 616 763 580	100 135 831 494 197 248
23	751 018 458 383 613 488	751 018 547 919 978 496
24	3 099 441 423 652 148 001	3 099 441 608 404 238 336
25	27 418 136 574 859 993 489	27 418 137 392 016 523 264
26	96 857 546 842 298 241 268	96 857 548 285 626 286 080

k	$Q_n(n_k)$	n_k
27	730 156 894 087 760 708 214	730 156 899 527 949 287 424
28	4 656 612 859 882 900 146 794	4 656 612 877 230 338 473 984
29	26 822 090 128 376 729 645 915	26 822 090 178 337 156 628 480
30	212 341 546 852 956 928 476 256	212 341 547 050 716 436 103 168
31	1 108 273 863 395 419 430 927 682	1 108 273 863 911 501 459 357 696
32	7 613 562 045 841 775 337 850 444	7 613 562 047 614 449 998 626 816
33	25 262 124 833 820 295 233 777 308	25 262 124 836 761 198 170 996 736
34	175 787 135 949 513 502 233 670 439	175 787 135 959 745 670 777 143 296
35	948 784 872 863 616 282 998 894 234	948 784 872 891 229 576 043 167 744
36	9 880 750 439 995 132 481 304 280 769	9 880 750 440 138 916 389 931 974 656
37	49 622 030 928 687 145 579 945 860 190	49 622 030 929 048 193 483 916 967 936
38	190 993 887 372 205 292 034 200 123 988	190 993 887 372 900 123 890 134 548 480
39	1 711 669 028 733 996 013 101 990 741 984	1711 669 028 737 109 521 350 009 028 608
40	12 878 444 977 093 042 082 277 489 144 174	12 878 444 977 104 754 960 810 526 113 792
41	53 887 561 080 022 183 204 170 287 473 304	53 887 561 080 046 688 431 294 635 835 392
42	374 711 817 130 362 649 038 419 878 188 058	374 711 817 130 447 848 644 937 399 664 640
43	2 810 338 628 478 039 366 316 870 491 079 793	2 810 338 628 478 358 864 837 030 497 484 800

k	$Q_n(n_k)$	n_k
44	9 129 053 069 045 285 647 147 120 161 235 982	9 129 053 069 045 804 573 743 796 707 131 392
45	52 096 993 385 929 460 407 350 299 177 991 715	52 096 993 385 930 941 092 976 146 004 312 064
46	581 081 849 304 604 721 761 373 050 523 576 363	581 081 849 304 612 979 431 176 722 400 149 504
47	3 616 662 525 018 885 461 322 898 969 131 549 402	3 616 662 525 018 911 159 255 883 470 776 303 616
48	22 353 674 467 012 678 623 674 139 681 398 618 292	22 353 674 467 012 758 039 879 470 330 199 146 496
49	106 954 445 300 288 890 870 463 750 358 352 526 465	106 954 445 300 289 080 859 724 610 735 207 088 128
50	640 199 004 919 849 713 224 546 255 855 658 002 606	640 199 004 919 850 281 835 487 630 929 749 344 256
51	4 250 377 827 474 948 103 177 098 537 867 437 674 904	4 250 377 827 474 949 990 724 031 478 972 897 296 384
52	28 146 374 120 296 962 704 626 151 320 896 083 874 135	28 146 374 120 296 968 954 376 677 290 656 474 857 472
53	155 164 769 921 611 860 949 566 606 541 838 459 210 499	155 164 769 921 611 878 176 316 632 310 781 191 389 184
54	1 034 505 814 345 720 816 588 532 261 646 987 753 750 714	1 034 505 814 345 720 874 015 140 989 198 210 275 737 600
55	4 347 355 808 675 956 606 623 280 506 363 617 446 164 270	4 347 355 808 675 956 727 286 643 407 033 185 240 875 008
56	26 245 672 302 138 700 388 440 222 632 024 902 084 262 579	26 245 672 302 138 700 752 672 093 764 688 544 398 311 424
57	224 313 623 231 608 970 320 217 187 708 873 504 968 074 143	224 313 623 231 608 971 876 705 620 656 898 856 703 754 240
58	1 292 854 712 175 994 788 473 982 471 501 025 055 401 652 610	1 292 854 712 175 994 792 959 473 312 253 381 309 097 836 544
59	4 752 101 490 090 865 346 229 545 886 029 648 639 775 176 977	4 752 101 490 090 865 354 473 127 901 450 942 827 961 778 176

k	$Q_n(n_k)$	n_k
60	45 369 957 790 697 412 708 967 510 350 004 955 325 769 420 605	45 369 957 790 697 412 748 319 675 792 875 042 264 378 769 408
61	226 849 788 953 487 063 643 217 965 358 091 885 107 791 131 942	226 849 788 953 487 063 741 598 378 964 375 211 321 893 847 040
62	1 441 186 579 798 081 623 691 977 022 964 454 018 964 834 267 015	1 441 186 579 798 081 624 004 484 547 123 038 279 659 977 441 280
63	9 918 931 995 200 885 863 918 169 838 800 856 053 807 764 762 609	9 918 921 995 200 885 864 993 582 600 602 800 146 637 658 259 456
64	46 898 249 173 032 979 794 912 462 909 389 414 781 510 858 116 634	46 898 249 173 032 979 797 454 822 091 361 516 493 685 200 519 168
65	332 064 020 377 997 870 516 666 134 986 911 324 093 053 522 016 491	332 064 020 377 997 870 525 666 748 294 395 962 690 152 372 895 744
66	1 992 384 122 267 987 223 126 998 649 843 954 096 284 137 490 727 674	1 992 384 122 267 987 223 154 000 489 766 375 776 140 914 237 374 464
67	10 711 714 176 249 442 775 842 490 621 525 929 005 859 791 304 709 265	10 711 714 176 249 442 775 915 076 020 156 878 430 286 136 158 978 048
68	72 227 515 851 910 895 072 546 510 984 548 819 851 947 461 500 127 744	72 227 515 851 910 895 072 791 227 327 048 682 674 446 154 223 058 944
69	484 504 539 895 354 302 547 182 982 411 390 143 288 400 181 568 182 252	484 504 539 895 354 302 548 003 765 028 162 025 351 465 729 976 696 832
70	2 580 045 416 677 653 746 793 287 325 520 051 094 567 582 723 781 309 860	2 580 045 416 677 653 746 795 472 708 993 390 547 711 242 654 533 550 080

k	$Q_n(n_k)$	n_k
71	15 481 983 506 619 376 167 450 475 756 940 595 595 391 083 576 728 701 855	15 481 983 506 619 376 167 457 032 632 000 074 795 982 120 027 168 964 608
72	79 724 435 061 469 256 513 775 209 426 621 989 319 474 734 478 061 022 890	79 724 435 061 469 256 513 792 091 732 425 167 953 656 961 220 772 626 432
73	44 635 565 825 867 831 030 166 773 394 546 228 926 061 036 211 284 733 894	44 635 565 825 867 831 030 171 499 368 905 460 606 154 328 014 352 652 697
74	3 703 344 512 426 048 548 703 143 823 299 003 324 157 052 495 257 779 500 644	3 703 344 512 426 048 548 703 339 876 725 069 484 658 804 966 586 383 859 712
75	21 518 289 689 844 262 782 547 057 828 694 599 081 556 375 014 229 295 503 138	21 518 289 689 844 262 782 547 627 413 080 094 761 002 777 135 389 903 683 584

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