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On a geometric property of Lemniscates

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Let $f_n(z)=z^n+\cdots$, $g_m(z)=z^m+\cdots$ be two polynomials, denote by $E(f)$ respectively $E(g)$ the regions where $|f_n(z)|\le 1$ and $|g_m(z)|\le 1$. It is known [1] that $E(f)$ can not properly contain $E(g)$ and if the regions coincide then $f_n(z)^m \equiv g_m(z)^n$. We try to extend this result to higher dimensional Euclidean spaces and to our surprise find that the result does not generalize.

In the Euclidean space R^3, we define the product

$$p_n(w, w_k)=\prod_{k=1}^{n}|w-w_k|,$$

where $w=(w_1, w_2, w_3)$, $w_k=(w_{k1}, w_{k2}, w_{k3})$, and $|w-w_k|$ is the distance between w and w_k. Let $C(n)$ be the class of all such products with the same degree n. For any product p, we call $E(p)=\{w: p(w)\le 1\}$ the lemniscate of p. With the help of those definitions, we prove the following

THEOREM. *Let $p_n(w, w_k)$ and $p_n^*(w, w_k^*)$ be two products in $C(n)$ such that $E(p_n)\subseteq E(p_n^*)$. If all zeros w_k of p_n lie on the same plane, then we have $p_n(w, w_k)\equiv p_n^*(w, w_k^*)$.*

Proof. Since all zeros w_k of p_n lie on the same plane, we may, without loss of generality, assume that the plane is $M=\{w: w_3=0\}$. For points in M, p_n and p_n^* can be represented by

$$p_n(z, z_k)=\prod_{k=1}^{n}|z-z_k|,$$

and

$$p_n^*(z, w_k^*)=\prod_{k=1}^{n}\{|z-z_k^*|^2+w_{k3}^{*2}\}^{1/2},$$

where $z=w_1+iw_2$, $z_k=w_{k1}+iw_{k2}$, $z_k^*=w_{k1}^*+iw_{k2}^*$ and $w_k^*=(w_{k1}^*, w_{k2}^*, w_{k3}^*)$.

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With respect to the products p_n and p_n^*, we associate with the following two polynomials

$$q_n(z, z_k) = \prod_{k=1}^{n} (z - z_k),$$

and

$$q_n^*(z, z_k^*) = \prod_{k=1}^{n} (z - z_k^*).$$

It is obvious that

$$|q_n(z, z_k)| = p_n(z, z_k) \quad \text{and} \quad |q_n^*(z, z_k^*)| \leq p_n^*(z, w_k^*).$$

Now, we consider the function defined by

$$\begin{aligned} f(z) &= q_n^*(z, z_k^*)/q_n(z, z_k) \\ &= 1 + \frac{1}{z} \sum_{k=1}^{n} (z_k - z_k^*) + \cdots \end{aligned}$$

Clearly, $f(z)$ is holomorphic at the point ∞ with value $f(\infty) = 1$. Let $L(p_n) = E(p_n) \cap M$ and $L(p_n^*) = E(p_n^*) \cap M$, then by the given condition, we can see $L(p_n) \subseteq L(p_n^*)$. Moreover, by virtue of $|q_n^*(z, z_k^*)| \leq p_n^*(z, w_k^*)$, we also have $L(p_n^*) \subseteq L(q_n^*)$. It follows that

$$L(p_n) \subseteq L(p_n^*) \subseteq L(q_n^*).$$

For convenience, we let G denote the complement of $L(p_n)$, ∂G the boundary of G, and $\bar{G}$ the closure of G. Since $\bar{G}$ contains no zero of q_n, $f(z)$ is holomorphic on $\bar{G}$. In view of $L(p_n) \subseteq L(q_n^*)$, we find that for any point $z \in \partial G$

$$\begin{aligned} |f(z)| &= |q_n^*(z, z_k^*)|/p_n(z, z_k) \\ &= |q_n^*(z, z_k^*)| \leq 1. \end{aligned}$$

Since the point ∞ lies in the interior of G, it follows from the maximum principle that $|f(z)| \equiv 1$. This yields the desired result.

In the proof of the above theorem, we can see the condition of the inclusion $E(p_n) \subseteq E(p_n^*)$ is absolutely necessary. This important geometric property will naturally lead us to ask whether the condition that all zeros of p_n lie on the same plane is also necessary. In other words, we may ask whether the inclusion $E(p_n) \subseteq E(p_n^*)$ implies $p_n \equiv p_n^*$. The answer is no. As a matter of fact, the following example will show that if two zeros of p_n do not lie on the same plane as the remaining zeros of p_n, then the theorem is no longer true.

Example. For $a>0$, $\varepsilon>0$, we denote $p_6^\varepsilon(w, a)$ the product containing the six points $(\pm(a+\varepsilon), 0, 0)$, $(0, \pm(a+\varepsilon), 0)$, and $(0, 0, \pm(a+\varepsilon))$. It is sufficient to find some fixed $a>0$ and $\varepsilon>0$ such that $E(p_6^\varepsilon) \subseteq E(p_6^0)$. To do this, we need the following expansion

$$\begin{aligned}(p_6^\varepsilon(w, a))^2 &= \{[w_1-(a+\varepsilon)]^2+w_2^2+w_3^2\}\{[w_1+(a+\varepsilon)]^2+w_2^2+w_3^2\}\cdot\\ &\quad \{w_1^2+[w_2-(a+\varepsilon)]^2+w_3^2\}\{w_1^2+[w_2+(a+\varepsilon)]^2+w_3^2\}\cdot\\ &\quad \{w_1^2+w_2^2+[w_3-(a+\varepsilon)]^2\}\{w_1^2+w_2^2+[w_3+(a+\varepsilon)]^2\}\\ &= \prod_{k=1}^{3} \{(S-2aw_k)+2\varepsilon(a-w_k)+\varepsilon^2\}\{(S+2aw_k)+2\varepsilon(a+w_k)+\varepsilon^2\}\\ &= (p_6^0(w, a))^2+\varepsilon q(w, a)+\varepsilon^2\gamma(w, a, \varepsilon),\end{aligned}$$

where $S = w_1^2+w_2^2+w_3^2+a^2$,

$$\begin{aligned}\frac{1}{4a} q(w, a) &= (S-2w_1^2)(S^2-4a^2w_2^2)(S^2-4a^2w_3^2)\\ &\quad +(S-2w_2^2)(S^2-4a^2w_3^2)(S^2-4a^2w_1^2)\\ &\quad +(S-2w_3^2)(S^2-4a^2w_1^2)(S^2-4a^2w_2^2),\end{aligned}$$

and $\gamma(w, a, \varepsilon)$ is a polynomial of w_1, w_2, w_3, a, and ε. Let $Q(w, a) = (1/4a)q(w, a)$ and $R^* = \{w : \frac{1}{2} \le |w| \le 2\}$, then for $a=0$ and any point $w \in R^*$, we have

$$Q(w, 0) = (w_1^2+w_2^2+w_3^2)^5 \ge \frac{1}{2^{10}}.$$

By the continuity of Q and the compactness of R^*, there exists $0<\delta<\frac{1}{2}$ such that

$$Q(w, a) > \frac{1}{2^{11}} \quad \text{for any} \quad 0<a<\delta \quad \text{and} \quad w \in R^*.$$

Fixing a, we obtain.

$$q(w, a) > \frac{4a}{2^{11}}, \quad \text{for any} \quad w \in R^*.$$

We then can choose a small $\varepsilon>0$ for which

$$\varepsilon q(w,a)+\varepsilon^2\gamma(w,a,\varepsilon)>\frac{\varepsilon a}{2^{11}},\quad w\in R^*.$$

Now, for any point $w\in\partial E(p_6^0)$, we have $p_6^0(w,a)=1$. If we can prove that $\partial E(p_6^0)\subseteq R^*$, we shall have

$$(p_6^\varepsilon(w,a))^2>1+\frac{\varepsilon a}{2^{11}}>1,\quad w\in\partial E(p_6^0).$$

This implies that $\partial E(p_6^0)\subseteq R^3-E(p_6^\varepsilon)$ and therefore we conclude that $E(p_6^\varepsilon)\subseteq E(p_6^0)$. To prove this inclusion $\partial E(p_6^0)\subseteq R^*$, we need only observe that the choice of $a<\frac{1}{2}$ guarantees that $p_6^0(w,a)<1$ for any $|w|<\frac{1}{2}$, and $p_6^0(w,a)>1$ for any $|w|>2$. This yields the above inclusion and therefore $E(p_6^\varepsilon)\subseteq E(p_6^0)$ and $p_6^\varepsilon\neq p_6^0$.

Remark. 1. By the same argument, our theorem can be extended from R^3 to R^n.

2. We notice that the reason the same type of example can not hold in R^2 is that the corresponding polynomial

$$Q(w,0)=(w_1^2+w_2^2-2w_1^2)S^2+(w_1^2+w_2^2-2w_2^2)S^2=0.$$

3. Between the positive theorem and the negative example, there remain many problems. One of the most interesting is the following

Problem. Under the same hypothesis of the above theorem, if all zeros w_k of p_n with one exception lie on the same plane, is it still true that $p_n(w,w_k)\equiv p_n^*(w,w_k^*)$?

REFERENCE

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