INTERSECTION PROPERTIES OF SYSTEMS OF FINITE SETS

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Abstract

Let X be a finite set of cardinality n. If $L = \{l_1, \ldots, l_r\}$ is a set of non-negative integers with $l_1 < l_2 < \ldots < l_r$, and k is a natural number, then by an (n, L, k)-*system* we mean a collection of k-element subsets of X such that the intersection of any two different sets has cardinality belonging to L. We prove that if $\mathscr{A}$ is an (n, L, k)-system, with $|\mathscr{A}| > cn^{r-1}$ ($c = c(k)$ is a constant depending on k), then

(i) there exists an l_1-element subset D of X such that D is contained in every member of $\mathscr{A}$,

(ii) $(l_2 - l_1) \mid (l_3 - l_2) \mid \ldots \mid (l_r - l_{r-1}) \mid (k - l_r)$,

(iii) $\prod_{i=1}^{r} (n - l_i)/(k - l_i) \geqslant |\mathscr{A}|$ (for $n \geqslant n_0(k)$).

Parts of the results are generalized for the following cases: (a) we consider t-wise intersections, where $t \geqslant 2$; (b) the condition $|A| = k$ is replaced by $|A| \in K$ where K is a set of integers; (c) the intersection condition is replaced by the following: among $q+1$ different members $A_1, \ldots, A_{q+1}$ there are always two, A_i, A_j, such that $|A_i \cap A_j| \in L$.

We consider some related problems. An open question: let $L' \subset L$; do there exist an (n, L, k)-system of maximal cardinality ($\mathscr{A}$) and an (n, L', k)-system of maximal cardinality ($\mathscr{A}'$) such that $\mathscr{A} \supset \mathscr{A}'$?

1. Introduction

Throughout this paper lower case latin letters denote integers, capital letters stand for sets, and capital script letters for families of sets.

Let $L = \{l_1, \ldots, l_r\}$, where $l_1 < l_2 < \ldots < l_r$, and K be sets of integers. By an (n, L, K)-system we mean a family $\mathscr{A}$ of subsets of a set X, with $|X| = n$, such that for $A_1, A_2 \in \mathscr{A}$ we have $|A_1|, |A_2| \in K$, $|A_1 \cap A_2| \in L$. If $K = \{k\}$ then the notation (n, L, k)-system is applied, too.

A family $B = \{B_1, B_2, \ldots, B_c\}$ of sets is called a Δ-*system* of cardinality c if there exists a set $D \subset B_i$, with $i = 1, \ldots, c$, such that the sets $B_1 \setminus D, \ldots, B_c \setminus D$ are pairwise disjoint. D is called the *kernel* of the Δ-system.

THEOREM 1 (Erdös and Rado [7]). *There exists a function* $\varphi_c(k)$ *such that any family of* $\varphi_c(k)$ *distinct k-element sets contains a* Δ*-system of cardinality c.*

An old conjecture of Rado and the second author is that there exists an absolute constant c' such that $\varphi_c(k) < (cc')^k$. The best existing upper bound (of order about $c^k k!$) is due to Spencer [**16**].

THEOREM 2 (Erdös, Ko, and Rado [8]). *If* $\mathscr{A}$ *is an* $(n, \{l, l+1, \ldots, k-1\}, k)$*-system of maximal cardinality, then for* $n \geq n_0(k, l)$ *there exists a set D of cardinality l such that for every* $A \in \mathscr{A}$, $D \subseteq A$ *holds. In particular, for* $l = 1$, $n_0(k, l) = 2k+1$ *is the best possible value for* $n_0(k, l)$.

(For $l \geq 2$ the best existing upper bound on $n_0(k, l)$ is due to Frankl [**10**].)

THEOREM 3 (Deza [**1**]). *An* $(n, \{l\}, k)$*-system of cardinality more than* $k^2 - k + 1$ *is a* Δ*-system.*

The object of this paper is to generalize Theorems 2 and 3 for (n, L, K)-systems. In the proofs heavy use is made of Theorem 1.

The next four theorems express properties of (n, L, k)-systems.

Throughout the paper we assume that $n > n_0(k, \varepsilon)$ for $\varepsilon > 0$. Let us set $c(k, L) = \max(k - l_1 + 1, l_2{}^2 - l_2 + 1) + \varepsilon$. $\mathscr{A}$ is an (n, L, k)-system.

THEOREM 4. *If* $|\mathscr{A}| \geq c(k, L) \prod_{i=2}^{r} (n - l_i)/(k - l_i)$ *then there exists a set D of cardinality* l_1 *such that* $D \subseteq A$ *for every* $A \in \mathscr{A}$.

THEOREM 5. *If* $|\mathscr{A}| \geq k^2 2^{r-1} n^{r-1}$ *then*

$$(l_2 - l_1) \,|\, (l_3 - l_2) \,|\, \ldots \,|\, (l_r - l_{r-1}) \,|\, (k - l_r).$$

THEOREM 6.

$$|\mathscr{A}| \leq \prod_{i=1}^{r} \frac{n - l_i}{k - l_i}.$$

The following result is a generalization of Theorems 4, 5, and 6 for (n, L, K)-systems. Let $K = \{k_1, \ldots, k_s\}$, with $k_1 < \ldots < k_s$. Let us define $K_0 = K \cap \{0, \ldots, l_1\}$, $K_i = \{l_i + 1, \ldots, l_{i+1}\} \cap K$, for $i = 1, \ldots, r-1$, and

$$K_r = K \cap \{l_r + 1, \ldots, k_s\}.$$

Let us set $k_i^* = \min\{k \mid k \in K_i\}$, for $i = 0, \ldots, r$.

THEOREM 7. *Let* $\mathscr{A}$ *be an* (n, L, K)*-system.*

(i) *If* $|\mathscr{A}| > k_s c(k_s, L) \prod_{i=2}^{r} (n - l_i)/(k^* - l_i)$ *then there exists a set D of cardinality* l_1 *such that* $D \subseteq A$ *for every* $A \in \mathscr{A}$.

(ii) *If* $|\mathscr{A}| > k_s{}^3 2^{r-1} n^{r-1}$ *then there exists a* $k \in K_r$ *such that*

$$(l_2-l_1)\,|\,(l_3-l_2)\,|\,\dots\,|\,(l_r-l_{r-1})\,|\,(k-l_r).$$

(iii)

$$|\mathscr{A}| \leqslant \sum_{i=0}^{r} \varepsilon_i \prod_{\substack{j=1\\ l_j<k_i^*}}^{r} \frac{n-l_j}{k_i^*-l_j}$$

where $\varepsilon_i = 0$ *if* $K_i = \emptyset$, $\varepsilon_i = 1$ *otherwise.*

The next theorem is a common generalization of Theorems 4 and 6 and a theorem of Hajnal and Rothschild [**11**].

THEOREM 8. *Let* $\mathscr{A}$ *be a family of k-element subsets of the n-element set X such that whenever* $A_1, \dots, A_{q+1}$ *are* $q+1$ *different sets belonging to* $\mathscr{A}$ *we can find two of them* A_i, A_j *such that* $|A_i \cap A_j| \in L$ ($q \geqslant 1$ *is fixed*).

(i) *There exists a constant* $c = c(k,q)$ *such that*

$$|\mathscr{A}| > (q-1) \prod_{i=1}^{r} \frac{n-l_i}{k-l_i} + cn^{r-1}$$

implies the existence of sets $D_1, D_2, \dots, D_s$ *such that for every* $A \in \mathscr{A}$ *there exists an* i, *with* $1 \leqslant i \leqslant s$, *satisfying* $D_i \subset A$, $|D_1| = \dots = |D_s| = l_1$. *Further, if* q_i *denotes the maximum number of sets* $A_1, \dots, A_{q_i}$ *such that, for* $1 \leqslant j \leqslant q_i$, $D_i \subset A_j$ *but for* $i' \neq i$, $D_i \not\subset A_j$, *and* $|A_{j_1} \cap A_{j_2}| \notin L$ *for* $1 \leqslant j_1 < j_2 \leqslant q_i$, *then* $\sum_{i=1}^{s} q_i = q$.

(ii)

$$|\mathscr{A}| \leqslant q \prod_{i=1}^{r} \frac{n-l_i}{k-l_i} + O(n^{r-1}) \quad (n > n_0(k,q)).$$

In the next theorem we generalize Theorems 4, 5, and 6 for the case of t-wise intersections.

THEOREM 9. *Let* $\mathscr{A}$ *be a family of k-subsets of X. Suppose that, for any t different members* $A_1, \dots, A_t$ *of* $\mathscr{A}$, $|A_1 \cap \dots \cap A_t| \in L$. *Then*

(i) *there exists a constant* $c = c(k,t)$ *such that*

$$|\mathscr{A}| > cn^{r-1}$$

implies the existence of an l_1*-element set D such that* $D \subset A$ *for every* $A \in \mathscr{A}$,

(ii) $|\mathscr{A}| > cn^{r-1}$ *implies that* $(l_2-l_1)\,|\,\dots\,|\,(l_r-l_{r-1})\,|\,(k-l_r)$,

(iii) $|\mathscr{A}| \leqslant (t-1) \prod_{i=1}^{r} (n-l_i)/(k-l_i) \quad (n > n_0(k,t))$.

First versions of Theorems 4, 5, 6, and 7 were announced in [**2**], the case where $|L| = 2$ was considered in [**4**].

2. The proof of Theorems 4, 5, and 6

In the case where $r=1$ the statements of the theorems follow from Theorem 3.

Now suppose $r \geqslant 2$. We apply induction on k. The case where $k=1$ is trivial.

Let us first consider the case where $l_1=0$. Then the statement of Theorem 4 is evident. Let x be an arbitrary element of X. Let us define $\mathscr{A}_x=\{A\setminus\{x\} \mid x\in A\}$. Then $\mathscr{A}_x$ is an $(n-1,\{l_2-1,\ldots,l_r-1\},k-1)$-system. Hence, by the induction hypothesis,

$$|\mathscr{A}_x| \leqslant \prod_{i=2}^{r} \frac{(n-1)-(l_i-1)}{(k-1)-(l_i-1)} = \prod_{i=2}^{r} \frac{n-l_i}{k-l_i}. \tag{1}$$

Counting the number of pairs (x,A), for $x\in A\in\mathscr{A}$, in two different ways we obtain

$$k|\mathscr{A}| = \sum_{x\in X} |\mathscr{A}_x|. \tag{2}$$

From (1) and (2) it follows that

$$|\mathscr{A}| \leqslant \frac{|X|}{k}\prod_{i=2}^{r}\frac{n-l_i}{k-l_i} = \prod_{i=1}^{r}\frac{n-l_i}{k-l_i},$$

which proves Theorem 6 for this case.

Now we wish to prove Theorem 5. So we may suppose that

$$|\mathscr{A}| > k^2 2^{r-1} n^{r-1}.$$

Let us set $d=k^2 2^{r-2} n^{r-2}$ and $\mathscr{A}^0=\mathscr{A}$. If $\mathscr{A}^j$ is defined and there exists an element $x\in X$ such that $0<|\mathscr{A}^j_x|\leqslant d$ then define

$$\mathscr{A}^{j+1} = \mathscr{A}^j \setminus \{A\in\mathscr{A}^j \mid x\in A\}.$$

After finitely many steps the procedure stops, that is, we obtain a family $\mathscr{A}'$ in which every element of X has either degree 0 or degree more than d, and

$$|\mathscr{A}'| \geqslant |\mathscr{A}| - nd > k^2 2^{r-2} n^{r-1}.$$

Let X' be the set of elements of X which have non-zero degree in $\mathscr{A}'$. If $x\in X'$ then $\mathscr{A}'_x$ is an $(n-1,\{l_2-1,\ldots,l_r-1\},k-1)$-system, and

$$|\mathscr{A}'_x| > d = k^2 2^{r-2} n^{r-2},$$

whence by the induction hypothesis there exists a set $D_x\subset X\setminus\{x\}$, with $|D_x|=l_2-1$, such that $D_x\subset A$ for every $A\in\mathscr{A}'_x$.

We assert that for any $y\in D_x$, $D_y=(D_x\setminus\{y\})\cup\{x\}$.

Suppose that for some y it does not hold. As any member A of $\mathscr{A}'_x$ contains y so it has to contain D_y as well and consequently $A\supseteq((D_x\cup D_y)\setminus\{x\})$. $|(D_x\cup D_y)\setminus\{x\}|\geqslant l_2$, which implies that any two elements of $\mathscr{A}'_x$ intersect in at least l_2 elements. Hence $\mathscr{A}'_x$ is an

$(n-1,\{l_3-1,\ldots,l_r-1\},k-1)$-system. So Theorem 6 implies that

$$|\mathscr{A}'_x| \leqslant \prod_{i=3}^{r} \frac{n-l_i}{k-l_i} \leqslant n^{r-2},$$

a contradiction. So we have proved that, for $x \neq y$, $D_x \cup \{x\}$ and $D_y \cup \{y\}$ coincide or they are disjoint.

Consequently the sets $D_x \cup \{x\}$, for $x \in X'$, form a partition of the set X' such that any member A of $\mathscr{A}'$ is the union of some of them. Hence $l_2 | k$. We assert that for any $3 \leqslant i \leqslant r$ there are two sets $A, B \in \mathscr{A}'$ such that $|A \cap B| = l_i$. Indeed, otherwise $\mathscr{A}'$ is an $(n,\{l_1,\ldots,l_{i-1},l_{i+1},\ldots,l_r\},k)$-system, so by Theorem 6

$$|\mathscr{A}'| \leqslant \prod_{j\neq i} \frac{n-l_j}{k-l_j} \leqslant n^{r-1},$$

a contradiction. Now if $|A \cap B| = l_i$ then that $l_2 | l_i$ follows from the fact that A and B, whence $A \cap B$ too, are the unions of some of the pairwise disjoint l_2-element sets $D_x \cup \{x\}$. In particular it follows that

$$l_2 = (l_2 - l_1) \,|\, (l_3 - l_2).$$

Applying the induction hypothesis to $\mathscr{A}'_x$ we obtain that

$$((l_3-1)-(l_2-1)) \,|\, ((l_4-1)-(l_3-1)) \,|\, \ldots \,|\, ((k-1)-(l_r-1)),$$

that is, $(l_3-l_2)\,|\,(l_4-l_3)\,|\,\ldots\,|\,(k-l_r)$, which finishes the proof for the case where $l_1 = 0$.

Now we need a lemma.

LEMMA 1. (i) *Let the sets $A_1,\ldots,A_c$ form a Δ-system with kernel D, where $|D| = l_1$, and $c \geqslant k-l_1+2$. Then for any set B, with $|B| \leqslant k$, $|B \cap A_i| \geqslant l_1$ for $i = 1,\ldots,c$ implies $B \supset D$.*

(ii) *Let the sets $F_i^1, F_i^2, \ldots, F_i^t$ form a Δ-system with kernel E_i, where $|F_i^j| = k$ for $i = 1,\ldots,s$, $j = 1,\ldots,t$.*

Suppose that the sets E_i form a Δ-system with kernel D, where $|D| = l$, and that $t > (s-1)(k-1)$. Then there are indices $1 \leqslant j_i \leqslant t$ for $i = 1,\ldots,s$ such that the sets $F_i^{j_i}$ form a Δ-system with kernel D.

Proof. Let us set $|B \cap D| = l' \leqslant l_1$. Then $|B \cap A_i| \geqslant l_1$ implies that $|B \cap (A_i \setminus D)| \geqslant l_1 - l'$ for $i = 1,\ldots,c$. As the sets $A_i \setminus D$ are pairwise disjoint we obtain

$$k = |B| \geqslant l' + c(l_1 - l') \geqslant l' + (k - l_1 + 2)(l_1 - l')$$

or equivalently $(k-l_1) \geqslant (k-l_1+1)(l_1-l')$, which yields $l_1 = l'$ as desired.

Now we prove (ii). Suppose that for $i = 1,\ldots,s'$ we have chosen indices $1 \leqslant j_i \leqslant t$ such that the sets $F_i^{j_i}$ form a Δ-system with kernel D. Now we wish to choose the index $j = j_{s+1}$ in such a way that $F_{s'}^{j} \cap F_i^{j_i} = D$

for $i = 1, \ldots, s'$ and $F^j_{s'+1} \cap E_i = D$ for $i = s'+2, \ldots, s$. An index j does not satisfy the conditions if and only if $F^j_{s'+1} \setminus D$ is not disjoint from the set

$$H_{s'} = \left(\bigcup_{i \leqslant s'} (F^j_i \setminus D) \right) \cup \left(\bigcup_{i > s'+1} (E_i \setminus D) \right).$$

As the sets $F^j_{s'+1} \setminus E_{s'+1}$ are pairwise disjoint, for $j = 1, \ldots, t$, and $|H_{s'}| = s'(k-l) + \sum_{i=s'+2}^{s} |\otimes E_i \setminus D| \leqslant (s-1)(k-1) < t$, the appropriate choice of $F^j_{s'+1}$ is always possible, which proves the lemma.

Now we turn to the proof of Theorem 4 for the case where $l_1 > 0$. If we can find $k - l_1 + 2$ sets $A_1, \ldots, A_{k-l_1+2}$ belonging to $\mathscr{A}$ which form a Δ-system with kernel D, where $|D| = l_1$, then it follows that $D \subseteq A$ for every $A \in \mathscr{A}$, since $|A \cap A_i| \geqslant l_1$, for $i = 1, \ldots, k - l_1 + 2$, and from the lemma.

So we may assume that such a Δ-system does not exist. Now let us choose a set D^2_1 of cardinality l_2 such that D^2_1 is the kernel of a Δ-system formed by k^2 members of $\mathscr{A}$ $(A^{2,1}_1, \ldots, A^{2,k^2}_1)$, and let us define $\mathscr{A}^2_1 = \{A \in \mathscr{A} \mid D^2_1 \nsubseteq A\}$.

Now we choose a set D^2_2 of cardinality l_2 which is the kernel of a Δ-system formed by k^2 different members of $\mathscr{A}^2_1$ and define $\mathscr{A}^2_2 = \{A \in \mathscr{A}^2_1 \mid D^2_2 \nsubseteq A\}$, and so on. After a finite number of steps, say q_2, we cannot find a set $D^2_{q_2+1}$ of cardinality l_2 which is the kernel of a Δ-system formed by k^2 different members of $\mathscr{A}^2_{q_2}$.

Now we choose a set D^3_1 of cardinality l_3 which is the kernel of a Δ-system formed by k^3 different elements of $\mathscr{A}^2_{q_2}$ and define

$$\mathscr{A}^3_1 = \{A \in \mathscr{A}^2_{q_2} \mid D^3_1 \nsubseteq A\};$$

after say q_3 steps we cannot find such a $D^3_{q_3+1}$. Then we look for an l_4-element set which is the kernel of a Δ-system formed by k^4 members of $\mathscr{A}^3_{q_3}$, and so on. At last we obtain a family $\mathscr{A}^r_{q_r}$ which does not contain any Δ-system with kernel D_j, where $|D_j| = l_j$, and of cardinality k^j $(j = 1, \ldots, r)$. As $\mathscr{A}^r_{q_r}$ is an (n, L, k)-system it means that $\mathscr{A}^r_{q_r}$ does not contain any Δ-system of cardinality at least k^r, implying that

$$|\mathscr{A}^r_{q_r}| < \varphi_k r(k). \tag{3}$$

Now we assert that

$$q_j < \varphi_{k^{j-1}}(l_j) \quad (j = 3, \ldots, r) \tag{4}$$

and that

$$q_2 \leqslant c(k, L). \tag{5}$$

If it is not true then we could find among the kernels of cardinality l_j a Δ-system of cardinality k^i and kernel D_i, with $|D_i| = l_i$, for some $1 \leqslant i < j$, or, for $j = 2$, a Δ-system of cardinality $k - l_1 + 2$ and with a kernel D_1 of cardinality l_1.

Applying Lemma 1 we obtain that, for the corresponding j, $\mathscr{A}^{j-1}_{q_{j-1}}$ contains a Δ-system consisting of k^i sets and having a kernel D_i, with $|D_i| = l_i$, for $1 \leqslant i < j$, or for $j = 2$ that $\mathscr{A}$ contains a Δ-system of cardinality $k - l_1 + 2$ and with kernel D_1, with $|D_1| = l_1$.

The first possibility contradicts the choice of q_{j-1} while the second one is contrary to our assumptions. So (4) and (5) are proved.

If $1 \leqslant u \leqslant q_j$ then define $\mathscr{A}(j,u) = \{A \setminus D^j_u \mid A \in \mathscr{A}, D^j_u \subseteq A\}$. Then $\mathscr{A}(j,u)$ is an $(n-l_j, \{0, l_{j+1}-l_j, \ldots, l_r - l_j\}, k - l_j)$-system.

Hence by the induction hypothesis

$$|\mathscr{A}(j,u)| \leqslant \prod_{i=j}^{r} \frac{(n-l_j)-(l_i-l_j)}{(k-l_j)-(l_i-l_j)} = \prod_{i=j}^{r} \frac{n-l_i}{k-l_i}.$$

Consequently,

$$|\mathscr{A}^{j-1}_{q_{j-1}} \setminus \mathscr{A}^{j}_{q_j}| \leqslant q_j \prod_{i=j}^{r} \frac{n-l_i}{k-l_i} \tag{6}$$

for $j = 2, \ldots, r$, where $\mathscr{A}^1_{q_1} = \mathscr{A}$. From (3), (4), (5), and (6) we obtain

$$\begin{aligned} |\mathscr{A}| &= \sum_{j=2}^{r} |\mathscr{A}^{j-1}_{q_j} \setminus \mathscr{A}^{j}_{q_j}| + |\mathscr{A}^{r}_{q_r}| \\ &\leqslant c(k,L) \prod_{j=2}^{r} \frac{n-l_i}{k-l_i} + \sum_{j=3}^{r+1} \varphi_{k^{j-1}}(l_j) \prod_{i=j}^{r} \frac{n-l_i}{k-l_i}. \end{aligned} \tag{7}$$

In (7) we use the conventions that $l_{r+1} = k$ and that the empty product is 1.

From (7) we obtain

$$|\mathscr{A}| \leqslant (c(k,L) + o(1)) \prod_{i=2}^{r} \frac{n-l_i}{k-l_i},$$

which is a contradiction as $n > n_0(k)$. Now the proof of Theorem 4 is finished. So in proving Theorems 5 and 6 we may suppose that there exists a set D, with $|D| = l_1$, such that $D \subseteq A$ for every $A \in \mathscr{A}$.

Let us define $\mathscr{A}(D) = \{A \setminus D \mid A \in \mathscr{A}\}$. It follows then that $\mathscr{A}(D)$ is an $(n - l_1, \{0, l_2 - l_1, \ldots, l_r - l_1\}, k - l_1)$-system. We know that $k - l_1 < k$ as $l_1 > 0$. Hence both Theorem 5 and Theorem 6 follow from the induction hypothesis.

Equality in the estimation of Theorem 6 (briefly 'equality') is realizable by the hyperplane-family of any perfect matroid-design (cf. [**14**]) of rank $|L|+1$, such that for any j-flat F^j we have $|F^{|L|}| = k$, $|F^{|L|+1}| = n$, $|F^j| = l_{j+1}$, $0 \leqslant j < |L|$. For example, in the case when L is an arithmetic progression with difference $d = l_2 - l_1$, we may obtain equality by an $(l_2 - l_1)$-inflation of an $S(|L|, k/d, n/d)$ if this Steiner-system exists. The affine and projective geometries provide other examples when equality is possible. (The collection of all the j-flats F^j with $|F^j| = k_i^*$ for some $0 \leqslant i < |L|$ gives equality in the estimation (iii) of Theorem 7.)

In the case where $L = \{0, 1, 3\}$ the equality implies the existence of an $S(2, 3, k)$, whence $2 | (k-1)$, $6 | (k-1)k$. In the first case, $k = 5$, equality is not possible, moreover it can be proved that no $(n, L, 5)$-system has more than $2n^{11/4} = o(n^{|L|})$ elements though $(1-0) | (3-1) | (5-3)$. The collection of the 2-dimensional subspaces of $\mathrm{PG}(s, 2)$, $\mathrm{AG}(s, 3)$, respectively, provide equality for the cases where $k = 7, 9$. The first open problem is to decide whether there are infinitely many values of n for which we can have equality in the case where $k = 13$.

REMARK 1 (on Theorem 4). Without changing the argument we can prove the following: if $k' \geqslant k$ and

$$|\mathscr{A}| > c(k', L) \prod_{j=2}^{r} \frac{n-l_i}{k-l_i}$$

then there are sets $A_1, \ldots, A_{k'-l_1+2} \in \mathscr{A}$ which form a Δ-system with a kernel of cardinality l_1.

REMARK 2 (on Theorem 5). For the case where $L = \{0, l\}$ in [**4**] it was shown that $|\mathscr{A}| > n$ implies $l | k$ and this estimation is the best possible (this is a generalization of the Fisher–Majumdar inequality [**15**]).

REMARK 3 (on Theorem 6). In [**16**] it was shown by Ray–Chaudhuri and Wilson that $|\mathscr{A}| \leqslant \binom{n}{|L|}$ for any (n, L, k)-system $\mathscr{A}$ (this is another generalization of the Fisher–Majumdar inequality [**15**]). This estimation does not depend on k, but it is weaker than Theorem 6 for $|\mathscr{A}| > C(k) n^{|L|-1}$. Its proof (using, a propos, linear independence of certain systems of vectors) will be interesting to extend for the cases of Theorems 7, 8, and 9.

3. The proof of Theorem 7

We apply induction on r. If $\mathscr{A}'$ denotes $\{A \in \mathscr{A} \mid |A| > l_r\}$ then it follows from the induction hypothesis that

$$|\mathscr{A} \setminus \mathscr{A}'| \leqslant \sum_{i=0}^{r-1} \varepsilon_i \prod_{j=1}^{i} \frac{n-l_j}{k_i^* - l_j},$$

when $r \geqslant 2$, while the same inequality holds trivially for $r = 1$ as well. Hence

$$|\mathscr{A} \setminus \mathscr{A}'| \geqslant |\mathscr{A}| - r \prod_{j=1}^{r} \frac{n-l_j}{k_r^* - l_j}.$$

First we prove (i). As $|K_r| \leqslant k_s - l_r \leqslant k_s - r$ and $c(k_s, L) \geqslant 1$, there exists a $k \in K_r$ such that

$$|\{A \in \mathscr{A} \mid |A| = k\} = \mathscr{A}(k)| > c(k_s, L) \prod_{i=2}^{r} \frac{n-l_i}{k_r^* - l_i} \geqslant c(k_s, L) \prod_{i=2}^{r} \frac{n-l_i}{k-l_i}.$$

Hence by Remark 1 there exist $k_s - l_1 + 2$ elements $A_1, \ldots, A_{k_s - l_1 + 2} \in \mathscr{A}(k)$ such that for $D = A_1 \cap A_2$, $|D| = l_1$ and the sets $A_1 \setminus D, \ldots, A_{k_s - l_1 + 2} \setminus D$ are pairwise disjoint. So by Lemma 1, for every $A \in \mathscr{A}$, $A \supset D$ and hence (i) holds.

Now let us prove (ii). Now we can find a $k \in K_r$ for which

$$|\mathscr{A}(k)| > k_s{}^2\, 2^{r-1} n^{r-1}$$

holds. As $\mathscr{A}(k)$ is an (n, L, k)-system, Theorem 5 implies that (ii) is true.

To prove (iii) observe first that, for $x \in X \setminus D$, $\mathscr{A}'_x = \{A \setminus D \mid x \in A \in \mathscr{A}'\}$ satisfies the hypothesis of the theorem with

$$n' = n - l_1, \quad K' = \{k - l_1 \mid k \in K_r\}, \quad L = \{l_2 - l_1, \ldots, l_r - l_1\},$$

so by the induction hypothesis it follows that

$$|\mathscr{A}'_x| \leqslant \prod_{j=2}^{r} \frac{n - l_j}{k_r^* - l_j}.$$

Counting the number of pairs (x, A), where $x \in A$, $x \in X \setminus D$, and $A \in \mathscr{A}'$, in two different ways we obtain

$$\sum_{x \in X \setminus D} |\mathscr{A}'_x| \geqslant |\mathscr{A}'|\,(k_r^* - l_1)$$

and consequently

$$(n - l_1) \prod_{j=2}^{r} \frac{n - l_j}{k_r^* - l_j} \geqslant |\mathscr{A}'|\,(k_r^* - l_1),$$

and (iii) follows.

From the estimation (iii) of Theorem 7 it follows that in the case where $L = [l, k-1]$, $K = [g, h]$, and $n > n_0(k)$, any (n, L, K)-system satisfies

$$|\mathscr{A}| \leqslant \sum_{i=g}^{k} \binom{n-l}{i-l},$$

which generalizes Theorem 2 of Hilton [**13**] for the case where $l > 1$.

4. The proof of Theorem 8

We apply double induction on k, q. Let us first consider the case where $l_1 = 0$. In this case (i) holds automatically. To prove (ii) observe that if we define $\mathscr{A}_x = \{A \setminus \{x\} \mid A \in \mathscr{A},\, x \in A\}$, then $\mathscr{A}_x$ satisfies the hypothesis of the theorem with $n' = n - 1$, $L' = \{l_2 - 1, \ldots, l_r - 1\}$, $k' = k - 1$, and $q' = q$. Hence by the induction hypothesis $|\mathscr{A}_x| \leqslant q \prod_{i=2}^{r} (n - l_i)/(k - l_i)$, and this equality holds for the case where $L = \{l_1\}$ too. As $|\mathscr{A}| - k = \sum_{x \in X} |\mathscr{A}_x|$, it follows that $|\mathscr{A}| \leqslant q \prod_{i=1}^{r} (n - l_i)/(k - l_i)$.

Now suppose that $l_1 > 0$. Let us choose a set D_1, with $|D_1| = l_1$, such that there exist $A_1^1, \ldots, A_{kq}^1 \in \mathscr{A}$ satisfying $A_i^1 \cap A_j^1 = D_1$ for $1 \leqslant i < j \leqslant kq$.

Then let us set $\mathscr{A}^1 = \{A \in \mathscr{A} \mid A \supseteq D_1\}$. Now we choose D_2 in the same way and define $\mathscr{A}^2$, and so on. After a finite number, say p, of steps $\mathscr{A}^p$ does not contain any Δ-system of cardinality kq and with kernel D, where $|D| = l_1$. We assert that $p \leqslant q$. Otherwise we have at least $q+1$ Δ-systems $A_1^i, A_2^i, \ldots, A_{kq}^i$ with kernels D_i, where $|D_i| = l_1$, for $i = 1, \ldots, q+1$. As the sets $A_1^1 \setminus D_1, \ldots, A_{kq}^1 \setminus D_1$ are pairwise disjoint and

$$\left| \bigcup_{i=2}^{q+1} D_i \right| \leqslant l_1 q < kq,$$

we can find an index j_1 such that $(A_{j_1}^1 \setminus D_1) \cap D_i = \emptyset$ for $i = 1, \ldots, q+1$.

If we have chosen $A_{j_1}^1, \ldots, A_{j_s}^s$ then we want to choose $A_{j_{s+1}}^{s+1}$ in such a way that $(A_{j_{s+1}}^{s+1} \setminus D_{s+1}) \cap (A_{j_i}^i \setminus D_i) = \emptyset = (A_{j_{s+1}}^{s+1} \setminus D_{s+1}) \cap D_{i'}$, for $1 \leqslant i \leqslant s$, $s+2 \leqslant i' \leqslant q+1$. As

$$\left| \bigcup_{i=1}^{s} (A_{j_i}^i \setminus D_i) \right| + \left| \bigcup_{i'=s+2}^{q+1} D_{i'} \right| \leqslant s(k - l_1) + (q - s) l_1 < qk$$

and the sets $A_j^{s+1} \setminus D_{s+1}$ are pairwise disjoint, for $j = 1, \ldots, kq$, such a choice of $A_{j_{s+1}}^{s+1}$ is possible. But if $1 \leqslant s < s' \leqslant q+1$, then

$$A_{j_s}^s \cap A_{j_{s'}}^{s'} \subseteq D_s \cap D_{s'},$$

which implies that

$$|A_{j_s}^s \cap A_{j_{s'}}^{s'}| \notin L,$$

a contradiction.

Now we want to show that $|\mathscr{A}^p| = O(n^{r-1})$. We proceed in essentially the same way as in the proof of Theorem 4 for the case where $l_1 > 0$, so the proof will only be sketched.

Let us choose D_1^2, with $|D_1^2| = l_2$, in such a way that there exist $A_1, \ldots, A_{k^2 q}$ belonging to $\mathscr{A}^p$ which form a Δ-system with kernel D_1^2.

Now define $\mathscr{A}_1^2 = \{A \in \mathscr{A}^p \mid A \not\supseteq D_1^2\}$. Then choose D_2^2, and so on. When there are no more l_2-element sets which are kernels of a Δ-system of cardinality qk^2 then try to find an l_3-set which is the kernel of a Δ-system of cardinality qk^3, and so on.

By Lemma 1, among the Δ-systems of kernel l_i there are no qk^{i-1} which form a Δ-system, whence their number is less than $\varphi_{qk^{i-1}}(l_i)$. If D_i' is an l_i-element set, then $\mathscr{A}_{D_i'} = \{A \setminus D_i' \mid A \in \mathscr{A}, D_i' \subset A\}$ satisfies the hypothesis of the theorem with $n' = n - l_i$, $k' = k - l_i$, $L' = \{0, l_{i+1} - l_i, \ldots, l_r - l_i\}$, and $q' = q$.

The induction hypothesis yields

$$|\mathscr{A}_{D_i'}| \leqslant (q+1)\prod_{j=i}^{r}\frac{n-l_j}{k-l_j} = O(n^{r-1}) \quad \text{for } i \geqslant 2,$$

from which it follows that $|\mathscr{A}^p| = O(n^{r-1})$.

If E is a set of cardinality more than l_1 then the family

$$\mathscr{A}_E = \{A \in \mathscr{A} \mid E \subseteq A\}$$

satisfies the assumptions of the theorem with $n' = n-|E|$, $k' = k-|E|$, $L' = \{l_2-|E|, \ldots, l_r-|E|\} \cap \{0, 1, \ldots, l_r\}$, and $q' = q$.

Hence it follows by induction that $|\mathscr{A}_E| = O(n^{r-1})$. Let us set $\mathscr{B}_i = \{A \in \mathscr{A} \mid D_i \subset A,\, D_j \nsubseteq A \text{ for } j \neq i\}$. Now it follows that

$$\left|\mathscr{A} \setminus \bigcup_{j=1}^{p} \mathscr{B}_j\right| = O(n^{r-1})$$

as this family can be written as the union of the families $\mathscr{A}_{D_{1,2}}$, where $D_{1,2} = D_{i_1} \cup D_{i_2}$, for $1 \leqslant i_1 < i_2 \leqslant p$, and $|D_{i_1} \cup D_{i_2}| > l_1$.

Let c' be a sufficiently large constant and let us set

$$q_i - 1 = \left[\{|\mathscr{B}_i| - c'n^{r-1}\} \Big/ \left\{\prod_{j=1}^{r}\frac{n-l_j}{k-l_j} + c_0(k,q)n^{r-1}\right\}\right]$$

($[x]$ is the greatest integer not exceeding x).

As $|\mathscr{A}| \leqslant |\mathscr{A}^p| + |\mathscr{A} \setminus \bigcup_{j=1}^{p}\mathscr{B}_j| + \sum_{j=1}^{p}|\mathscr{B}_j|$, it follows for $c > c_0(c', k, q)$ that $\sum_{i=1}^{p} q_i \geqslant q$. Let q_i' denote the greatest integer such that there exist $A_1^i, \ldots, A_{q_i'}^i \in \mathscr{B}_i$ satisfying $|A_{j_1}^i \cap A_{j_2}^i| \notin L$ for $1 \leqslant j_1 < j_2 \leqslant q_i'$.

As $\bar{\mathscr{B}}_i = \{B \setminus D_i \mid B \in \mathscr{B}_i\}$ satisfies the assumptions of the theorem with $n' = n-l_1$, $k' = k-l_1$, $L' = \{0, l_2-l_1, \ldots, l_r-l_1\}$, and $q' = q$, by induction we obtain $q_i' \geqslant q_i$. If $q_i' = q_i$ for $i = 1, \ldots, p$ and $\sum_{i=1}^{p} q_i = q$ then we are done. So we may suppose that either $\sum q_i > q$ or, for some $1 \leqslant j \leqslant p$, $q_j' > q_j$. In the latter case we may assume that $q_1' > q_1$. Hence in any case $q_1' + \sum_{j=2}^{p} q_j > q$.

Let us choose q_1' sets $A_1, \ldots, A_{q_1'} \in \mathscr{B}_1$ such that $|A_{j_1} \cap A_{j_2}| \notin L$ for $1 \leqslant j_1 < j_2 \leqslant q_1'$. Suppose that $A_{q_1'+1}, \ldots, A_{q_1'+q_2+\ldots+q_{i-1}}$ are defined already. Let us set

$$\mathscr{B}_i' = \mathscr{B}_i \setminus \{B \in \mathscr{B}_i \mid \text{there exists } j,$$
$$1 \leqslant j \leqslant q_1' + q_2 + \ldots + q_{i-1}, \text{ such that } |B \cap A_j| \geqslant l_1\}.$$

It can be seen as above that $|\mathscr{B}_i'| = |\mathscr{B}_i| + O(n^{r-1})$. Hence by the induction hypothesis we can find q_i sets

$$A_{q_1'+q_2+\ldots+q_{i-1}+1}, \ldots, A_{q_1'+q_2+\ldots+q_{i-1}+q_i}$$

such that the cardinality of the intersection of any two different sets among them does not belong to L. Moreover, if

$$1 \leqslant j_1 \leqslant q_1' + q_2 + \ldots + q_{i-1} < j_2 \leqslant q_1' + q_2 + \ldots + q_i$$

then $|A_{j_1} \cap A_{j_2}| < l_1$, implying that $|A_{j_1} \cap A_{j_2}| \notin L$. Finally we obtain $q_1' + \sum_{i=2}^{p} q_i > q$ sets $A_1, \ldots, A_{q_1' + q_2 + \ldots + q_p}$ such that $|A_{j_1} \cap A_{j_2}| \notin L$ for $1 \leqslant j_1 < j_2 \leqslant q+1$, a contradiction. So necessarily $q_i' = q_i$, $q = \sum_{i=1}^{p} q_i$, and by the induction hypothesis

$$|\mathscr{B}_i| = |\bar{\mathscr{B}}_i| \leqslant q_i \prod_{j=1}^{r} \frac{n-l_j}{k-l_j} + c_0(k-l_1, q_i) n^{r-1},$$

yielding

$$|\mathscr{A}| \leqslant q \prod_{j=1}^{r} \frac{n-l_j}{k-l_j} + c_0(k,q) n^{r-1}$$

for an appropriate choice of $c_0(k,q)$, which proves (ii).

To finish the proof of (i) we have to show that $\mathscr{A}^p = \emptyset$. Suppose it is not the case and let $A_0 \in \mathscr{A}^p$. We define A_i recurrently. If, for some $i > 0$, $A_0, A_1, \ldots, A_{q_1}, A_{q_1+1}, \ldots, A_{q_1+\ldots+q_{i-1}}$ are defined then first define

$$\mathscr{B}_i' = \mathscr{B}_i \setminus \{B \in \mathscr{B}_i \mid \text{there exists } j,$$
$$0 \leqslant j \leqslant q_1 + \ldots + q_{i-1}, \text{ such that } |B \cap A_j| \geqslant l_1\}.$$

Then $|\mathscr{B}_i'| = |\mathscr{B}_i| + O(n^{r-1})$. So by the induction hypothesis we can define

$$A_{q_1+q_2+\ldots+q_{i-1}+1}, \ldots, A_{q_1+\ldots+q_{i-1}+q_i} \in \mathscr{B}_i'$$

such that, for $q_1 + \ldots + q_{i-1} + 1 \leqslant j_1 < j_2 \leqslant q_1 + \ldots + q_i$, $|A_{j_1} \cap A_{j_2}| \notin L$. But, as $\sum_{i=1}^{p} q_i = q$, this means that in the end we find $q+1$ sets $A_0, \ldots, A_q \in \mathscr{A}$ such that $|A_{j_1} \cap A_{j_2}| \notin L$ for $0 \leqslant j_1 < j_2 \leqslant q$, and this final contradiction concludes the proof of the theorem.

REMARK 4. We conjecture that the assumptions of Theorem 8 imply that

$$|\mathscr{A}| \leqslant q \prod_{i=1}^{r} \frac{n-l_i}{k-l_i},$$

that is, we may omit the last term in (ii). If this conjecture is true then it is the best possible in certain cases.

Let $k > 2r$ and $X = \{1, \ldots, n\}$. Let $\varpi_1, \ldots, \varpi_q$ be q random permutations of X and let $\mathscr{A}$ be an (n, L, k)-system of cardinality $\prod_{i=1}^{r} (n-l_i)/(k-l_i)$ if such a system exists. If $i \in X$ then $\varpi(i)$ is the image of i by ϖ. Further, set $\varpi(A) = \{\varpi(a) \mid a \in A\}$ for $A \subset X$ and $\varpi(\mathscr{A}) = \{\varpi(A) \mid A \in \mathscr{A}\}$. Then $\varpi_1(\mathscr{A}), \ldots, \varpi_q(\mathscr{A})$ are (n, L, k)-systems, and if $n > n_0(\varepsilon)$ it can be easily seen that they are pairwise disjoint with probability not less than $1-\varepsilon$. So for an appropriate choice of $\varpi_1, \ldots, \varpi_q$ the family

$$\mathscr{B} = \varpi_1(\mathscr{A}) \cup \varpi_2(\mathscr{A}) \cup \ldots \cup \varpi_q(\mathscr{A})$$

satisfies the assumptions of Theorem 8 and has cardinality

$$q \prod_{i=1}^{r} \frac{n-l_i}{k-l_i}.$$

REMARK 5. In the case where $L = \{l_1, l_1+1, \ldots, k-1\}$ Theorem 8 yields that for a system $\mathscr{A}$, of maximum cardinality there are q different l_1-sets $D_1, \ldots, D_q$ such that every element of $\mathscr{A}$ contains at least one of the D_i's. The maximality of $\mathscr{A}$ implies that $\mathscr{A} = \{A \subseteq X \mid \text{there exists } i,\ 1 \leqslant i \leqslant q, \text{ such that } D_i \subseteq A\}$, and the sets $D_1, \ldots, D_q$ are pairwise disjoint, that is, Theorem 8 is indeed a generalization of the Hajnal–Rothschild theorem [**11**].

5. The proof of Theorem 9

We proceed in essentially the same way as with the proof of Theorems 4, 5, and 6. Therefore the proof is only briefly sketched. We apply induction on k; the case where $k = 1$ is trivial.

(a) $l_1 = 0$. In this case (i) holds automatically with $D = \emptyset$. In proving (ii) we may suppose that $r \geqslant 2$ as otherwise we have nothing to prove. Choosing the constant $c(k,t)$ in such a way that it satisfies

$$c(k,t) \geqslant 2c(k-1,t)$$

we may, as in the proof of Theorem 5, successively omit the elements of X which are contained in at most $c(k-1,t)n^{r-2}$ members of $\mathscr{A}$. Finally, we obtain a family $\mathscr{A}'$ which consists of subsets of a set $X' \subseteq X$, where every element of X' has degree greater than $c(k-1,t)n^{r-2}$ and $|\mathscr{A}'| > c(k-1,t)n^{r-1}$.

Now using the induction hypothesis we obtain that for every $x \in X'$ there exists a set D_x such that $|D_x| = l_2 - 1$, for $x \notin D_x$, and $A \in \mathscr{A}'$, for $x \in A'$ imply $D_x \subseteq A$. It follows, as in the proof of Theorem 5, that the sets $\{x\} \cup D_x$ form a partition of X' and, by the induction hypothesis,

$$|\mathscr{A}'| > c(k-1,t)n^{r-1}$$

implies that for any $l \in L$ there exist $A_1, \ldots, A_l \in \mathscr{A}'$ such that

$$|A_1 \cap \ldots \cap A_l| = l.$$

But $A_1 \cap \ldots \cap A_l$ is the disjoint union of some of the l_2-element sets $x \cup D_x$, and it follows that $l_2 | l_i$ and $l_2 | k$. The property $l_3 | l_4 | \ldots | l_r | k$ follows from the induction hypothesis applied to one of the families

$$\mathscr{A}'_x = \{A \setminus x \mid x \in A \in \mathscr{A}'\}, \quad \text{for } x \in X'.$$

Now we prove (iii). Let $x \in X$. If $r > 1$ then we can use the induction hypothesis for the family $\mathscr{A}_x = \{A \setminus x \mid x \in A \in \mathscr{A}\}$ and obtain

$$|\mathscr{A}_x| \leqslant (t-1) \prod_{i=2}^{r} \frac{n-l_i}{k-l_i}.$$

If $r = 1$, that is, $L = \{0\}$, then it is obvious that $|\mathscr{A}_x| \leqslant t-1$. Counting the number of the incident pairs (x, A), where $x \in A \in \mathscr{A}$, in two different ways we obtain that $|\mathscr{A}_x| = k|\mathscr{A}|$, yielding

$$|\mathscr{A}| \leqslant (t-1) \prod_{i=1}^{r} \frac{n-l_i}{k-l_i},$$

as desired.

(b) $l_1 > 0$. First we prove (i). If there are $k+t$ sets $A_1, \ldots, A_{k+t} \in \mathscr{A}$ which form a Δ-system with kernel D, where $|D| \leqslant l_1$, then by the assumptions of the theorem $|D| = l_1$ and it follows that $D \subset A$ for every $A \in \mathscr{A}$. So, in the case where $r = 1$, the assertion follows from Theorem 1 for $c = \varphi_{t+k}(k)$.

Now we do the same thing as in the proof of Theorem 3. We select all the l_2-element subsets of X which are kernels of a Δ-system of cardinality $(k+t)^2$, consisting of members of $\mathscr{A}$.

As we may suppose that no $(k+t)$-element Δ-system with an l_1-element kernel exists, Lemma 1 yields that there are at most $\varphi_{k+t}(l_2)$ such l_2-element sets. Then we omit all the sets containing some of these l_2-element sets and look for Δ-systems of cardinality $(k+t)^3$ and with a kernel of cardinality l_3, and so on.

Finally, using the fact that by the induction assumption a given l_i-element subset of X is contained in at most $(t-1)\prod_{j=i}^{r}(n-l_j)/(k-l_j)$ members of $\mathscr{A}$, we obtain that

$$\varphi_{k+t}(l_2) = (t-1) \prod_{j=2}^{r} \frac{n-l_j}{k-l_j} + O(n^{r-2}),$$

which is a contradiction, for c sufficiently large.

To finish the proof of (ii) and (iii) it is sufficient to apply the induction hypothesis to the system $\mathscr{A}_D = \{A \setminus D \mid A \in \mathscr{A}\}$.

REMARK 6. It is possible to prove Theorem 9 when the condition $|A_i| = k$ is replaced by $|A_1 \cap \ldots \cap A_{t-1}| \leqslant k$ for any $t-1$ different members of $\mathscr{A}$.

REMARK 7. Let us introduce the following two functions:

$$f_{k,l}(n) = \max\{|\mathscr{A}| \mid \mathscr{A} \text{ satisfies the assumptions of the theorem with } L = \{l\}, \text{ and } |\textstyle\bigcap_{A \in \mathscr{A}} A| < l\};$$

$$g_{k,l}(n) = \max\{|\mathscr{A}| \mid \mathscr{A} \text{ satisfies the assumptions of the theorem with } L = \{0, l\} \text{ and } l \nmid k\}.$$

We know only that $f_{k,l}(n) \leqslant c_k$ ($c_k = k^2-k+1$ in the case where $t = 2$) and $g_{k,l}(n) \leqslant c'_k n$ ($c'_k = 1$ in the case where $t = 2$), where c_k, c'_k are constants depending only on k.

REMARK 8. It is proved in [**9**] that in the case where $L = \{1, 2, \ldots, k-1\}$ (iii) holds already for $k \geqslant (t-1)t^{-1}n$. It would be interesting to obtain better bounds for the general case as well.

6. Concluding remarks

(1) Let $L' \subset L$. Is it then true that there exist an (n, L, k)-system $\mathscr{A}$ and an (n, L', k)-system $\mathscr{A}'$, both of maximum cardinality, such that $\mathscr{A}' \subseteq \mathscr{A} (n > n_0(k))$? It is easy to prove that this is true whenever $L = \{l_1, l_1+1, \ldots, k-1\}$. In the case where $L = \{0, 2, 3, \ldots, k-1\}$ and $L' = \{2, 3, \ldots, k-1\}$ this is equivalent to a conjecture of Sós and the second author stating that for $k \geqslant 4$ an $(n, \{0, 2, 3, \ldots, k-1\}, k)$-system has cardinality at most $\binom{n-2}{k-2}$. In the case where $k = 3$ it is not true, which shows that the answer is negative in the case where $L' = \{2\}$ and $L = \{0, 2\}$.

(2) In the case where $L = \{1, 2, \ldots, k-1\}$ a theorem of Hilton and Milner [**12**] gives that Theorem 4 holds already for

$$|\mathscr{A}| \geqslant \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1,$$

and this bound is the best possible. It would be interesting to obtain best-possible bounds in the general case, too. The third author can prove that in the case where $L = \{l, l+1, \ldots, k-1\}$ the optimal bound is

$$\binom{n-l}{k-l} - \binom{n-k-1}{k-l} + l \quad \text{for} \quad k > k_0(l),\ n > n_0(k).$$

(3) Let s be a positive integer. Let B be an $m \times k$ matrix with entries $0, 1, \ldots, s$. Suppose that any two rows of B coincide in at least l positions. The authors can prove that, for $s > s_0(l)$, $m \leqslant (s+1)^{k-l}$. They conjecture [**3**] that if $s = k-1$ and every row of B is a permutation of $\{0, 1, \ldots, k-1\}$ then $m \leqslant (k-l)!$ for $k \geqslant k_0(l)$. This was proved by Deza and Frankl in [**5**] for the following cases: $l = 1$, k arbitrary; $l = 2$, $k = q$; $l = 3$, $k = q+1$ where q is the power of a prime.

(4) It is possible to generalize Theorems 7 and 9 simultaneously, that is, for families of sets $\mathscr{A}$ such that, for $i_1 < i_2 < \ldots < i_t$, $|A_{i_1}| \in K$,

$$|A_{i_1} \cap \ldots \cap A_{i_t}| \in L.$$

Such families are called quasi-block-designs by Sós in [**17**] where the problem of studying these objects was raised.

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