COMBINATORIAL PROBLEMS ON SUBSETS AND THEIR INTERSECTIONS

by

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ABSTRACT

Let $|S| = n$, $m(n ; \ell_1,\ell_2,k)$ respectively $m'(n,\ell_1,\ell, k)$ denote the cardinality of the largest family of subsets $A_i \subset S$ satisfying $|A_i| = k$ (respectively $|A_i| \le k$) and $|A_{i_1} \cap A_{i_2}| = \ell_1$ or ℓ_2. In this paper we prove

a) $m(n,0,\ell_2,k) \le \binom{n}{2}$, $m'(n,0,\ell_2,k) \le \binom{n}{2} + n+1$; equality, iff $k = 2$;

b) $m(n,0,\ell_2,k) \le n$, if $\ell_2 \nmid k$, with equality for an infinity of n.

For $n \ge n_o(k)$ we show that:

a) $m(n_1,\ell_1,\ell_2,k) \le \binom{n-\ell_1}{2}$, $m'(n,\ell_1,\ell_2,k) \le \binom{n-\ell_1}{2} + (n-\ell_1) + 1$;

b) more exactly, $m(n,\ell_1,\ell_2,k) \le \left[\frac{n-\ell_1}{k-\ell_1}\left[\frac{n-\ell_2}{k-\ell_2}\right]\right]$ with equality for an infinity of n.

Let integers $0 \le \ell_1 \le \ell_2 < k < n$ be given. Denote by $M(n,\ell_1,\ell_2,k)$ any maximal system $\alpha = \{A_i\}$ of different sets such that

$$\Big|\bigcup_{A_i\in\alpha} A_i\Big| \le n, \quad |A_i| = k\,(A_i\in\alpha), \quad |A_i \cap A_j| = \ell_1,\ell_2 \;(A_i,A_j \in \alpha,\ i \ne j), \tag{1}$$

by $m(n,\ell_1,\ell_2,k) = |M(n,\ell_1,\ell_2,k)|$, (2)

by $M'(n,\ell_1,\ell_2,k)$ any maximal system $\alpha = \{A_i\}$ such that.

$$\Big|\bigcup_{A_i\in\alpha} A_i\Big| \le n, \quad |A_i| \le k\,(A_i\in\alpha), \quad |A_i \cap A_j| = \ell_1,\ell_2\,(A_i,A_j \in \alpha_1 \;\; i \ne j), \tag{1'}$$

and by $m'(n,\ell_1,\ell_2,k) = |M(n,\ell_1,\ell_2.k)|$. (2')

Let $\ell > 0$ be a given integer. The *kernal* of the system $\alpha = \{A_i\}$ is the intersection $K(\alpha) = \bigcap_{A_i\in\alpha} A_i$. $|\alpha| > 2$. (3)

System α is an *ℓ-star*, if

$$|K(\alpha)| \ge \ell \quad . \tag{4}$$

System α is a *Δ-system*, if

all sets $A_i \setminus K(\alpha)$ are disjoint . (5)

Assume first $\ell_1 = \ell_2 = \ell$. Then Ryser proved the following (in other terms) Theorem 1 ([8])

$$m(n,\ell,\ell,k) \le n \,, \tag{6}$$

$$m'(n,\ell,\ell,k) \le n+1 \,, \tag{6'}$$

equality holds, if there exist an (n,k,ℓ)-design.

In fact, it was also shown in Theorem 1 of [8], that if $\alpha = \{A_1,A_2,\ldots,A_n\}$ satisfies $|\bigcup_{i=1} A_i| = n$, $|A_i \cap A_j| = \ell (\forall 1 \le i < j \le n)$ then it is either (n,k,ℓ)-design or a λ-design, $\lambda = \ell$. Theorem 1 is a generalization of the de Bruijn-Erdos's Theorem (case $\ell = 1$), which in turn is a generalization of Fisher's inequality for (b,v,r,k,λ)-design. Deza proved (in other terms)

Theorem 2 ([2])

There is an $r(k,\ell)$ such that

$$r(k,\ell) \le k^2 - k + 1 \,, \tag{7}$$

$n > \ell + r(k,\ell)(k-\ell) \Rightarrow m(n,\ell,\ell,k) > r(k,\ell) \Rightarrow$

$$\Rightarrow \text{any } M(n,\ell,\ell,k) \text{ is a } \Delta\text{-system} \Rightarrow m(n,\ell,\ell,k) = \left[\frac{n-\ell}{k-\ell}\right] \,, \tag{8}$$

$n > \ell + r(k,\ell) - 1 \Rightarrow m'(n,\ell,\ell,k) > r(k,\ell) \Rightarrow$

$$\Rightarrow \text{any } M'(n,\ell,\ell,k) \text{ is a } \Delta\text{-system} \Rightarrow m'(n,\ell,\ell,k) = n-\ell+1 \,. \tag{8'}$$

For $\ell = 1$ and infinitely many k (7) is best possible. We obtain from [1], [2] and [7] that

$$k^2-k+1 \geq \max(\ell+2,(k-\ell)^2 + k-\ell+1) \geq r(k,\ell) \geq \max (\ell+2, q^2+q+1), \quad (9)$$

where $q = \max q^*$, such that $q^* \leq k-\ell$ and $PG(2,q^*)$ exists. The function $r(k,\ell)$ and several generalizations of it were considered in detail in [3].

In this paper we consider the case $\ell_1 < \ell_2$. From now on we assume $\ell_1 < \ell_2$. It is evident that

$$m(n,\ell_1,\ell_2, k) \geq m(n-\ell_1, 0, \ell_2-\ell_1, k-\ell_1), \quad (10)$$

$$m'(n_1 \ell_1, \ell_2, k) \geq m(n-\ell_1, 0, \ell_2-\ell_1, k-\ell_1) , \quad (10')$$

since for example if $\alpha = \{A_i\} = M(n-\ell_1, 0, \ell_2-\ell_1, k-\ell_1\}$

and $|A| = \ell_1$, $A \cap (\bigcup_{A_i \in \alpha} A_i) = \emptyset$ then

$$|\{A_i \cup A\}| \leq m(n, \ell_1, \ell_2, k) .$$

Deza and Erdos proved the following theorem (this is inversion of (10), (10') and generalization of Theorem 2).

Theorem 3 ([4])

Let $0 < \ell_1 < \ell_2 < k < n$. There are $s(k)$ and $s'(k)$, such that

$$m(n, \ell_1, \ell_2, k) > \frac{\ell_2^2-\ell_2+1}{k}\ n + s(k) \Rightarrow \text{any } M(n,\ell_1,\ell_2,k) \text{ is an } \ell_1\text{-star} \Rightarrow$$

$$\Rightarrow m(n,\ell_1,\ell_2,k) = \max\left(\frac{\ell_2^2-\ell_2+1}{k}\ n + s(k),\ m(n-\ell_1,0,\ell_2-\ell_1,k-\ell_1)\right), \tag{11}$$

$$m'(n,\ell_1,\ell_2,k) > (\ell_2^2-\ell_2+1)\ n + s'(k) \Rightarrow \text{any } M'(n,\ell_1,\ell_2,k) \text{ is an } \ell\text{-star} \Rightarrow$$

$$\Rightarrow m'(n,\ell_1,\ell_2,k) = \max\left((\ell_2^2-\ell_2+1)n + s'(k),\ m'(n-\ell_1,0,\ell_2-\ell_1,k-\ell_1)\right). \tag{11'}$$

Assume now $\ell_1 = 0$, $\ell_2 = \ell > 0$.

<u>Theorem 4</u>. Let $0 < \ell < k < n$. Then

$$m(n,0,\ell,k) = \binom{n}{2} \quad \text{for} \quad k = 2, \tag{12}$$

$$m(n,0,\ell,k) \le \left[\frac{n^2}{k}\right] \quad \text{for} \quad k > 2,$$

$$m(n,0,\ell,k) \le \left[\frac{n}{k}\left[\frac{n-\ell}{k-\ell}\right]\right] \quad \text{for} \quad n > \ell + r(k,\ell)(k-\ell), \tag{13}$$

$$m(n,0,\ell,k) = \frac{n(n-\ell)}{k(k-\ell)} \quad \text{for the case} \quad \ell \mid k \quad \text{and}$$

$$n > f_0(k,\ell), \quad \ell \mid n, \quad \frac{k}{\ell} - 1 \ \Big|\ \frac{n}{\ell} - 1\ , \quad \frac{k}{\ell}\left(\frac{k}{\ell} - 1\right) \ \Big|\ \frac{n}{\ell}\left(\frac{n}{\ell} - 1\right)\ ;$$

$$m(n,0,\ell,k) \le n \quad \text{if} \quad \ell \nmid k, \tag{14}$$

$m(n,0,\ell,k) = n$ ~~for~~ $v \mid n$ where v is an integer, such that and there exists a (v, k, ℓ)-design.

In fact, equality (12) is trivial, because $m(n,0,\ell,2) = m(n,0,1,2) \le |\{A_i : |A_i| = 2\}| = \binom{n}{2}$. It is easy to see that $M(n^*,0,1,k^*)$ is a *pairwise balanced design* PBD$[k^*, n^*]$. R.M. Wilson proved in [9] that a PBD $[k^*,n^*]$ exists if $n^* > f_0(k^*)$, $k^*|n^*$, $k^*(k^*-1) \mid n^*(n^*-1)$. In this case, we have $m(n^*, 0, 1, k^*) = \frac{n^*(n^*-1)}{k^*(k^*-1)}$. Now we take a ℓ-multiple of PBD $[k^*, n^*]$ and put $n = \ell n^*$, $k = \ell k^*$. We obtain

$$m(n,0,\ell,k) \ge m(n^*,0,1,k^*) = \frac{\frac{n}{\ell}(\frac{n}{\ell} - 1)}{\frac{k}{\ell}(\frac{k}{\ell} - 1)} = \frac{n(n-\ell)}{k(k-\ell)}$$

for $n^* = \frac{n}{\ell} > f_0(k^*)$, i.e. $n > \ell f_0(k/\ell)$. If also $n > \ell + r(k,\ell)(k-\ell)$ holds then we have equality in (13). We obtain second inequality (14) by taking n/v (v,k,ℓ)-designs $\alpha_j = \{A_{ij}\}$, $1 \le j \le n/v$, such that

$$\Big(\bigcup_{A_{ij_1}\in\alpha_{j_1}} A_{ij_1}\Big) \cap \Big(\bigcup_{A_{ij_2}\in\alpha_{j_2}} A_{ij_2}\Big) = \emptyset \quad \text{for} \quad 1 \le j_1 < j_2 \le n/v) ;$$

It is evident that $m(n,0,\ell,k) \ge |\alpha_1|\, n/v = n$. Now we will prove the upper bounds (12), (13), (14). Let any $M(n,0,\ell,k) = \alpha = \{A_i\}$ be given. We have

$$|\alpha| k \le n\, m(n,\ell,\ell,k) \quad \text{and so} \quad |\alpha| \le \left[\frac{m(n,\ell,\ell,k)n}{k}\right] . \qquad (15)$$

Now inequality (12) follows from (15) and (6) of Theorem 1; inequality (13) follows from (15) and (8) of Theorem 2.

To prove (14), assume that there exists $M(n,0,\ell,k) = \{A_1 A_2,\ldots,A_b\}$, $b > n$. Let $\bigcup_{i=1}^{b} A_i = \{x_1,x_2,\ldots,x_n\}$.

Define $n \times b$ incidence matrix N as follows:

$$N = (n_{ij}) \quad \text{where} \quad b_{ij} = \begin{cases} 1 & \text{if} \quad x_i \in A_j \\ 0 & \text{if} \quad x_i \notin A_j \end{cases} .$$

Clearly, $N^T N = (b_{ij})$, where

$$b_{ij} = \begin{cases} k & \text{if} \quad i = j \\ 0 & \text{if} \quad |A_i \cap A_j| = 0 \\ \ell & \text{if} \quad |A_i \cap A_j| = \ell \end{cases} .$$

Since N is $n \times b$ matrix and $b > n$, $N^T N$ is singular. Hence there exists a rational vector $(y_1, y_2, \ldots, y_b)^T$ such that

$$N^T N (y_1, y_2, \ldots, y_b)^T = 0 . \tag{16}$$

Now by choosing $(y_1, y_2, \ldots, y_b)$ suitably we can assume that $y_1, y_2, \ldots, y_b$ are integers and if $y_{i_1}, y_{i_2}, \ldots, y_{i_2}$ are the nonzero integers among these, then

$$\text{g.c.d.} \ (y_{i_1}, y_{i_2}, \ldots, y_{i_r}) = 1 .$$

Now from (16) we have $ky_i + \ell(\Sigma y_j) = 0, \quad i = 1,2,\ldots,b$ (17)

where terms in the sum Σy_j are those for which $b_{ij} = \ell$.

Hence from (17), $\ell | ky_i$ for each i, in particular, $\ell \mid k \, y_{i_j}$, $j = 1,2,\ldots,r$. Since g.c.d. $(y_{i_1}, y_{i_2}, \ldots, y_{i_2}) = 1$ we have a contradiction and so $\ell | k$.

<u>Theorem 5.</u> Let $0 < \ell < k < n$. Then

$$m'(n,0,\ell,k) = \binom{n}{2} + n + 1 \quad \text{for} \quad \ell = 1 , \tag{18}$$

$$m'(n,0,\ell,k) = 9 < \binom{n}{2} + n + 1 \quad \text{for} \quad n = 4, k = 3, \ell = 2,$$

and $\quad m'(n,0,\ell,k) \le \left[\frac{n(n+1)}{\ell+1}\right] + n + 1 < \binom{n}{2} + n + 1$ otherwise;

$$m'(n,0,\ell,k) \le \left[\frac{n(n-\ell+1)}{\ell+1}\right] + n + 1 \quad \text{for} \quad n > \ell + r(k,\ell) - 1. \tag{19}$$

In fact, the proof is analogous to the proof of Theorem 4. But instead of (15) we have $|\alpha| \le \left[\frac{nm'(n,\ell,\ell,k)}{\ell+1}\right] + n + 1$ for (15') $M'(n,0,\ell,k) = \alpha = \{A_i\}$ because denoting $\alpha^* = \{A_i \in \alpha : |A_i| \ge \ell + 1\}$, we obtain

$$|\alpha^*| \ (\ell+1) \le nm'(n,\ell,\ell,k) ,$$

$$|\alpha^*| \ge |\alpha| - m'(n,0,0,\ell) .$$

Now we return to the general case.

Theorem 6. Let $0 \le \ell_1 < \ell_2 < k \le n$. Then

$$m(n,\ell_1,\ell_2,k) \le \binom{n-\ell_1}{2} \text{ for } n \le k + \sqrt{k^2 + 2s(k,\ell)} , \tag{20}$$

$$m'(n,\ell_1,\ell_2,k) \le \binom{n-\ell_1}{2} + (n-\ell_1) + 1 \quad \text{for} \quad n \le (\ell_2^2-\ell_2+1) + \sqrt{(\ell_2^2-\ell_2+1)^2\ 2s'(k,\ell)} \tag{20'}$$

$$m(n,\ell_1,\ell_2,k) \le \left[\frac{(n-\ell_1)}{(k-\ell_1)}\left[\frac{(n-\ell_2)}{(k-\ell_2)}\right]\right] \text{ for } n \ge n_0(k,\ell) , \tag{21}$$

$$m'(n,\ell_1,\ell_2,k) \le \left[\frac{(n-\ell_1)\ (n-\ell_2\ 1)}{\ell_2 - \ell_1 + 1}\right] + (n-\ell_1) + 1 \quad \text{for} \quad n \ge n_0(k,\ell) ; \tag{21'}$$

$$m(n,\ell_1,\ell_2,k) \le n \quad \text{for} \quad \ell_2-\ell_1 \mid k-\ell_1, \quad n \ge n_0(k,\ell) . \tag{22}$$

In fact, (20), (21), (22) follow from Theorem 3 and Theorem 4, applied to the case $m(n-\ell_1, \ell_1-\ell_1, \ell_2-\ell_1, k-\ell_1)$. Similarly, we obtain (20'), (21').

This paper was initiated by the following problem of R.Lemmon communicated to P.Erdos by A.Stone:

Estimate $f(m,\ell, k) = \min |\bigcup_{i=1} A_i|$ if there exists a family $A_1,A_2,\ldots,A_m$ such that $|A_i| = k (1 \le i \le m)$, $|A_i \cap A_j| = 0,\ell\ (1 \le i < j \le m)$. A.Stone and R.Lemmon considered $f(m,\ell,k)$ for small n ; it is easy to show that $f(m,\ell,k) \ge mk - \ell\binom{m}{2}$ with equality for $m = k/\ell + 1$, if $\ell \mid k$).

The following problems are still open:

1) Does $m(n,\ell_1,\ell_2,k) \le \binom{n}{2}$ hold for $\ell_1 > 0$ and all n (not only for the case $n \ge n_0(k)$ as in Theorem 6) ? This is a conjecture of Erdos and Lovasz ;

2) Does a maximal system $\alpha = \{A_i\}$ of subsets of an n-set such that

$$|A_i| = k \ (\forall \ A_i \in \alpha), \quad (A_i \cap A_j) = 0, \ \ell_2, \ \ell_3 \quad (\forall \ A_i, A_j \in \alpha, \quad i \ne j)$$

contain at most $\binom{n}{3}$ sets? Also, it would be interesting to find analog of equality (13) for this case.

3) Find an analog of (14) for $m'(n, 0, \ell, k)$; we proved only $m'(n, 0, \ell, k) \le n$ for $\ell > k/2$.

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