Since K is convex and contains no points of Λ_1 in its interior, K is bounded by the tangents to D at $a, b, a+b$; that is, K is bounded by the sides of the triangle $\triangle$. Hence $K \subseteq \triangle$, and $\delta(K) \leqslant \delta(\triangle) = 2\sqrt{3}$.

To complete the proof of our result, we notice that no closed, convex, proper subset of the equilateral triangle has the same minimal width as the triangle. For, no such subset can contain all three vertices of the triangle, and removal of any vertex of the triangle decreases the minimal width.

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A Property of 70

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It is well known (see, e.g., [3]) that 30 is the largest integer with the property that all smaller integers relatively prime to it are primes. In this note I will consider a related situation in which the corresponding special number turns out to be 70. (For a while I believed 30 to be the key figure in the new context,too, but E. G. Straus showed me that the correct value was indeed 70.) Following the proof of this special property of 70, I will mention a few related problems, some of which seem to me to be very difficult. I hope to convince the reader that there are very many interesting and new problems left in what is euphemistically called "elementary" number theory. Although these problems are easy to comprehend, their solutions will undoubtedly require either remarkable ingenuity or extensive application of known techniques.

Throughout this paper we will be studying sequences of positive integers related to a given integer n. The basic sequence $\{a_i\}_{i=0}^{\infty}$ begins with $a_0 = n$; once $a_0, a_1, \ldots, a_{k-1}$ are known, a_k is chosen to be the smallest integer greater than a_{k-1} that is relatively prime to the product $a_0 a_1 \cdots a_{k-1}$. Clearly each prime greater than n is an a_k. Moreover, each a_k greater than n^2 is a prime. TABLE 1 contains examples of the sequences $\{a_k\}$ corresponding to certain integers n.

THEOREM 1. *70 is the largest integer for which all the a_k (for $k \geqslant 1$) are primes or powers of primes.*

Proof. I will try to make the proof as short as possible; thus it is not as elementary as it might be. We begin with the difficult but useful result [5] that for $x > 17/2$, there are at least *three* primes in the interval $(x, 2x)$. Hence, for $n > 17^2 = 289$ there are at least three primes in the interval $\left(\frac{1}{2}n^{1/2}, n^{1/2}\right)$. Furthermore, at least one of these primes does not divide n since their product exceeds $n^{3/2}/8$ (which in turn is greater than n). Thus, if p_1 is the greatest prime satisfying $p_1 < n^{1/2}$ and $p_1 \nmid n$ we know that $p_1 > \frac{1}{2}n^{1/2}$. Also, for $n > 289$, there are at least three primes in $(2n^{1/2}, n/4)$ since $n > 16n^{1/2}$. At least one of these three primes does not divide n since their product exceeds $4n$. Hence, if q_1 is the least prime satisfying $q_1 \geqslant n^{1/2}$ and $q_1 \nmid n, q_1 < n/4 < p_1^2$.

n	non-prime a_k	$f(n)$	a_k that are not prime powers
3	2^2	0	
4	3^2	0	
5	$2\cdot 3$	1	6
6	5^2	0	
7	$2^3, 3^2, 5^2$	0	
8	$3^2, 5^2, 7^2$	0	
9	$2\cdot 5, 7^2$	1	10
10	$3\cdot 7$	1	21
11	$4\cdot 3, 5^2, 7^2$	1	12
12	$5^2, 7^2, 11^2$	0	
15	$2^4, 7^2, 11^2, 13^2$	0	
18	$5^2, 7^2, 11^2, 13^2, 17^2$	0	
22	$5^2, 3^3, 7^2, 13^2, 17^2, 19^2$	0	
24	$5^2, 7^2, 11^2, 13^2, 17^2, 19^2, 23^2$	0	
30	$7^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2$	0	
31	$2^5, 3\cdot 11, 5\cdot 7, 11^2, 13^2, 19^2, 23^2, 29^2$	2	33, 35
46	$7^2, 3\cdot 17, 5\cdot 11, 13^2, 19^2, 29^2, 31^2, 37^2, 41^2, 43^2$	2	51, 55
70	$9^2, 11^2, 13^2, 17^2, 19^2, 23^2, 29^2, 31^2, 37^2, 41^2, 43^2, 47^2, 53^2, 59^2, 61^2, 67^2$	0	
71	$2^3\cdot 3^2, 7\cdot 11, 5\cdot 17, 13^2, 19^2, \ldots, 67^2$	3	72, 77, 85
97	$2\cdot 7^2, 3^2\cdot 11, 5\cdot 23, 13^2, 17^2, 19^2, 29^2, \ldots, 97^2$	3	98, 99, 115
272	$3\cdot 7\cdot 13, 5^2\cdot 11, 19^2, 23^2, 29^2, \ldots, 271^2$	2	273, 275

SAMPLE SEQUENCES generated from integers n by counting upwards from n, omitting every integer that contains a prime factor in common with any previous terms in the sequence. Since every prime larger than n will automatically be included, we record here only the non-prime numbers that occur in the sequences. (No non-primes occur beyond n^2—as observed in the text—so our record terminates before that point.) The column headed "$f(n)$" records the number of members of the sequence that are neither prime nor a power of a prime. Those numbers for which $f(n)=0$ have the property that all members of the sequence are primes or powers of prime; they are 3, 4, 6, 7, 8, 12, 15, 18, 22, 24, 30, 70. It is proved in the accompanying article that no other numbers have this property.

TABLE 1.

Now consider p_1q_1. If it is one of the a_k's, then the property stated in our theorem—namely, that all a_k are primes or powers of primes—is satisfied for $n > 289$. If not, then there must be an a_i with $n < a_i < p_1q_1$ and $(a_i, p_1q_1) > 1$. We only have to prove that this a_i must have at least two distinct prime factors. If this does *not* hold, then a_i would have to be a power of p_1 or a power of q_1. Clearly it cannot be a power of q_1 since $q_1^2 > p_1q_1$. However, it cannot be a power of p_1, either, since $p_1^2 < n$ and $p_1^3 > p_1q_1$ (because $p_1^2 > q_1$). Thus all a_k corresponding to $n > 289$ are primes or powers of primes. The same conclusion holds for $70 < n \leqslant 289$, and may be verified by direct computation.

By more complicated methods, we can prove the following related result:

THEOREM 2. *For all sufficiently large n, at least one of the a_k's is the product of exactly two distinct primes.*

I shall not give the proof since it is fairly complicated and uses deep results in analytical number theory. Although I was fairly sure that this result held for every n greater than 70, and thus strengthened Theorem 1, I could not prove this. Recently C. Pomerance found a proof of Theorem 2 for n greater than 6000; he also observed that the result fails for $n=272$ (see TABLE 1).

The following conjecture, related to Theorem 2, seems very difficult. Denote by $p(x)$, the least prime factor of x. Then for sufficiently large n, there are always composite numbers x satisfying

$$n < x < n + p(x) \qquad (1)$$

The inequality (1) is a slight modification of an old conjecture that J. L. Selfridge and I proposed in

[2]. In fact, I expect that for sufficiently large n there are squarefree x's satisfying (1) which have exactly k distinct prime factors. I am sure that this conjecture is very deep. It would of course imply Theorem 2 since the integers x satisfying (1) must be a_k's corresponding to the given value of n.

I wish now to state a few simple facts and pose some difficult problems about our a_k's. We have already noted that each prime greater than n is an a_k, and that each a_k greater than n^2 is prime. Let p be a prime less than n and let $a(n,p)$ be the least a_k which is a multiple of p. It is easy to see that $a(n,p) \leqslant p^{\alpha+1}$ where $p^\alpha \leqslant n < p^{\alpha+1}$, for if none of the $a_k < p^{\alpha+1}$ are multiples of p, then $p^{\alpha+1}$ is an a_k.

Denote by $f(n)$ the number of those a_k which are not powers of primes. Clearly, $f(n) \leqslant \pi(n^{1/2})$ (where $\pi(x)$ denotes the number of primes $\leqslant x$) since each such a_k must have a prime factor exceeding $n^{1/2}$. (If $n^{1/2} < p$, then p^2 is an a_i and so no a_k can equal pt for $p \leqslant t$). I do not have any good upper or lower bounds for $f(n)$. I conjecture that $f(n) > n^{1/2-\varepsilon}$ for $n > n_0(\varepsilon)$. I am not sure whether $\lim_{n\to\infty} f(n)/\pi(n^{1/2}) = 0$.

Denote by $P(n)$ the largest prime which is less than n. It is not difficult to show that the largest a_k which is not a prime is just $P^2(n)$. On the other hand, I cannot determine the largest a_k which is not a power of a prime. In fact, I cannot even get an asymptotic formula for it and, in fact, have no guess as to its order of magnitude. It may be true that if $n \geqslant n_0(\varepsilon)$ and a_k is not a power of a prime, then $a_k < (1+\varepsilon)n$. Pomerance informs me that he can prove this, and in fact Penney, Pomerance and I are writing a longer joint paper on this subject.

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Rencontre as an Odd-Even Game

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In [3], Schuster and Philippou looked at some nonintuitive aspects of four examples of games that end with either an odd or an even integer. Their analysis of how one should bet in these "odd-even" games is simplified by the statistical independence which is inherent in the Bernoulli and Poisson probability models. We consider here an interesting variation of what is sometimes called the Matching Problem as an example of an odd-even game based on sampling without replacement. The Matching Problem was first published in 1708 by Montmort under the name "*Treize*", later became known as "*Rencontre*", and can be stated as follows:

> Two equivalent decks of different cards are each put into random order and then compared against each other. If a card occupies the same position in both decks, then a match has occurred. What is the probability of no matches?