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A MEASURE OF THE NONMONOTONICITY OF THE EULER PHI FUNCTION

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1. Introduction. Let f be a real valued arithmetic function satisfying $\lim_{n\to\infty} f(n) = +\infty$. Define another arithmetic function $F = F_f$ by setting

$$F_f(n) = \sharp\{j < n: f(j) \geqq f(n)\} + \sharp\{j > n: f(j) \leqq f(n)\} .$$

The size of the values assumed by the function F provides a measure of the nonmonotonicity of f. In particular, F is identically zero if and only if f is strictly increasing.

Here we shall take f to be φ, Euler's function, and study the associated function F_φ, which we henceforth call F.

We shall show that $F(n)/n$ is asymptotically representable as a function of $\varphi(n)/n$. Then we shall prove that $F(n)/n$ has a distribution function. We shall study $\max_{n\leqq x} F(n)$ and $\min_{n>x} F(n)$ and investigate conditions on $\varphi(n)/n$ which lead to large and small values of $F(n)/n$.

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2. An asymptotic formula for F. For $0 \leqq a, b \leqq \infty$, let

$$\Phi(a, b) = \sharp\{n \leqq a: \varphi(n) \leqq b\} .$$

We have

$$\sharp\{j < n: \varphi(j) \geqq \varphi(n)\} = n - \Phi(n, \varphi(n)) + \sharp\{j < n: \varphi(j) = \varphi(n)\} ,$$
$$\sharp\{j > n: \varphi(j) \leqq \varphi(n)\} = \Phi(\infty, \varphi(n)) - \Phi(n, \varphi(n)) .$$

Thus

$$F(n) = n + \Phi(\infty, \varphi(n)) - 2\Phi(n, \varphi(n)) + \sharp\{j < n: \varphi(j) = \varphi(n)\} .$$

It is known that

$$\Phi(\infty, y) = \zeta y + O(ye^{-\sqrt{\log y}}) ,$$

where ζ denotes the constant $\zeta(2)\zeta(3)/\zeta(6) \approx 1.9436$ [1]; and

$$\Phi(x, y) = xg(y/x) + O(ye^{-\sqrt{\log y}}) ,$$

where g is a continuous, increasing function on $[0, 1]$ which is determined by a contour integral [2].

Moreover, g is strictly concave, as we now indicate. We have from [2, Eq. (12)] that

$$(0) \qquad \alpha g'(\alpha) = g(\alpha) - D_\varphi(\alpha)\,, \quad 0 < \alpha \leqq 1\,.$$

Here

$$D_\varphi(\alpha) = \lim_{x\to\infty} \frac{1}{x} \#\{n \leqq x : \varphi(n) \leqq \alpha n\}\,.$$

It is known that this limit exists and defines a continuous function of α (cf. [6, Ch 4], [7, § 5]). Clearly D_φ is nondecreasing. In fact, it is known to be strictly increasing on (0, 1) [8, pp. 319, 323].

If we integrate the differential equation for g and use the fact that $g(1) = 1$, we obtain

$$g(\alpha) = \alpha + \alpha \int_\alpha^1 t^{-2} D_\varphi(t)dt\,,$$

and differentiating again, and differencing, we get for $0 < u < v \leqq 1$

$$\begin{aligned} g'(v) - g'(u) &= -\frac{1}{v} D_\varphi(v) + \frac{1}{u} D_\varphi(u) - \int_u^v t^{-2} D_\varphi(t)dt \\ &= -\int_u^v t^{-1} dD_\varphi(t) < \{D_\varphi(u) - D_\varphi(v)\}/v < 0\,. \end{aligned}$$

Thus g is strictly concave on (0, 1).

Noting that

$$\begin{aligned} \#\{j < n : \varphi(j) = \varphi(n)\} &\leqq \Phi(\infty, \varphi(n)) - \Phi(\infty, \varphi(n) - 1) \\ &= O\{\varphi(n) e^{-\sqrt{\log \varphi(n)}}\}\,, \end{aligned}$$

we have

$$\frac{F(n)}{n} = 1 + \zeta \frac{\varphi(n)}{n} - 2g\Big(\frac{\varphi(n)}{n}\Big) + O\Big\{\frac{\varphi(n)}{n}\, e^{-\sqrt{\log \varphi(n)}}\Big\}\,.$$

If we set

$$(1) \qquad h(u) = 1 + \zeta u - 2g(u)$$

and enlarge the error we obtain the asymptotic formula

$$(2) \qquad \frac{F(n)}{n} = h(\varphi(n)/n) + O(e^{-\sqrt{\log n}})\,.$$

Below is an approximate graph of h. Note that h is strictly convex.

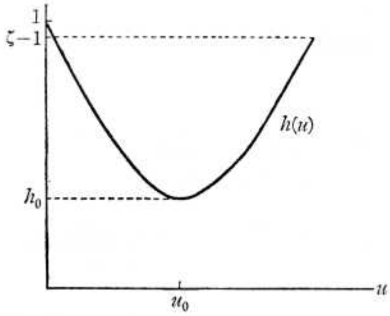

FIGURE 1

3. A distribution function.

THEOREM 1. *$F(n)/n$ has a continuous distribution function.*

Proof. Let h_0 denote the minimal value of h and u_0 the point at which the minimum is achieved. Let h^* denote the branch of the inverse function of h which maps $[h_0, 1]$ onto $[0, u_0]$, and let h^{**} denote the branch which maps $[h_0, \zeta - 1]$ onto $[u_0, 1]$. Also, let $h^{**}(\alpha) = 1$ for $\zeta - 1 < \alpha \leqq 1$. Note that h^* and h^{**} are well defined, even at u_0, on account of the strict convexity of h.

Since D_φ and h are continuous, for $h_0 \leqq \alpha \leqq 1$ we have

$$\begin{aligned} D_\varphi(h^{**}(\alpha)) - D_\varphi(h^*(\alpha)) &= \lim_{x\to\infty} \frac{1}{x} \#\{n \leqq x \colon h^*(\alpha) \leqq \varphi(n)/n \leqq h^{**}(\alpha)\} \\ &= \lim_{x\to\infty} \frac{1}{x} \#\{n \leqq x \colon h(\varphi(n)/n) \leqq \alpha\} , \end{aligned}$$

a continuous function of α which vanishes at $\alpha = h_0$ and equals 1 for $\alpha = 1$.

Given $\varepsilon > 0$ we have

$$\begin{aligned} \lim_{x\to\infty} \frac{1}{x} \#\left\{n \leqq x \colon h\left(\frac{\varphi(n)}{n}\right) \leqq \alpha - \varepsilon\right\} &\leqq \varliminf_{x\to\infty} \frac{1}{x} \#\left\{n \leqq x \colon \frac{F(n)}{n} \leqq \alpha\right\} \\ \leqq \varlimsup_{x\to\infty} \frac{1}{x} \#\left\{n \leqq x \colon \frac{F(n)}{n} \leqq \alpha\right\} &\leqq \lim_{x\to\infty} \frac{1}{x} \#\left\{n \leqq x \colon h\left(\frac{\varphi(n)}{n}\right) \leqq \alpha + \varepsilon\right\} . \end{aligned}$$

It follows that if $h_0 \leqq \alpha \leqq 1$, then

$$D_F(\alpha) = \lim_{x\to\infty} \frac{1}{x} \#\left\{n \leqq x \colon \frac{F(n)}{n} \leqq \alpha\right\} = D_\varphi(h^{**}(\alpha)) - D_\varphi(h^*(\alpha)) .$$

Further, $D_F(\alpha) = 0$ for $\alpha < h_0$ and $D_F(\alpha) = 1$ for $\alpha > 1$. Thus $F(n)/n$ has a continuous distribution function.

4. Upper estimates. We shall exploit the observation, based on the graph of h, that $F(n)/n$ is near its largest when $\varphi(n)/n$ is near 0.

LEMMA 1. *For all large x there exists an integer $n_0 = n_0(x)$ such that $x - x\log^{-1} x < n_0 \leqq x$ and*

$$(3) \qquad \varphi(n_0)/n_0 \sim e^{-\gamma}/\log\log x \sim \min_{1\leq m\leq x} \varphi(m)/m .$$

Proof. Let p_r denote the rth prime (in the usual order) and $P(r)$ the product of the first r primes. Choose $r' = r'(x)$ to be the largest integer for which $P(r') \leqq x/\log x$. The prime number theorem implies that

$$\sum_{p\leqq p_{r'}} \log p \sim p_{r'} ,$$

and hence, by an easy calculation, $p_{r'} \sim \log x$.

Set $n_0 = [x/P(r')]P(r')$. Then $x - P(r') < n_0 \leqq x$ and

$$\frac{\varphi(n_0)}{n_0} \leqq \prod_{p\leqq p_{r'}} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log p_{r'}} \sim \frac{e^{-\gamma}}{\log\log x} .$$

It is known (cf. [5, Th. 328]) that

$$\min_{1\leq m\leq x} \varphi(m)/m \sim e^{-\gamma}/\log\log x .$$

THEOREM 2. *As $x \to \infty$,*

$$\max_{n\leq x} F(n) = x - (\zeta e^{-\gamma} + o(1))x/\log\log x .$$

Proof. Let α_0 (presently to be specified) be a small positive number such that $h(\alpha) \leqq h(\alpha_0) < 1$ for $\alpha_0 < \alpha < 1$. Suppose first that $\varphi(n)/n \geqq \alpha_0$. Then there exists an $\varepsilon > 0$ such that $F(n) < (1-\varepsilon)n$ for all sufficiently large n and if x is large, $F(n) < (1-\varepsilon)x$ for *all* $n \leqq x$ and satisfying $\varphi(n)/n \geqq \alpha_0$.

For small positive values of α we use the approximation

$$g(\alpha) = \zeta\alpha + O\{\exp(-\exp 1/(k\alpha))\} ,$$

which holds for some absolute constant k [2, Lemma 4]. If we combine this estimate with (1) and (2) we obtain

$$(4) \qquad \frac{F(n)}{n} = 1 - \zeta\frac{\varphi(n)}{n} + O\left\{\exp\left(-\exp\frac{n}{k\varphi(n)}\right)\right\} + O(e^{-\sqrt{\log n}}) .$$

The function $\alpha \mapsto 1 - \zeta\alpha + c\exp\{-\exp 1/(k\alpha)\}$ is decreasing for small positive α. Choose α_0 to be positive but so small that the function is decreasing for $0 < \alpha < \alpha_0$ and $h(\alpha_0) > \zeta - 1$.

Now for $\varphi(n)/n < \alpha_0$ we use the inequality

$$\varphi(n)/n \geqq (e^{-\gamma} + o(1))/\log\log x\ , \quad 1 \leqq n \leqq x,$$

to obtain the bound

$$F(n) \leqq x\{1 - (\zeta e^{-\gamma} + o(1))/\log\log x\}\ , \quad 1 \leqq n \leqq x\ .$$

The $o(1)$ term tends to zero as $x \to \infty$ (independently of n).

On the other hand, taking n_0 as in the lemma yields

$$\begin{aligned} F(n_0) &= n_0\{1 - (\zeta e^{-\gamma} + o(1))/\log\log x\} \\ &= x\{1 - (\zeta e^{-\gamma} + o(1))/\log\log x\}\ . \end{aligned}$$

Define a sequence $\{n_k\}$ of "new highs" of F by the condition $F(n) < F(n_k)$ for all $n < n_k$.

We note for later use that $\varphi(n_k)/n_k \sim e^{-\gamma}/\log\log n_k$ as $k \to \infty$. We can see this by noting first that $\varphi(n_k)/n_k \to 0$ by the first paragraph of the proof of Theorem 2. Then we write (4) with $n = n_k$ and Theorem 2 with $x = n_k$ and equate the expressions to obtain

$$1 - \frac{\zeta\varphi(n_k)}{n_k}(1 + o(1)) + O(e^{-\sqrt{\log n}}) = 1 - \frac{\zeta e^{-\gamma} + o(1)}{\log\log n_k}\ .$$

Theorem 2 has two immediate consequences.

COROLLARY 1. *$F(n) < n$ for all sufficiently large n.*

COROLLARY 2.

$$n_{k+1} - n_k = o(n_k/\log\log n_k)\ , \quad k \longrightarrow \infty\ .$$

Proof. For $n_k \leqq x < n_{k+1}$ we have

$$\max_{n \leqq x} F(n) = F(n_k)$$

or

$$x\left\{1 - \frac{\zeta e^{-\gamma} + o(1)}{\log\log x}\right\} = n_k\left\{1 - \frac{\zeta e^{-\gamma} + o(1)}{\log\log n_k}\right\}\ .$$

Let $x \to n_{k+1}-$ to obtain the corollary.

REMARK. The size of n or n_k plays a vital role in the two corollaries. The first corollary is false for small n as the examples $F(13) = 13$ and $F(73) = 75$ show.

The proof of Theorem 2 implies that $\varphi(n_k)/n_k \to 0$ as $k \to \infty$.

Numerical computation shows that the n_k's are primes for all $n_k \leqq 500$ (the limit of the calculation). The explanation of this anomaly (apart from the effect of the error term) is as follows. Let u_1 be the number in $(0, 1)$ for which $h(u_1) = \zeta - 1$ (cf. (Fig. 1)). It appears from (4) that $u_1 \approx .03$. Simple estimates show that $\varphi(n)/n > .03$ for all $n < e^{e^{18}}$. Thus for n of modest size, the largest values of $h(\varphi(n)/n)$ occur for $\varphi(n)/n$ near 1.

We conclude this section by establishing a lower bound inequality for $n_{k+1} - n_k$.

THEOREM 3. *For any* $\varepsilon > 0$

$$n_{k+1} - n_k > n_k^{1-\varepsilon}\,, \quad k \longrightarrow \infty\ .$$

Proof. Given $\varepsilon > 0$ and n_k, let $p^* = p^*(k)$ denote the largest prime such that $\prod_{p \leqq p^*} p \leqq n_k$. The prime number theorem and simple estimates imply that $p^* \sim \log n_k$. We shall show that at most $\varepsilon p^*/\log p^*$ primes $p \leqq p^*$ fail to divide n_k. Similar estimates apply for n_{k+1} and thus n_k and n_{k+1} have at least $\pi(p^*) - 2[\varepsilon p^*/\log p^*]$ prime factors in common.

Let w be an integer such that

$$\pi(w) = \pi(p^*) - 2[\varepsilon p^*/\log p^*]\ .$$

Then we have

$$n_{k+1} - n_k \geqq \prod_{p \leqq w} p = \prod_{p \leqq p^*} p \prod_{w < p \leqq p^*} p^{-1}\ .$$

Also,

$$\sum_{w < p \leqq p^*} \log p \leqq (\log p^*)[\pi(p^*) - \pi(w)] \leqq 2\varepsilon p^*\ ,$$

and so

$$n_{k+1} - n_k \geqq \frac{n_k}{2p^*} \exp\left[-2\varepsilon p^*\right] \geqq n_k^{1-3\varepsilon}\ .$$

We introduce the integer

$$N = \left[n_k \prod_{p < p^*} p^{-1} \right] \prod_{p < p^*} p\ .$$

Since $N \leqq n_k$ we have $F(N) \leqq F(n_k)$. We can estimate $F(N)$ and $F(n_k)$ because of the special form of N and n_k. Also, N is not much smaller than n_k. These facts will enable us to show that

$$\#\{p \leqq p^* : p \nmid n_k\} \leqq \varepsilon p^*/\log p^*\ .$$

Let ν denote the number of primes $p \leqq p^*$ such that $p \nmid n_k$. We suppose that $\nu > \varepsilon p^*/\log p^*$ and shall deduce a contradiction.

At most $\nu + 1$ prime divisors of n_k (counting multiplicity) can exceed p^*, as we now indicate. Suppose that there were at least $\nu + 2$ prime divisors of n_k exceeding p^*. For each of the ν primes $p_i \leqq p^*$ with $p_i \nmid n_k$ associate a prime $p'_i > p^*$ with $p'_i | n_k$. Each of the p''s can be used at most as many times as it occurs in the factorization of n_k. We have

$$n_k > n' = n_k \prod_{i=1}^{\nu} p_i/p'_i \ ;$$

further n' is divisible by each prime not exceeding p^* and by at least two primes exceeding p^*. Thus $n_k > n' > p^{*2} \prod_{p \leqq p^*} p$. On the other hand the definition of p^* implies that $n_k < 2p^* \prod_{p \leqq p^*} p$, contradicting the last inequality.

Let y and z denote composite numbers such that $\pi(p^*) - \pi(y) = \nu$, $\pi(z) - \pi(p^*) = \nu + 1$. Then

$$\frac{\varphi(n_k)}{n_k} = \prod_{p \leqq p^*}\left(1 - \frac{1}{p}\right) \prod_{\substack{p \leqq p^* \\ p \nmid n_k}} \left(1 - \frac{1}{p}\right)^{-1} \prod_{\substack{p > p^* \\ p | n_k}} \left(1 - \frac{1}{p}\right)$$
$$\geqq \prod_{p \leqq p^*}\left(1 - \frac{1}{p}\right) \prod_{y < p \leqq p^*} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p^* < p < z} \left(1 - \frac{1}{p}\right) .$$

Letting $\nu = \eta p^*/\log p^*$, $\varepsilon < \eta \leqq 1$, we have

$$\pi(y) = \pi(p^*) - \nu = (1 - \eta + o(1))p^*/\log p^* \ ,$$

and so $y = (1 - \eta + o(1))p^*$. Similarly $z = (1 + \eta + o(1))p^*$. Thus

$$\prod_{y < p \leqq p^*} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p^* < p < z} \left(1 - \frac{1}{p}\right) = \frac{(\log p^*)^2}{(\log y)(\log z)} (1 + O(e^{-\sqrt{\log p^*}})) \ .$$

Differentiation shows that, for fixed q, the function

$$\eta \longmapsto \frac{\log^2 q}{\log((1 - \eta)q) \log((1 + \eta)q)}$$

is increasing for $0 < \eta < 1$. Thus

$$\frac{(\log p^*)^2}{(\log y)(\log z)} \geqq \frac{(\log p^*)^2}{\log((1 - \varepsilon)p^*) \log((1 + \varepsilon)p^*)}$$
$$\geqq \left\{1 - \frac{\varepsilon + \varepsilon^2/2 + O(\varepsilon^3)}{\log p^*}\right\}^{-1} \left\{1 + \frac{\varepsilon - \varepsilon^2/2 + O(\varepsilon^3)}{\log p^*}\right\}^{-1}$$
$$\geqq 1 + \frac{\varepsilon^2}{\log p^*} + O\left(\frac{\varepsilon^3}{\log p^*} + \frac{\varepsilon^2}{\log^2 p^*}\right) .$$

Thus

$$\prod_{y < p \leqq p^*} \left(1 - \frac{1}{p}\right)^{-1} \prod_{p^* < p < z} \left(1 - \frac{1}{p}\right) \geqq 1 + \frac{\varepsilon^2}{2 \log p^*} \ ,$$

provided that k is sufficiently large and ε sufficiently small. It follows that

$$\frac{\varphi(n_k)}{n_k} \geqq \left(1 + \frac{\varepsilon^2}{2 \log p^*}\right) \prod_{p \leqq p^*} \left(1 - \frac{1}{p}\right).$$

We have $\varphi(N)/N \sim e^{-\gamma}/\log\log N$ because of the form of N, and $\varphi(n_k)/n_k \sim e^{-\gamma}/\log\log n_k$ by the argument following the proof of Theorem 2. It follows from (4), that for some $\alpha > 0$,

$$\frac{F(x)}{x} = 1 - \zeta\frac{\varphi(x)}{x} + O\{\exp(-\log^\alpha x)\}$$

holds for $x = N$ and $x = n_k$.

We combine the formulas for $F(n_k)$ and $F(N)$ with the bound we obtained for $\varphi(n_k)/n_k$, the inequalities

$$n_k \geqq N = \left[\frac{n_k}{\prod_{p<p^*} p}\right] \prod_{p<p^*} p > n_k - \prod_{p<p^*} p \geqq n_k\left(1 - \frac{1}{p^*}\right)$$

and $\varphi(N)/N \leqq \prod_{p<p^*}(1 - p^{-1})$ to obtain

$$F(n_k) \leqq \frac{N}{1 - \frac{1}{p^*}}\left\{1 - \zeta\left(1 + \frac{\varepsilon^2}{2\log p^*}\right) \prod_{p \leqq p^*} \left(1 - \frac{1}{p}\right) + ce^{-\log^\alpha N}\right\}$$

$$< N\left\{1 - \zeta \prod_{p<p^*} \left(1 - \frac{1}{p}\right) - c\exp(-\log^\alpha N)\right\} \leqq F(N),$$

where c is a positive constant. This inequality is impossible, since the n_k's are the new highs of F. It follows that at most $\varepsilon p^*/\log p^*$ primes $p \leqq p^*$ fail to divide n_k and hence our lower bound for $n_{n+1} - n_k$ holds.

5. **Small values of $F(n)/n$.** We have shown in § 2 that $F(n)/n \sim h(\varphi(n)/n)$. The function h attains a minimal value h_0 at an interior point u_0 of $(0, 1)$, as we presently shall show. The point u_0 is unique by the strict convexity of h. Thus $F(n)/n$ is, asymptotically, near its minimal value h_0 when $\varphi(n)/n$ is near u_0.

Numerical data suggest that u_0 is near 1/2 and h_0 is near 1/3. We shall show that $.473 < u_0 < .475$ and $.321 < h_0 < .324$.

LEMMA 2. $h'(0) = -\zeta$, $h'(1) = \zeta$.

Proof. We have by (1) that $h'(u) = \zeta - 2g'(u)$. The estimate (cf. [2], Lemma 4)

$$g(u) = \zeta u + O\{\exp(-\exp 1/(ku))\}$$

implies that $g'(0) = \zeta$, and hence $h'(0) = -\zeta$. Equation (0) implies that $g'(1) = 0$, and hence $h'(1) = \zeta$.

Thus the minimum of h is achieved in the open interval $(0, 1)$.

We shall now establish a formula which will lead to estimates for $g(1/2)$. This will be useful because of the close connection between g and h and the proximity of u_0 to $1/2$.

LEMMA 3.

$$g(1/2) = \frac{1}{2} + \frac{\zeta}{6} - \left\{\left(\frac{\zeta}{4} - g\left(\frac{1}{4}\right)\right) - \left(\frac{\zeta}{8} - g\left(\frac{1}{8}\right)\right) + \left(\frac{\zeta}{16} - g\left(\frac{1}{16}\right)\right) - \cdots\right\}.$$

Proof. We estimate

$$\#\{n \leqq x : n \text{ odd}, \varphi(n) \leqq y\},$$

a problem closely related to the main theorem of [2]. The generating function

$$\begin{aligned} F(s, z) &\overset{\text{def}}{=} \sum_{n=1}^{\infty} n^{-s}\varphi(n)^{-z} \\ &= \prod_p \{1 + p^{-s}(p-1)^{-s}(1 + p^{-s-z} + p^{-2s-2z} + \cdots)\} \\ &= \prod_p \{1 - p^{-s-z} + p^{-s}(p-1)^{-z}\}\zeta(s+z) \\ &\overset{\text{def}}{=} \prod (s, z)\zeta(s+z) \end{aligned}$$

was used in [2], and the function g was represented by

$$g(\alpha) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\prod(1-z, z)}{z(1-z)}\alpha^z dz, \quad 0 \leqq \alpha \leqq 1.$$

The formula is valid at the end points by uniform convergence of the integral.

We delete the even integers and write

$$\begin{aligned} F_0(s, z) &= \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} n^{-s}\varphi(n)^{-z} \\ &= \prod (s, z)\zeta(s+z)\left\{\frac{1 - 2^{-s-z}}{1 - 2^{-s-z} + 2^{-s}}\right\}. \end{aligned}$$

The functions $F(s, z)$ and $F_0(s, z)$ have the same singularities in the region

$$\{(s, z) \in \boldsymbol{C} \times \boldsymbol{C} : \operatorname{Re} s + z > 0\},$$

because any singularity of the new factor $(1-2^{-s-z})/(1-2^{-s-z}+2^{-s})$ is cancelled by a zero of $\prod(s, z)$, and the new factor has no zeros in this region.

It now follows, *mutatis mutandis*, that

$$\begin{aligned} g_0(\alpha) &\overset{\text{def}}{=} \lim_{x\to\infty} \frac{1}{x} \#\{n \leqq x: n \text{ odd}, \varphi(n) \leqq \alpha x\} \\ &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\prod(1-z, z)}{z(1-z)} \alpha^z (1+2^z)^{-1} dz \\ &= \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\prod(1-z, z)}{z(1-z)} \left\{\left(\frac{\alpha}{2}\right)^z - \left(\frac{\alpha}{4}\right)^z + \left(\frac{\alpha}{8}\right)^z - \cdots\right\} dz \\ &= g(\alpha/2) - g(\alpha/4) + g(\alpha/8) - \cdots . \end{aligned}$$

If we note that $g_0(1) = 1/2$ and sum the series $\zeta/4 - \zeta/8 + \zeta/16 - \cdots$ we obtain the lemma.

Now g is concave and $g(\varepsilon) \sim \zeta\varepsilon$ as $\varepsilon \to 0$. Thus the series in the formula for $g(1/2)$ is alternating with terms decreasing to zero, indeed at a geometric rate. To further exploit our formula we must first estimate $D_\varphi(t)$ for t near 0.

LEMMA 4. $D_\varphi(t) < 12t^3$, $0 < t < 1$.

Proof. By Chebychev's inequality

$$t^{-3}\#\left\{n \leqq x: \frac{\varphi(n)}{n} \leqq t\right\} = t^{-3} \sum_{\substack{n \leqq x \\ n/\varphi(n) \geqq 1/t}} 1 \leqq \sum_{n\leqq x} \left(\frac{n}{\varphi(n)}\right)^3 ,$$

and we estimate the last sum by writing

$$(n/\varphi(n))^3 = (1 * \beta)(n) ,$$

where $*$ denotes multiplicative convolution and β is a nonnegative multiplicative function satisfying $\beta(p) = (p^3-(p-1)^3)/(p-1)^3$, $\beta(p^\alpha) = 0$ for all primes p and all exponents $\alpha \geqq 2$.

Thus

$$\begin{aligned} \sum_{n\leqq x}\left(\frac{n}{\varphi(n)}\right)^3 &= \sum_{n\leqq x}\left[\frac{x}{n}\right]\beta(n) \\ &\leqq x\sum_{n=1}^{\infty}\frac{\beta(n)}{n} = x\prod_p\left(1+\frac{\beta(p)}{p}\right) \\ &= x\prod_p\left\{1+\frac{1}{p}\frac{p^3-(p-1)^3}{(p-1)^3}\right\} \overset{\text{def}}{=} \gamma x . \end{aligned}$$

Now

$$\gamma = \zeta(2)^3 \prod_p \left\{1 + \frac{3p^2 - 3p + 1}{p(p-1)^3}\right\}\left\{1 - \frac{1}{p^2}\right\}^3$$
$$= \zeta(2)^3 \prod_p \left\{1 + \frac{6p^4 + 4p^3 - 3p^2 - p + 1}{p^7}\right\}.$$

It is easy to check that for all $p \geqq 3$

$$6p^4 + 4p^3 - 3p^2 - p + 1 < 7p^4 .$$

We have

$$\gamma \leqq \zeta(2)^3\left(1 + \frac{115}{128}\right)\left\{\left(1 + \frac{7}{3^3}\right)\left(1 + \frac{7}{5^3}\right)\left(1 + \frac{7}{7^3}\right)\right\}\exp\left\{\sum_{p \geqq 11} 7p^{-3}\right\},$$

and

$$7 \sum_{p \geqq 11} p^{-3} < 7 \int_{10}^{\infty} t^{-3} dt = .035 .$$

Thus $\gamma \leqq 12$, and $D_\varphi(t)$ satisfies the claimed bound.

We combine the last two lemmas with numerical data of Charles R. Wall [10] on the density function D_φ to obtain upper and lower estimates for $g(1/2)$.

LEMMA 5.

$$\frac{1}{2} + \frac{\zeta}{6} - .00154 < g(1/2) < \frac{1}{2} + \frac{\zeta}{6} - .00075 .$$

Proof. The alternating series representation of $g(1/2)$ leads to the inequalities

$$\frac{1}{2} + \frac{\zeta}{6} - \left\{\left(\frac{\zeta}{4} - g\left(\frac{1}{4}\right)\right) - \left(\frac{\zeta}{8} - g\left(\frac{1}{8}\right)\right) + \left(\frac{\zeta}{16} - g\left(\frac{1}{16}\right)\right)\right\}$$
$$\leqq g(1/2) \leqq \frac{1}{2} + \frac{\zeta}{6} - \left\{\left(\frac{\zeta}{4} - g\left(\frac{1}{4}\right)\right) - \left(\frac{\zeta}{8} - g\left(\frac{1}{8}\right)\right)\right\}.$$

The differential equation (0) has the solution

$$(5) \qquad u^{-1}g(u) = \zeta - \int_0^u D_\varphi(t)t^{-2}dt .$$

The constant is evaluated here by noting that $g'(0) = \zeta$. The integral converges at zero by the preceding lemma. Thus we have

$$2^{-k}\zeta - g(2^{-k}) = 2^{-k}\int_0^{2^{-k}} D_\varphi(t)t^{-2}dt .$$

It follows that

$$\left(\frac{\zeta}{4} - g\left(\frac{1}{4}\right)\right) - \left(\frac{\zeta}{8} - g\left(\frac{1}{8}\right)\right) + \left(\frac{\zeta}{16} - g\left(\frac{1}{16}\right)\right)$$
$$= \frac{1}{4}\int_{1/8}^{1/4} D_\varphi(t)\frac{dt}{t^2} + \frac{1}{8}\int_{1/16}^{1/8} D_\varphi(t)\frac{dt}{t^2} + \frac{3}{16}\int_0^{1/16} D_\varphi(t)\frac{dt}{t^2} .$$

We estimate the three integrals from above, using the bound of the preceding lemma for $0 \leqq t \leqq .007$ and the upper bounds of Wall for $.007 < t \leqq .25$. We obtain the upper bound .00154.

Similar treatment of

$$\left(\frac{\zeta}{4} - g\left(\frac{1}{4}\right)\right) - \left(\frac{\zeta}{8} - g\left(\frac{1}{8}\right)\right)$$

leads to the lower bound .00075.

LEMMA 6. (*Main formula.*)

$$2D_\varphi(1/2) - 1 + \zeta/6 + 2R = \int_{u_0}^{1/2} t^{-1} dD_\varphi(t) ,$$

where $.00075 < R < .00154$.

Proof. We have by (5)

$$\frac{g(u_0)}{u_0} - \frac{g(1/2)}{1/2} = \int_{u_0}^{1/2} D_\varphi(t) t^{-2} dt .$$

From (1) and the fact that $h'(u_0) = 0$ we get $g'(u_0) = \zeta/2$. Combining this with (0) we obtain

$$g(u_0) = u_0\zeta/2 + D_\varphi(u_0) .$$

This expression, Lemma 5, and the preceding integral yield

$$\frac{D_\varphi(u_0)}{u_0} - 1 + \frac{\zeta}{6} + 2R = \int_{u_0}^{1/2} D_\varphi(t) t^{-2} dt .$$

Integrating by parts we get the desired expression.

THEOREM 4. $u_0 > .473$ *and* $h_0 < .324$.

Proof. Starting from Lemma 6, we write

$$2D_\varphi\left(\frac{1}{2}\right) - 1 + \frac{\zeta}{6} + 2R = \left\{\int_{.475}^{.5} + \int_{u_0}^{.475}\right\} t^{-1} dD_\varphi(t)$$
$$\geqq \frac{1}{.5}\{D_\varphi(.5) - D_\varphi(.499)\} + \frac{1}{.499}\{D_\varphi(.499) - D_\varphi(.498)\}$$
$$+ \cdots + \frac{1}{.476}\{D_\varphi(.476) - D_\varphi(.475)\} + \frac{1}{.475}\{D_\varphi(.475) - D_\varphi(u_0)\} ,$$

Note that this inequality is valid regardless of whether $u_0 \leqq .475$ or not.

We rearrange terms, isolating $D_\varphi(u_0)$:

$$\frac{D_\varphi(u_0)}{.475} \geqq 1 - \frac{\zeta}{6} - 2R + \left(\frac{1}{.499} - \frac{1}{.5}\right)D_\varphi(.499) + \cdots + \left(\frac{1}{.475} - \frac{1}{.476}\right)D_\varphi(.475) .$$

If we use the upper estimate for R and the lower estimates of [10] for $D_\varphi(.475), \cdots, D_\varphi(.499)$, we find that $D_\varphi(u_0) > .3380$.

The stated inequalities follow at once from this bound. First, we have from [10] that $D_\varphi(.473) < .3362$, and thus $u_0 > .473$. Next, it follows from Equations (0) and (1) that $h_0 = 1 - 2D_\varphi(u_0)$. Thus, $h_0 < .324$.

We also have bounds for u_0 and h_0 in the opposite directions.

THEOREM 5. *$u_0 < .475$ and $h_0 > .321$.*

Proof. Using Lemma 6 again, we write

$$2D_\varphi\left(\frac{1}{2}\right) - 1 + \frac{\zeta}{6} + 2R = \left\{\int_{.475}^{.5} + \int_{u_0}^{.475}\right\} t^{-1} dD_\varphi(t) .$$

This time we express the first integral as an upper Riemann-Stieltjes sum and sum by parts to obtain

$$\int_{.475}^{.5} t^{-1} dD_\varphi(t) \leqq \frac{D_\varphi(.5)}{.499} + \left(\frac{1}{.498} - \frac{1}{.499}\right)D_\varphi(.499) + \cdots + \left(\frac{1}{.475} - \frac{1}{.476}\right)D_\varphi(.476) - \frac{D_\varphi(.475)}{.475} .$$

Thus

$$\int_{u_0}^{.475} t^{-1} dD_\varphi(t) \geqq \frac{D_\varphi(.475)}{.475} - I ,$$

where

$$I = 1 - \frac{\zeta}{6} - 2R + \left(\frac{1}{.499} - \frac{1}{.5}\right)D_\varphi(.5) + \cdots + \left(\frac{1}{.475} - \frac{1}{.476}\right)D_\varphi(.476) .$$

We estimate I from above by using the upper bounds for $D_\varphi(.476), \cdots, D_\varphi(.500)$ from [10] and the lower bound for R from Lemma 6. We obtain the inequality

$$\int_{u_0}^{.475} t^{-1} dD_\varphi(t) \geqq \frac{D_\varphi(.475)}{.475} - .7145 , \tag{6}$$

from which both assertions of the theorem will follow.

The bound $D_\varphi(.475) \geqq .33969$ from [10] implies that

$$\int_{u_0}^{.475} t^{-1} dD_\varphi(t) > .0006 > 0$$

and hence $u_0 < .475$.

Next, since $u_0 > .473$, we obtain from (6)

$$\frac{1}{.473}\{D_\varphi(.475) - D_\varphi(u_0)\} \geqq \frac{D_\varphi(.475)}{.475} - .7145 .$$

This inequality and the bound $D_\varphi(.475) < .34166$ from [10] yield $D_\varphi(u_0) < .3394$. Thus, we finally obtain $h_0 = 1 - 2D_\varphi(u_0) > .321$.

6. **Lower estimates for F.** The sequence $F(n)$ tends to infinity with n, since

$$F(n)/n \sim h(\varphi(n)/n) \geqq h_0 > 0 .$$

In this section we are going to establish

THEOREM 6. *As $x \to \infty$,*

$$\min_{n>x} F(n) \sim h_0 x .$$

This estimate follows easily from the following

LEMMA 7. *Let $\alpha \in (0, 1)$ and let $\varepsilon > 0$ be given. Then there exists an X (depending on ε and α) such that for each $x \geqq X$, the interval $(x, x + \varepsilon x]$ contains an integer j with $|\varphi(j)/j - \alpha| < \varepsilon$.*

Proof. The argument proceeds in two steps. First we obtain *some* integer j_0 (not necessarily in $(x, x + \varepsilon x]$) composed of at least two distinct prime factors, for which $|\varphi(j_0)/j_0 - \alpha| < \varepsilon$. Then we show that a suitable multiple of j_0 lies in $(x, x + \varepsilon x]$ and satisfies the same φ estimate.

Let $\alpha = \alpha_0$. Let q_1 be the smallest prime p_ν for which $1 - p_\nu^{-1} > \alpha_0$. Set $\alpha_1 = \alpha_0(1 - q_1^{-1})^{-1}$ and $j_1 = q_1$. Repeat the foregoing, choosing q_2 to be the smallest prime p_ν exceeding q_1 for which $1 - p_\nu^{-1} > \alpha_1$. Let $j_2 = q_1 q_2$ and $\alpha_2 = \alpha_1(1 - q_2^{-1})^{-1}$. If $1 > \alpha_2 > 1 - \varepsilon/(\alpha + \varepsilon)$, we can stop here. Otherwise we continue until we obtain an integer $j_r = q_1 q_2 \cdots q_r$, $r = r(\alpha, \varepsilon)$, such that

$$\alpha \leqq \varphi(j_r)/j_r < \alpha + \varepsilon .$$

This is possible to achieve since $1 - p_\nu^{-1} \to 1$ as $\nu \to \infty$ and $\prod_{\nu=1}^\infty (1 - p_\nu^{-1}) = 0$.

Set $j_\nu = j^*$ and consider the sequence $\{j^* q_1^a q_2^b \colon a, b = 0, 1, 2, 3, \cdots\}$. Clearly

$$\varphi(j^*)/j^* = \varphi(j^* q_1^a a_2^b)/(j^* q_1^a q_2^b) .$$

It suffices to show that for each large x the interval $(x, x + \varepsilon x]$ contains some $q_1^a q_2^b$, $a, b \geqq 0$.

It is well known that the sequence $\{q_1^a q_2^b \colon a, b \in \boldsymbol{Z}\}$ is dense in the positive reals for q_1, q_2 distinct primes. Choose $a > 0$ and $-b < 0$ such that $1 < q_1^a q_2^{-b} < 1 + \varepsilon$. Given x, set

$$\begin{aligned} s &= [(\log x)/(\log q_1 q_2)] , \\ t &= [(\log q_1 q_2)/(\log q_1^a q_2^{-b})] + 1 , \end{aligned}$$

and $a_k = q_1^{s+ka} q_2^{s-kb}$, $(0 \leqq k \leqq t)$.

We have

$$a_0 = (q_1 q_2)^s \leqq x < (q_1 q_2)^{s+1} < a_t$$

and

$$1 < a_{k+1}/a_k = q_1^a q_2^{-b} < 1 + \varepsilon .$$

Thus there exists some $k \in [1, t]$ such that $x < q_1^{s+ka} q_2^{s-kb} < x + \varepsilon x$.

Finally, we must insure that the exponent $s - kb \geqq 0$. This we do by noting that a, b, and t depend only on ε and are fixed, while $s \to \infty$ with x.

LEMMA 8. *Given $\varepsilon > 0$ there exists an $X = X(\varepsilon)$ such that for each $x \geqq X$ the interval $(x, x + \varepsilon x]$ contains an integer j with $h(\varphi(j)/j) < h_0 + 2\varepsilon$.*

Proof. Since h is convex and differentiable we have

$$|h'(x)| \leqq \max \{|h'(0)|, |h'(1)|\} = \zeta , \quad 0 \leqq x \leqq 1 .$$

The mean value theorem and Lemma 7 imply that there exists an integer j in each far out interval $(x, x + \varepsilon x]$ such that

$$|h(\varphi(j)/j) - h_0| \leqq \zeta \left| \frac{\varphi(j)}{j} - u_0 \right| < \zeta\varepsilon < 2\varepsilon .$$

Proof of Theorem 6. On the one hand,

$$\begin{aligned} \min_{n>x} F(n) &= \min_{n>x} \{n h(\varphi(n)/n) + O(n e^{-\sqrt{\log n}})\} \\ &\geqq x h_0 - c x e^{-\sqrt{\log x}} = h_0 x + o(x) . \end{aligned}$$

On the other hand, for given $\varepsilon > 0$ and all sufficiently large x there exists an integer m such that

$$x < m \leqq x + \varepsilon x\,, \quad h(\varphi(m)/m) < h_0 + 2\varepsilon\,.$$

For this integer m we have

$$F(m) < (h_0 + 2\varepsilon)m + cme^{-\sqrt{\log m}}\,,$$

and hence

$$\begin{aligned}\min_{n>x} F(n) &\leqq F(m) \leqq (h_0 + 2\varepsilon)(x + \varepsilon x) + 2cxe^{-\sqrt{\log x}} \\ &\leqq h_0 x + o(x)\,.\end{aligned}$$

Let $\{m_k\}_{k=1}^{\infty}$ be the sequence of discontinuities of $x \mapsto \min_{n>x} F(n)$. (Set $m_1 = 2$.) We can deduce from Theorem 6 the following

COROLLARY 3. *$m_{k+1}/m_k \to 1$ as $k \to \infty$.*

Proof. For $m_k \leqq x < m_{k+1}$ we have

$$\min_{n>m_k} F(n) = \min_{n>x} F(n)\,.$$

Thus $h_0 m_k \sim h_0 x$. Let $x \to m_{k+1}-$.

7. General arithmetic functions. We conclude by showing that rather general arithmetic functions ψ possess an associated monotonicity measuring function $F = F_\psi$. Our argument is related to one occurring in [4]. It appears unlikely that there are general analogues of our numbered theorems in §§ 3–6 which are valid without more specific arithmetic information.

It is convenient to estimate the two components of F separately. Let

$$\begin{aligned}F_1(n) &= \#\{m < n\colon \psi(m) \geqq \psi(n)\}\,, \\ F_2(n) &= \#\{m > n\colon \psi(m) \leqq \psi(n)\}\,.\end{aligned}$$

In both cases we assume that ψ is positive valued and that $\psi(n)/n$ has a distribution function D_ψ.

THEOREM 7. *Let ψ be as above. Then, as $n \to \infty$,*

$$F_1(n) = \psi(n)\int_{t=\psi(n)/n}^{\infty}\{1 - D_\psi(t)\}t^{-2}dt + o(n)\,. \tag{7}$$

Further, assume that there exist positive numbers c and δ such that

$$\#\{m \in (x, 2x]\colon \psi(m)/m < y\} \leqq cxy^{1+\delta} \tag{8}$$

holds for all $y \in (0, 1)$ and all $x \geqq 1$. Then

$$F_2(n) = \psi(n)\int_{t=0}^{\psi(n)/n} D_\psi(t)t^{-2}dt + o(n + \psi(n)) . \tag{9}$$

REMARKS. A. It is a simple consequence of hypothesis (8) that there exist at most a finite number of integers n for which $\psi(n)$ assumes any one value. Also, (8) implies that the integral in (9) converges at the origin.

B. For application to the Euler φ function, the estimate

$$\sum_{m=1}^{n} (m/\psi(m))^2 \ll n$$

(cf. [4]) guarantees that (8) holds with $\delta = 1$. Condition (8) is vacuous for the sum of divisors function σ, since $\sigma(n) \geqq n$ for all $n \geqq 1$.

C. Can we replace the equal sign in (7) or in (9) by "$\sim$" and drop the o-term? This is not generally permissible for (7) as one can see by the case in which $D_\psi(\alpha) = 1$ for some finite α, $\psi(n)/n \geqq \alpha$, and there exists at least one integer $m < n$ such that $\psi(m) \geqq \psi(n)$. The conjecture is also generally false for (9) as well, as we can see in the case where $D_\varphi(t) > 0$ for all $t > 0$. By Remark A there exists an infinite number of integers n for which $F_2(n) = 0$, and for these n the asymptotic relation would fail.

Proof. We shall show that (9) holds. The proof of (7) is similar but simpler, and is omitted.

Proof. We introduce a partition of (n, ∞). Let $\varepsilon > 0$, $K \in \boldsymbol{Z}^+$ with $\varepsilon K > 1$ and let $n' = n + \psi(n)$. Write

$$(n, \infty) = \bigcup_{i=1}^{K} (n + (i-1)\varepsilon n', n + i\varepsilon n'] \cup (n + K\varepsilon n', \infty) .$$

For the finite intervals we use the following estimates, which are valid for $1 \leqq x < y < \infty$:

$$\begin{aligned} &\#\{m \in (x, y]: \psi(m) \leq m\psi(n)/y\} \\ &\quad \leqq \# \overset{\text{def}}{=} \#\{m \in (x, y]: \psi(m) \leq \psi(n)\} \\ &\quad \leqq \#\{m \in (x, y]: \psi(m) \leq m\psi(n)/x\} , \end{aligned}$$

and hence

$$(y - x)D_\psi(\psi(n)/y) + o(y) \leqq \# \leqq (y - x)D_\psi(\psi(n)/x) + o(y) .$$

If we set

$$\Sigma = \varepsilon n' \sum_{i=1}^{K} D_\psi(\psi(n)/(n + i\varepsilon n'))$$

and

$$F_2(a, b) = \#\{m \in (a, a + b]: \psi(m) \leqq \psi(n)\} ,$$

then we obtain

$$\begin{aligned}\textstyle\sum + Ko(K\varepsilon n') &\leqq F_2(n, K\varepsilon n') \\ &\leqq \textstyle\sum + \varepsilon n' D_\psi(\psi(n)/n) - \varepsilon n' D_\psi(\psi(n)/(n + K\varepsilon n')) \\ &\quad + Ko(K\varepsilon n') .\end{aligned}$$

Now $\sum$ is an approximating sum for the Riemann integral

$$\begin{aligned}I &= \varepsilon n' \int_{t=0}^{K} D_\psi(\psi(n)/(n + t\varepsilon n'))dt \\ &= \psi(n) \int_{s=\psi(n)/(n+K\varepsilon n')}^{\psi(n)/n} D_\psi(s) s^{-2} ds ,\end{aligned}$$

and since the integrand in the first expression is monotone, we get $|I - \sum| < \varepsilon n'$. The hypotheses on $\psi(n)/n$ imply that

$$D_\psi(y) \leqq C y^{1+\delta} , \quad 0 < y < 1 .$$

Thus

$$\int_0^{\psi(n)/(n+K\varepsilon n')} D_\psi(t) t^{-2} dt \leqq \frac{C}{\delta}\left(\frac{\psi(n)}{n + K\varepsilon n'}\right)^{\delta} \leqq \frac{C}{\delta}(K\varepsilon)^{-\delta} .$$

Combining these estimates we find that

$$\begin{aligned}F_2(n, K\varepsilon n') &= \psi(n) \int_0^{\psi(n)/n} D_\psi(t) t^{-2} dt \\ &\quad + O(\varepsilon n') + Ko(K\varepsilon n') + O((K\varepsilon)^{-\delta} n') .\end{aligned}$$

Now we treat the unbounded interval. For each $x \geqq 1$ we have

$$\begin{aligned}F_2(x, x) &\leqq \#\{m \in (x, 2x]: \psi(m)/m \leqq \psi(n)/x\} \\ &\leqq Cx(\psi(n)/x)^{1+\delta} .\end{aligned}$$

Thus

$$\begin{aligned}F_2(n + K\varepsilon n', \infty) &\leqq C\psi(n)^{1+\delta}(n + K\varepsilon n')^{-\delta}(1 + 2^{-\delta} + 4^{-\delta} + \cdots) \\ &\ll \psi(n)(K\varepsilon)^{-\delta} .\end{aligned}$$

It follows that

$$\begin{aligned}F_2(n) &= \psi(n) \int_0^{\psi(n)/n} D_\psi(t) t^{-2} dt \\ &\quad + O(\varepsilon n') + K^2 \varepsilon o(n') + O((K\varepsilon)^{-\delta} n') .\end{aligned}$$

If we first choose ε small and then K so large that $(K\varepsilon)^{-\delta}$ is small, we obtain the desired asymptotic.

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