Some Extremal Problems in Geometry V

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Continuing our work of [3] we obtain lower bounds on the number of simplices of different volumes and on the number of hyperplanes determined by n points in E^k, not all of which lie on an E^{k-1} and no k of which lie on an E^{k-2}. We also determine the minimum number of triangles t determined by n noncollinear points in the plane and discuss what values of t can be achieved.

L. M. Kelly and W. O. J. Moser [4] proved a number of results that we use in this paper. They proved that if n points are given in the real projective plane, with no more than n - k points collinear, and if

$$n \geq 1/2\{3(3k-2)^2 + 3k-1\} \tag{1}$$

then the number of lines t containing two or more of the points satisfies

$$t \geq kn - 1/2(3k+2)(k-1). \tag{2}$$

The result is therefore true in the euclidean plane also. Putting $k = [c\sqrt{n}]$, we see that (1) holds for n sufficiently large if $c^2 < 2/27$, and for such c, we have

$$t \geq cn^{3/2} + O(n), \tag{3}$$

the requirement that n be sufficiently large now being redundant. Kelly and Moser also show that if t_i is the number of lines containing exactly i of the points, then

(4) $$t_2 \geq 3 + \sum_{i \geq 4} (i-3)t_i \; .$$

Adding $t_2 + t_3$ to both sides of (4) we obtain $3 \max(t_2,t_3) \geq 2t_2 + t_3$ $3 + t_2 + t_3 + t_4 + \ldots = 3 + t$. Consequently,

(5) $$\max(t_2, t_3) \geq 1 + \tfrac{1}{3}t.$$

This useful inequality first appears in P.D.T.A. Elliott's [2] as (6).

In [3] we proved the following

Theorem 1. Given n points in E^2, not all on a line, the number of different areas determined by the triangles formed from these points is at least $cn^{3/4}$, where c is a positive absolute constant.

Here we prove the following:

Theorem 2. Given a set S of n points in E^3, no three on a line, not all on a plane, there are at least $cn^{3/4}$ distinct volumes among the simplices.

Proof: Fix a point P of S and project the remaining points of S from P onto a plane π, obtaining $n - 1$ distinct projected points. Suppose that there is a line in π containing at least $\frac{1}{2}$ n points of S. By theorem 1 there are $cn^{3/4}$ different areas among the triangles, and fixing a point Q of S not on the plane yields $cn^{3/4}$ different volumes. We may therefore suppose that no line of π contains more than $n/2$ points, and by (3) there are at least $cn^{3/2}$ connecting lines in π. These give rise to $cn^{3/2}$ planes through P containing three or more points of S. Let π_0 be one of these planes. Let $\pi_1,\ldots,\pi_s$ be the connecting planes parallel to π_0, and let $\pi_{s+1},\ldots,\pi_r$ be the planes parallel to π_0 containing one or two points of S. We may suppose that $r < \varepsilon n^{3/4}$ for any $\varepsilon > 0$. To see this, pick a point X_i from π_i for $1 \leq i \leq r$ and 3 points A, B and C from π_0. The simplices X_iABC determine at least $r/2$ different volumes. Hence we may indeed assume $r < \varepsilon n^{3/4}$ for any positive ε. Now $\sum_{i=1}^{s} |\pi_i| \geq n - 2\varepsilon n^{3/4} > \frac{1}{2} n$, and by a well-known inequality

$$\sum_{i=1}^{s} \binom{|\pi_i|}{3} \geq \frac{s}{6}\left(\frac{n}{2s} - 3\right)^3 \geq \frac{cn^3}{s^2} \geq \frac{cn^3}{\varepsilon^2 n^{3/2}} = \frac{cn^{3/2}}{\varepsilon^2} \quad .$$

Taking the triples from all the $cn^{3/2}$ parallel families, we obtain

$$\binom{n}{3} \geq \frac{cn^{3/2} cn^{3/2}}{\varepsilon^2} = \frac{cn^3}{\varepsilon^2} \quad ,$$

which is impossible if ε is sufficiently small, and the result follows.

Remark. Erdös has made the following conjecture: There exists a constant $c > 0$, independent of n and k so that if there are given n points in the plane, no $n - k$ on a line, then the points determine at least ckn lines.

If true, this conjecture with $k = n/2$ together with the proof of theorem 2 implies the existence of at least cn distinct volumes.

In E^k we have

Theorem 3 Given n points in E^k, $k \geq 3$, no k on an E^{k-2} and not all on an E^{k-1}, the k-dimensional simplices with those points as vertices have at least $d_k n^{\alpha_k}$ distinct volumes where $\alpha_k = \frac{k-2}{k-1}$ and $d_k > 0$.

To prove this we need the following

Lemma 1. Given n points in E^k, no k on an E^{k-2}, not all on an E^{k-1}, there are at least $c_k n^{k-1}$ distinct hyperplanes containing exactly k points.

Proof. If $k = 2$ this follows from a result of L. M. Kelly and W. O. J. Moser [4] stating that n noncollinear points in the plane determine at least $\frac{3n}{7}$ lines with two points. Let $k > 2$ and use induction on k. Let P be one of the points, and project the other $n - 1$ points from P onto a hyperplane H. No three points are collinear, and so H has $n - 1$ distinct points, not all on an E^{k-2}, and no $k - 1$ on an E^{k-3}. By the induction hypothesis H contains $c_{k-1}n^{k-2}E^{k-2}$ spaces with exactly $k - 1$ points. When joined to P these become the same number of hyperplanes through P having exactly k points on them. If we do this for the n choices for P, each hyperplane is counted k times. Thus we get

$$\frac{c_{k-1}n^{k-1}}{k} = c_k n^{k-1}$$

hyperplanes as claimed.

Remarks. The example of $k - 1$ skew lines with $\frac{n}{k-1}$ points on each shows that the hypothesis of no k points on an E^{k-2} is necessary. Dirac [1] proved the existence of one such hyperplane when $k = 3$.

Proof of Theorem 3. Project the system from one of the points P onto a hyperplane H. Then H has $n - 1$ points, not all on an E^{k-2}, and no $k - 1$ on an E^{k-3}. By lemma 1 there are at least $c_{k-1}n^{k-2}$ distinct E_{k-2} spaces in H containing exactly $k - 1$ points. This leads to the same number of E_{k-1} spaces through P containing exactly k points, and no two of these hyperplanes are parallel. Let H_0 be one of these, let $H_1,\ldots,H_s$ be the connecting hyperplanes parallel to H_0, and let $H_{s+1},\ldots,H_r$ be the hyperplanes parallel to H_0 containing fewer than k points. We may suppose that

$r \in \varepsilon n^{\alpha_k}$ for any fixed $\varepsilon > 0$. To see this, pick a point X_i from H_i for $1 \le i \le r$ and k points $y_1, \ldots, y_k$ from H_0. The simplices $X_i Y_1 \ldots Y_k$ determine at least $\frac{r}{2}$ different volumes. Now $\sum_{i=1}^{s} |H_i| \ge n - k\varepsilon n^{\alpha_k} > \frac{1}{2} n$ if ε is sufficiently small, and by a well known inequality

$$\sum_{i=1}^{s} \binom{|H_i|}{k} \ge \frac{s}{k!} \left(\frac{n}{2s} - k\right)^k \ge \frac{cn^k}{s^{k-1}} \ge \frac{cn^k}{(\varepsilon n^{\alpha k})^{k-1}} = \frac{cn^{k-\alpha_k(k-1)}}{\varepsilon^{k-1}} .$$

Taking the k-tuples from all the $c_{k-1} n^{k-2}$ parallel families, we obtain

$$\binom{n}{k} \ge \frac{c_{k-1} n^{k-2} cn^{k-\alpha_k(k-1)}}{\varepsilon^{k-1}} \quad ,$$

which is a contradiction if $\alpha_k = \frac{k-2}{k-1}$ and ε is sufficiently small. The theorem follows.

If the condition of no k on an E^{k-2} is dropped, then we can only prove the following:

Theorem 4. Given n points in E^k, not all on an E^{k-1}, there are at least $c_k n^{\varepsilon_k}$ distinct volumes, where $\varepsilon_k = 3(3k-2)^{-1}$.

Proof. If k = 2, this follows from theorem 1. Let k > 2 and use induction on k. If there is an E^{k-1} containing m points we get $cm^{\varepsilon_{k-1}}$ volumes, otherwise we get at least $\frac{n}{m}$ volumes. Putting $m = n^{\alpha}$, where $\alpha = (1 + \varepsilon_{k-1})^{-1}$ gives

$$\min\left(\frac{n}{m}, cm^{\varepsilon_{k-1}}\right) \ge cn^{\varepsilon_k} \text{ , and the theorem follows.}$$

and the theorem follows.

It seems natural to ask the question: Given n points in space, no three on a line, not all on a plane, how many planes do they determine? We are able to answer this question by the following:

Theorem 5. Under the above conditions at least $\binom{n-1}{2} + 1$ planes are determined, provided $n \geq 552$.

Remark. The example of $n - 1$ points on a plane and one point off the plane shows that the result is best possible.

We need the following lemma:

Lemma 2. Given n points in E^3, no three on a line and $n - 2$ on a plane at least $2\binom{n-2}{2} - \left[\frac{n-2}{2}\right]$ planes are determined.

Proof. Let π be the plane with $n - 2$ points and let P and Q be the other two points. The points of π and P determine $\binom{n-2}{2}$ planes. It is enough to show that Q can lie on at most $\left[\frac{n-2}{2}\right]$ of these. Firstly, suppose that the line PQ is parallel to π. The m planes through P and Q intersect π in m parallel lines generated by the $n - 2$ points, and so $m \leq \left[\frac{n-2}{2}\right]$. Secondly, suppose that PQ intersects π in a point A. Since no three points are collinear, A is not one of the $n - 2$ points on π, and so the number of planes through P and Q is again at most $\left[\frac{n-2}{2}\right]$.

Proof of Theorem 5. If $n - 1$ points are coplanar, then $\left[\frac{n-1}{2}\right] + 1$ planes are determined. We suppose that there is a plane π containing at least $n - 7$ and at most $n - 2$ of the points. Then by lemma 2 the number of planes determined is at least $2\binom{n-7}{2} - \left[\frac{n-7}{2}\right]$, and this is greater than $\binom{n-1}{2} + 1$ for $n \geq 23$.

We may therefore suppose that at most $n - 6$ points are coplanar. From one of the points P we project all of the points onto a fixed plane π.

Since no three points are collinear the plane π contains $n-1$ points, and at most $n-7$ of them are collinear. It follows from (2) and (5) that there are at least $\frac{7}{3}n - 22$ lines in π containing either two or three projected points, provided $n \geq 552$. Hence there are at least $\frac{7}{3}n - 22$ planes through P having four or fewer points on them. There are n choices for P, and each plane is counted at most four times in this way. Hence the number of planes is at least $\frac{n}{4}(\frac{7}{3}n - 22)$, and this is greater than $\binom{n-1}{2} + 1$ for $n \geq 55$.

We next show that you get nearly as many planes if you restrict yourself to planes containing only three or four points.

Theorem 6. Given n points in E^3, no three on a line, not all on a plane, the points determine at least $\frac{1}{2}n^2 - cn$ planes having three or four points on them.

Proof. If $n-1$ points are coplanar, then $\binom{n-1}{2}$ planes through three points are determined. If at most $n-6$ points are coplanar, then the result follows from the proof of theorem 5. Suppose that there is a plane π containing at least $n-7$ points and at most $n-2$ points. Let P and Q be two points not on π. From the proof of lemma 2 we see that there are at least $2\binom{n-7}{2} - \left[\frac{n-7}{2}\right]$ planes through P or Q. There are at most five other points, and each can be on at most $n-7$ planes. Hence the number of planes through three or four points is at least

$$2\binom{n-7}{2} - \left[\frac{n-7}{2}\right] - 5(n-7),$$

and this is greater than $\frac{1}{2}n^2 - cn$ for suitable c.

P. D. T. A. Elliot proves in [2]

<u>Theorem 7</u>. Let S be a set of $n \geq 3$ points in the plane, not all on a line. Then S determines at least $\binom{n-1}{2}$ distinct triangles.

Elliott's proof seems only to be correct for $n \geq 16$. Here we present a simpler proof which is true for $n \geq 3$.

<u>Proof of Theorem 7</u>. Let k_i, $1 \leq i \leq r$, be the number of points on the ith line of the r lines determined by S. Then $\sum_{i=1}^{r} \binom{k_i}{2} = \binom{n}{2}$ and the number of triangles t is

$$t = \frac{1}{3} \sum_{i=1}^{r} \binom{k_i}{2} (n - k_i).$$

If $n - k_i \geq 3$ for all i the theorem follows. Suppose $n - k_i = 1$ for some i. Then there is a line with $n - 1$ points on it and $t = \binom{n-2}{2}$. We may therefore suppose that $n - k_i = 2$ for some i. There is a line with $n - 2$ points on it. The worst case occurs when the two points not on the line are collinear with a point on the line, and $t = 2\binom{n-3}{2} + 3(n+3)$. This is greater than or equal to $\binom{n-2}{2}$ for $n \geq 3$, and the theorem follows.

<u>Remarks</u>. The above proof does not use the fact that the n points lie in a plane. In fact the same argument proves the following theorem about sets:

<u>Theorem 8</u>. Let $|S| = n$ be a set, and let $2 \leq |A_k| < n$, $1 \leq k \leq m$ be a family of subsets so that every pair (x,y) of elements of S is contained in one and only one of the A_i. Then the number t of unordered triples (x,y,z) so that x, y and z do not lie in the same A_i satisfies $t \geq \binom{n-1}{2}$.

Remark. The method of theorem 7 can also be used to show that n noncoplanar points in E^3, no three of which are collinear determine at least $\binom{n-1}{3}$ simplices.

It seems natural to ask what values of t between $\binom{n}{3}$ and $\binom{n-1}{2}$ can be achieved in the plane. We have

Theorem 9. If $cn^{3-\frac{1}{3}} < m \leq \binom{n}{3}$, where c is a certain constant, then there exist configurations of n points in the plane with exactly m triangles. To prove this we need

Lemma 3. Every integer $t < \binom{n}{3} - cn^{8/3}$ can be written in the form

$$t = \sum_i \alpha_i \binom{n_i}{3},$$

where $\sum_i \alpha_i n_i \leq n$, $n_i \geq 3$ and the α_i are positive integers.

Proof. Let n_1 be the largest integer such that $t \geq \binom{n_1}{3}$. We then have $\frac{1}{6}(n_1 - 2)^3 < \binom{n_1}{3} \leq t < \frac{n^3}{6} - cn^{8/3}$. Hence $n_1 - 2 < n(1-6cn^{-1/3})^{1/3} < n(1-2cn^{-1/3})$, or $n_1 < 2 + n - 2cn^{2/3} \leq n - cn^{2/3}$ for c sufficiently large. Also $t - \binom{n_1}{3} < \binom{n_1+1}{3} - \binom{n_1}{3} = \binom{n_1}{2}$. Let n_2 be defined by $\binom{n_2}{3} \leq t - \binom{n_1}{3} < \binom{n_2+1}{3}$. Then $\frac{1}{6}(n_2-2)^3 < \binom{n_2}{3} \leq t - \binom{n_1}{3} < \binom{n_1}{2} < 1/2(n-cn^{2/3})^2$, $n_2-2 < 3^{1/3}n^{2/3}(1-cn^{-1/3})^{2/3} < 3^{1/3}(n^{2/3} - 2/3cn^{1/3})$. Thus $n_2 < 3^{1/3}n^{2/3}$ for c large enough. Thus $t - \binom{n_1}{3} - \binom{n_2}{3} \leq \binom{n_2+1}{3} - \binom{n_2}{3} = \binom{n_2}{2} < \frac{3^{2/3}}{2} n^{4/3}$. Let n_3 be defined by $\binom{n_3}{3} \leq t - \binom{n_1}{3} - \binom{n_2}{3} < \binom{n_3+1}{3}$. Then $\frac{1}{6}(n_3-2)^3 < t - \binom{n_1}{3} - \binom{n_2}{3} < \frac{3^{2/3}}{2} n^{4/3}$, $n_3 < 2 + 2n^{4/9} \leq 4n^{4/9}$.

Put $\alpha_1 = \alpha_2 = \alpha_3 = 1$, $\alpha_4 = t - \binom{n_1}{3} - \binom{n_2}{3} - \binom{n_3}{3} < \binom{n_3+1}{3} - \binom{n_3}{3} =$

$\binom{n_3}{2} < 8n^{8/9}$. Then putting $n_4 = 3$, we have $t = \sum_{i=1}^{4} \alpha_i \binom{n_i}{3}$ and $\sum \alpha_i n_i \leq n$ for $n \geq n_0$. Taking c sufficiently large will force $n \geq n_0$.

Proof of Theorem 9. Let α_i and n_i be given by lemma 3. Place the points so that there are α_i lines having n_i points on them, with the points in general position otherwise. The number of triangles is then

$$\binom{n}{3} - \sum_i \alpha_i \binom{n_i}{3} ,$$

and the result follows from lemma 3.

The proof of theorem 9 can be modified slightly to show

Theorem 10. If $cn^{3-\frac{1}{3}} < m \leq \binom{n}{3}$, where c is a certain constant, and $m \neq \binom{n}{3} - r_i$, where the r_i, $1 \leq i \leq k$ are certain integers, then there exists a configuration of n points in E^3, no three of which are collinear, which determine exactly m planes.

References

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