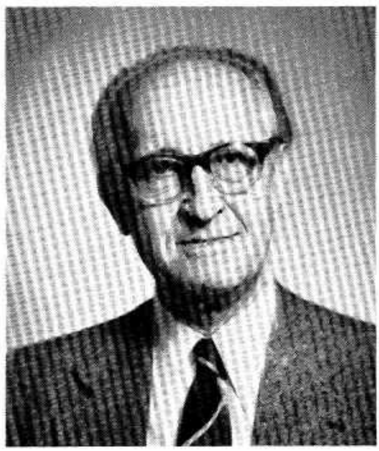
The Editors dedicate this issue of the

Journal of Graph Theory

to the memory of



Paul Turán

Academician, two-time winner of the Kossuth prize, Professor of the Eötvös Loránd University, Chairman of the Department of Complex Analysis of the Mathematical Institute, President of the Bolyai János Mathematical Society, holder of the Golden Degree of the Labour Order

who died after a long illness in his 66th year on **26 September 1976**

His death represents an irreplaceable loss to the world of mathematics.

The Graph Theory Bibliography of Paul Turán

[1] An extremal problem in graph theory. *Mat. Fiz. Lapok* 48 (1941) 435–452 (in Hungarian).

[2] On the theory of graphs. *Coll. Math.* 3 (1954) 19–30.

[3] On a problem of K. Zarankiewicz. *Coll. Math.* 3 (1954) 50–57 (with V. T Sós and T. Kövári).

[4] Application of graph theory to geometry and potential theory. *Combinatorial Structures and Their Applications.* Gordon and Breach, New York (1970) 423–434.

[5] On some applications of graph theory, II. *Studies in Pure Mathematics.* Academic Press, New York (1971) 89–99 (with P. Erdös, A. Meir and V. T. Sós).

[6] On some applications of graph theory to analysis. *Proceedings of the International Conference on Constr. Theory of Functions.* Varna (1970).

[7] On some applications of graph theory, III. *Canad. Math. Bull.* 15 (1972) 27–32 (with P. Erdös, A. Meir and V. T. Sós).

[8] On some applications of graph theory, I. *Discrete Math.* 2 (1972) 207–228 (with P. Erdös, A. Meir and V. T. Sós).

[9] A note of welcome. *J. Graph Theory* 1 (1977) 7–9.

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Paul Turán, 1910–1976: His Work in Graph Theory

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In this short note, I will restrict myself to Turán's work in graph theory, even though his main work was in analytic number theory and various other branches of real and complex analysis. Turán had the remarkable ability to write perhaps only one paper or to state one problem in various fields distant from his own; later others would pursue his idea and a new subject would be born.

In this way Turán initiated the field of extremal graph theory. He started this subject in 1941 (see [18] and [19]). He posed and completely solved the following problem: Define $f(n; K_p)$ as the smallest integer for which every graph of n vertices and $f(n; K_p)$ edges, denoting such a graph by $G(n; f(n, K_p))$, contains K_p. Turán determined $f(n; K_p)$ explicitly. Since his formula for $f(n; K_p)$ is somewhat complicated, it is better to state his stronger theorem:

The only $G(n; f(n; K_p)-1)$ which does not contain K_p is obtained by setting

$$n = \sum_{i=1}^{p-1} x_i,$$

where the x_i are as nearly equal as possible, i.e., they differ by at most one.

Let $|S| = n$ and $|S_i| = x_i$. Join two vertices whenever they belong to different sets S_i.

Denote by $K_p^{(r)}$ the complete r-graph of p vertices and $\binom{p}{r}$ edges. The basic elements of an r-graph are its vertices and unordered r-tuples (the edges). Denote by $f^{(r)}(n; K_p^{(r)})$ the smallest integer for which every r-graph $G^{(r)}(n; f^{(r)}(n; K_p^{(r)}))$ contains $K_p^{(r)}$. Turán asked in 1940 for the determination of $f^{(r)}(n; K_p^{(r)})$. This fascinating problem is unsolved for

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every $r>2$ and $p>r$. Turán made some plausible conjectures for $r=3$, $p=4$, and $r=3$, $p=5$. Here are his conjectures: First let $r=3$, $p=4$, $n=3m$. Let the vertices of $G^{(m)}(3m)$ be $X_1, \ldots, X_m$; $Y_1, \ldots, Y_m; Z_1, \ldots, Z_m$. The edges (each a set of 3 vertices) are

$$(X_{i_1}, Y_{i_2}, Z_{i_3}),$$
$$1 \leq i_1, i_2, i_3 \leq n$$

and

$$(X_{i_1}, X_{i_2}, Y_{i_3}), (Y_{i_1}, Y_{i_2}, Z_{i_3}), (Z_{i_1}, Z_{i_2}, X_{i_3}).$$

This graph has $n^3+3n\binom{n}{2}$ triples and no $K_4^{(3)}$. Turán conjectured that

$$f^{(3)}(3n, K_4^{(3)}) = n^3 + 3n\binom{n}{2} + 1.$$

He also conjectured that

$$f^{(3)}(2n, K_5^{(3)}) = n^2(n-1)+1.$$

No progress has been made with these conjectures since 1940.

It is easy to see that

$$\lim_{n\to\infty} f^{(r)}(n; K_p^{(r)})/\binom{n}{r} = c(r, p)$$

exists. Turán's original results imply that

$$c(2, p) = 1 - 1/(p-1),$$

$c(r, p)$ is unknown for all $p>r>2$. In Turán's memory I offer \$500 for the determination of any $c(2, p)$, $p>r>2$!

Turán also formulated several other extremal problems on graphs, some of which were solved by Gallai and myself [10]. I began a systematic study of extremal problems in graph theory in 1958 on the boat from Athens to Haifa and have worked on it since then. The subject has grown enormously and has a very large literature; Bollobás has written a comprehensive book on extremal problems in graph theory which will appear soon.

Turán wrote one more paper on extremal graph problems, with V. T. Sós (Mrs. Turán) and T. Kövári [16]. They prove that

$$f(n; K_{m,m}) < c_m n^{2-1/m}. \tag{1}$$

They conjecture that (1) is the best possible; in other words,

$$f(n; K_{m,m}) > c'_m n^{2-1/m}. \tag{2}$$

The inequality (2) is known only for $m=2$ and 3. (See also [3] and [13].)

For further remarks and generalizations of (1) for r-graphs see [4], [5], and [6].

Turán's theorem had many applications. The Turáns, A. Meir, and I applied it to various problems of geometry and potential theory in [11] and [12]. Katona [15] has the following pretty application:

Let $f_i(x)$, $1 \le i \le n$, be real functions such that $\int f_i^2(x) \ge 1$. Then there are at most $[n^2/4]$ pairs $1 \le i < j \le n$ for which $\int (f_i(x) + f_j(x))^2 \le 1$.

The following interesting conjecture of Witsenhausen is connected with Turán's theorem:

Let there be given n points $(X_i, 1 \le i \le n)$ in d-dimensional Euclidean space, $\|X_i - X_j\| \le 1$. Prove that

$$\sum_{1 \le i < j \le n} \|X_i - X_j\|^2$$

is a maximum when the n points are distributed as evenly as possible among the $d+1$ vertices of a regular simplex of edge-length 1.

Witsenhausen [20] proves this if $n \equiv 0 \bmod (d+1)$ but the general case is open even for $d = 2$. Many further applications are discussed in [9].

V. T. Sós and I in a joint paper [13] investigated a set of problems connecting the theorems of Turán and Ramsey. Denote by $f(n; m, k)$ the largest integer q for which there is a graph G of n vertices and q edges which contains no K_m and the complementary graph contains no K_k. If m and k are fixed and $n \ge r(m, k)$, then by Ramsey's theorem there is no such graph and we set $f(n; m, k) = 0$. The general determination of $f(n; m, k)$ is probably hopeless. In our paper we investigated the cases m fixed with $k = O(n)$, and obtained several inequalities. Trivially $f(n; 3, k) = O(n^2)$, since $f(n; 3, k) \le nk/2$. The cases of equality are not completely settled. We proved that $f(n; 5, k) = (1 + O(1))n^2/4$ and later Szemerédi [17] and Bollobás and I [2] proved $f(n; 4, k) = (1 + O(1))\, n^2/8$; see also [1] and [7]. For some recent problems in extremal graph theory, see [8].

Turán's untimely death is a great loss to mathematics and to me both personally and mathematically. He was one of my oldest friends and we have published more than 20 joint papers and collaborated for more than 40 years. He left many half-finished papers and his book on his method of power sums is not completely finished, but his pupils and collaborators will, I hope, be able to finish them soon in his spirit. His collected works will eventually be published by the Hungarian Academy of Sciences.

Halász and I will write a paper for *Acta Arithmetica* on his work in number theory which certainly was his main interest.

In a paper dedicated to Turán's memory which will soon appear in the proceedings of the 1976 Manitoba conference on number theory and computing, I gave a short history of our lifelong collaboration. In the last paragraph I wrote:

> It is always sad if a great man dies when he is still in his prime mentally, but there is one consolation: He never knew the two greatest evils—old age and stupidity. I hope I too will be spared them as Turán was.

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