APPROXIMATION BY RATIONAL FUNCTIONS

P. ERDŐS, D. J. NEWMAN AND A. R. REDDY

Introduction

Recently approximation of e^{-x} by rational functions has attracted the attention of several mathematicians (*cf.* [2]–[5], [7]–[10]). In this paper we present several new results. Some of the methods used here may be applied successfully to several related problems.

As usual we use throughout our work $\|\cdot\|$ to mean the maximum modulus within the set of points under consideration.

Lemmas

LEMMA 1 [8]. *Let $p(x)$ be a polynomial of degree at most n having only real zeros and suppose that $p(x) > 0$ on $[a, b]$. Then $[p(x)]^{1/n}$ is concave on $[a, b]$.*

LEMMA 2 [1; p. 10]. *Let $f(x)$ be a function which is $(n+1)$ times continuously differentiable on $[a, b]$ and satisfies the further assumption that $|f^{(n+1)}(x)| \geqslant M > 0$ for all $x \in [a, b]$. Then for any polynomial $p(x)$ of degree at most n,*

$$\|f(x)-p(x)\|_{L_\infty[a,b]} \geqslant \frac{2(b-a)^{n+1} M}{4^{n+1}(n+1)!}.$$

LEMMA 3. *Let $P(x)$ be any polynomial of degree at most $2n$ satisfying the assumption that $|P(k)|$ is bounded by 1, for $k = 0, 1, 2, \ldots, n, n+1, \ldots, 2n$. Then*

$$\max_{[0,\,2n]} |P(x)| \leqslant n4^n. \tag{1}$$

Proof. It is well known that $P(x)$ can be written as

$$\sum_{i=0}^{2n} P(x_i)\, l_i(x), \tag{2}$$

where

$$l_i(x) = \frac{(x-x_0)\,(x-x_1)\ldots(x-x_{i-1})\,(x-x_{i+1})\ldots(x-x_{2n})}{(x_i-x_0)\,(x_i-x_1)\ldots(x_i-x_{i-1})\,(x_i-x_{i+1})\ldots(x_i-x_{2n})}, \tag{3}$$

and $x_k = k$.

From (3), we obtain for $0 \leqslant x \leqslant 2n$, $n \geqslant 1$,

$$|l_i(x)| \leqslant \left| \frac{(2n)\,(2n-1)\,(2n-2)\ldots(2n-n)\,(0-(n+1))\,(0-(n+2))\ldots(0-2n)}{(2n-i)\,(i)\,(i-1)\,(i-2)\ldots(1)\,(-1)\,(-2)\ldots(i-2n)} \right|$$

$$= \frac{(n!)^{-2}\, n(2n)!\,(2n)!}{(2n-i)\,(i)!\,(2n-i)!} \leqslant \frac{n(2n)!}{i!(2n-i)!} \binom{2n}{n}. \tag{4}$$

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Hence we get from (2) and (4),

$$|P(x)| \leqslant \sum_{i=0}^{2n} \operatorname{Max} |P(x_i)| \, |l_i(x)| \leqslant \binom{2n}{n} \sum_{i=0}^{2n} \frac{n(2n)!}{(2n-i)!i!} = n4^{2n}.$$

LEMMA 4. *Let $p(x)$ be a polynomial of degree at most n. If this polynomial is bounded by M on an interval $[a, b] \subset [c, d]$, then throughout $[c, d]$ we have the relation*

$$|P(x)| \leqslant M \left| T_n\left(\frac{2(d-c)}{(b-a)} - 1\right)\right|, \tag{5}$$

where

$$2T_n(x) = (x+\sqrt{(x^2-1)})^n + (x-\sqrt{(x^2-1)})^n.$$

Proof. The inequality (5) follows easily from [**11**; (9), p. 68].

LEMMA 5. *If $Q(x)$ be a polynomial and Δ denotes the difference operator with increment* 1, *then*

$$\Delta^{n+1}(a^x Q(x)) = a^x(a\Delta + a - 1)^{n+1} Q(x). \tag{6}$$

Proof. It is well known [**6**; (10), p. 97] that

$$\Delta^m(a^x Q(x)) = \sum_{i=0}^{m} \binom{m}{i} \Delta^i Q(x) \, \Delta^{m-i} E^i a^x, \tag{7}$$

where $E = 1 + \Delta$. A little computation based on (7), along with the well-known fact that

$$\Delta^m(f(x)) = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} f(x+k),$$

will give us the required result.

LEMMA 6 [**6**; p. 13]. *If $f(x)$ is a polynomial of degree at most $n+1$, then*

$$(1-\Delta)^{-n-1} f(x) = \sum_{i=0}^{n+1} \binom{n+i}{i} \Delta^i f(x). \tag{8}$$

Henceforth we let N denote the set of non-negative integers.

Theorems

THEOREM 1. *Let $p(x)$ and $q(x)$ be any polynomials of degree at most $(n-1)$ having only non-negative coefficients. Then*

$$\left\| e^{-x} - \frac{p(x)}{q(x)} \right\|_{L_\infty(N)} \geqslant (4ne^{n+1})^{-1}. \tag{9}$$

Proof. Let us assume that (9) is false. Let $f(x) = e^x$; then there exist polynomials $p(x)$ and $q(x)$ such that at the origin and each positive integer

$$\left\| \frac{1}{f(x)} - \frac{p(x)}{q(x)} \right\| < \frac{1}{4ne^{n+1}}. \tag{10}$$

Now at $x = n$,

$$f(x) = f(n) = e^n. \tag{11}$$

At this point

$$\left|\frac{q(x)}{p(x)}\right| = \left|\frac{q(n)}{p(n)}\right| < \left(\frac{n+1}{n}\right) e^n. \tag{12}$$

If (12) were not valid, then (10) would be contradicted.

At $x = n+1$,

$$f(x) = f(n+1) = e^{n+1}. \tag{13}$$

From (12), and the assumption that $p(x)$ and $q(x)$ have non-negative coefficients, we have that

$$\left|\frac{q(n+1)}{p(n+1)}\right| \leqslant \left|\frac{(n+1)^{n-1} q(n)}{n^{n-1}\, p(n+1)}\right| < \left(\frac{n+1}{n}\right)^n e^n. \tag{14}$$

From (13) and (14), we get easily for $x = n+1$, that

$$\frac{1}{4ne^{n+1}} \leqslant \left(\frac{n}{n+1}\right)^n e^{-n} - e^{-n-1} < \frac{p(x)}{q(x)} - \frac{1}{f(x)}. \tag{15}$$

The relation (15) clearly contradicts (10) at $x = n+1$, and hence the result is established.

THEOREM 2. *The rational function*

$$r_{m,n}(x) = \frac{\int_0^\infty t^n(t-x)^m e^{-t}\,dt}{\int_0^\infty t^m(t+x)^m e^{-t}\,dt},$$

satisfies

$$\|e^{-x} - r_{m,n}(x)\|_{L_\infty[0,1]} \leqslant \frac{m^m n^n}{(m+n)^{m+n}\,(m+n)!}. \tag{16}$$

Proof. It is easy to check that for $0 \leqslant x \leqslant 1$

$$\left|\frac{\int_0^\infty t^n(t-x)^m e^{-t}\,dt}{\int_0^\infty t^m(t+x)^n e^{-t}\,dt} - e^{-x}\right| = \left|\frac{\int_0^\infty t^n(t-x)^m e^{-t}\,dt - \int_0^\infty t^m(t+x)^n e^{-(t+x)}\,dt}{\int_0^\infty t^m(t+x)^n e^{-t}\,dt}\right|$$

$$= \left| \frac{\int_0^\infty t^n(t-x)^m e^{-t}\,dt - \int_x^\infty t^n\,(t-x)^m e^{-t}\,dt}{\int_0^\infty t^m(t+n)^n e^{-t}\,dt} \right|$$

$$\leqslant \left| \frac{\int_0^x t^n(x-t)^m e^{-t}(-1)^m\,dt}{\int_0^\infty t^m(t+x)^n e^{-t}\,dt} \right|$$

$$\leqslant \left| \frac{\int_0^x t^n(1-t)^m e^{-t}\,dt}{\int_0^\infty t^{m+n} e^{-t}\,dt} \right| \leqslant \left| \frac{\int_0^x t^n(1-t)^m e^{-t}\,dt}{(m+n)!} \right|. \tag{17}$$

It is easy to verify that $t^n(1-t)^m$ attains its maximum on $[0, 1]$ for

$$t = \frac{n}{m+n}. \tag{18}$$

From (17) and (18), we get the relation

$$\left| \frac{\int_0^x t^n(1-t)^m e^{-t}\,dt}{(m+n)!} \right| \leqslant \frac{m^m n^n}{(m+n)^{m+n}\,(m+n)!}.$$

Hence the result (16) is proved.

THEOREM 3.

$$\left\| e^{-x} - \frac{1}{\sum_{k=0}^{n} \frac{x^k}{(k)!}} \right\|_{L_\infty[0,\infty)} \leqslant 2^{-n}. \tag{19}$$

Remark. This theorem is already known (*cf.* [**2**]). But the proof presented below is very simple.

Proof. It is known that

$$S_n(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = \frac{1}{n!} \int_0^\infty e^{-t}(t+x)^n\,dt.$$

Therefore

$$0 \leqslant \frac{1}{S_n(x)} - e^{-x} = \frac{\int_0^\infty e^{-t} t^n dt - \int_x^\infty e^{-t} t^n dt}{\int_0^\infty e^{-t}(t+x)^n dt}$$

$$= \frac{\int_0^x e^{-t} t^n dt}{\int_0^\infty e^{-t}(t+x)^n dt} \leqslant \frac{\int_0^x e^{-t} t^n dt}{\int_0^x e^{-t}(2t)^n dt} = 2^{-n}.$$

Hence (19) is proved.

THEOREM 4. *Let $p(x)$ be any polynomial of degree at most n having only real negative zeros. Then*

$$\left\| e^{-x} - \frac{1}{p(x)} \right\|_{L_\infty(N)} \geqslant \frac{1}{4ne^5}. \tag{20}$$

Proof. Let us assume that $p(x) > 0$ on $[0, 2]$. Then according to our Lemma 1,

$[p(x)]^{1/n}$ is concave on $[0, 2]$.

Therefore

$$2[p(1)]^{1/n} \geqslant [p(0)]^{1/n} + [p(2)]^{1/n}. \tag{21}$$

Let us write for $p(x)$ at $x = 0$, 1 and 2,

$$\|e^x - p(x)\| = \varepsilon. \tag{22}$$

Then

$$\left.\begin{aligned} &p(0) \geqslant 1-\varepsilon, \\ &p(1) \leqslant e+\varepsilon \leqslant \frac{e}{1-\varepsilon}, \\ &p(2) \geqslant e^2-\varepsilon \geqslant e^2 - e^2\varepsilon = e^2(1-\varepsilon). \end{aligned}\right\} \tag{23}$$

From (21) and (23), we have

$$\frac{2e^{1/n}}{(1-\varepsilon)^{1/n}} \geqslant (1-\varepsilon)^{1/n} + e^{2/n}(1-\varepsilon)^{1/n}. \tag{24}$$

From (24), we get

$$\frac{1}{(1-\varepsilon)^{2/n}} \geqslant \frac{e^{-1/n}+e^{1/n}}{2} \geqslant 1 + \frac{1}{2n^2}. \tag{25}$$

From (25), we obtain

$$\frac{1}{(1-\varepsilon)} \geqslant \left(1+\frac{1}{2n^2}\right)^{n/2} \geqslant \left(1+\frac{1}{4n}\right) = \frac{4n+1}{4n}, \text{ that is, } \varepsilon \geqslant (1+4n)^{-1}. \tag{26}$$

Let us assume that $[p_n(x)]^{-1}$ deviates least from e^{-x} at $x = 0, 1, 2$, and let

$$\left\| e^{-x} - \frac{1}{p_n(x)} \right\| = \delta. \tag{27}$$

Then we get from (27), for $x = 0, 1, 2$, by noting the fact that $p_n(x)$ has non-negative coefficients and $\delta \leqslant (en)^{-1}$ (*cf.* [9; Theorem 1]),

$$\|e^x - p_n(x)\| \leqslant \delta e^2\, p_n(2) \leqslant \delta e^4 (1 - e^2\delta)^{-1}. \tag{28}$$

But from (22) and (26), we have for every $p_n(x)$, at $x = 0, 1, 2$,

$$\|e^x - p_n(x)\| \geqslant (1 + 4n)^{-1}. \tag{29}$$

Hence $1/(1+4n) \leqslant \delta e^4 (1 - e^2\delta)^{-1}$, which implies that $\delta \geqslant e^{-5}(4n)^{-1}$.

THEOREM 5. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function. Then there is a polynomial* $p(x)$ *of degree at most* n *for which, for all* $n \geqslant 2$,

$$\left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{L_\infty(N)} \leqslant \frac{2}{f(n)}. \tag{30}$$

Proof. Let

$$p_n(x) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k f(x)|_{x=0}. \tag{31}$$

Then, clearly,

$$f(x) = p_n(x), \qquad x = 0, 1, 2, \ldots, n. \tag{32}$$

Therefore, for $x = 0, 1, 2, \ldots, n$,

$$\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\| = 0. \tag{33}$$

For $x \geqslant n+1$,

$$p_n(x) = \sum_{k=0}^{n} \binom{x}{k} \Delta^k f(x)|_{x=0} > f(n).$$

Therefore for $x = n+1, n+2, n+3, \ldots, 2n, 2n+1, \ldots,$

$$\left\| \frac{1}{f(x)} - \frac{1}{p_n(x)} \right\| \leqslant \frac{1}{f(n)} + \frac{1}{f(n)} = \frac{2}{f(n)}. \tag{34}$$

The relation (30) follows from (33) and (34).

THEOREM 6. *Let* $0 = a_0 < a_1 < a_2 < \ldots < a_n < a_{n+1} < \ldots$ *be any given sequence of real numbers. Let* $f(x)$ *be any continuous, non-vanishing and monotonic increasing* *function of x. Then there exists a sequence of polynomials $p_{2n}(x)$ for which at $x = a_0, a_1, a_2, \ldots, a_n, a_{n+1}, \ldots$, for all n,*

$$\left\| \frac{1}{f(x)} - \frac{1}{p_{2n}(x)} \right\| \leqslant \frac{2}{f(a_n)}. \tag{35}$$

Proof. Set

$$p_{2n}(x) = \sum_{k=0}^{n} l_k^2(x) f(x_k), \tag{36}$$

where

$$l_k(x) = \frac{(x-a_0)(x-a_1)\ldots(x-a_{k-1})(x-a_{k+1})\ldots(x-a_n)}{(x_k-a_0)(x_k-a_1)\ldots(x_k-a_{k-1})(x_k-a_{k+1})\ldots(x_k-a_n)}.$$

Therefore, for $x = a_0, a_1, a_2, \ldots, a_k, \ldots, a_n$,

$$f(x) = p_{2n}(x). \tag{37}$$

For $x = a_{n+1}, a_{n+2}, a_{n+3}, \ldots$ and so on, it is easy to check that

$$p_{2n}(x) > f(x). \tag{38}$$

Now we get from (37) and (38) at $x = \{a_j\}_{j=0}^{\infty}$ that

$$\left\| \frac{1}{f(x)} - \frac{1}{p_{2n}(x)} \right\| \leqslant \frac{1}{f(a_n)} + \frac{1}{f(a_n)} = \frac{2}{f(a_n)}.$$

Hence the result (35) is established.

THEOREM 7. *Let $p(x)$ be any polynomial of degree at most n having only non-negative coefficients and $q(x)$ be any polynomial of degree most n. Then we have, for all $n \geqslant 1$,*

$$\left\| e^{-x} - \frac{p(x)}{q(x)} \right\|_{L_\infty[0,1]} \geqslant [e + 2^{-1} e^2 4^n (n+1)!]^{-1}. \tag{39}$$

Proof. Let us assume that p/q deviates least from e^{-x} in the interval [0, 1]; then set

$$\left\| e^{-x} - \frac{p(x)}{q(x)} \right\| = \varepsilon. \tag{40}$$

We assume without loss of generality that $q(x) > 0$, on [0, 1]. From (40), it follows that, on [0, 1],

$$\left| e^{x} - \frac{q(x)}{p(x)} \right| \leqslant \frac{\varepsilon e^x |q|}{|p|} \leqslant \frac{\varepsilon e |q|}{|p|}. \tag{41}$$

It is well known that e^x can be approximated by its nth partial sum on [0, 1] with an error $(n!)^{-1}$. Hence, clearly,

$$\varepsilon \leqslant \frac{4}{n!}. \tag{42}$$

From (40),

$$\frac{|p|}{|q|} \geqslant \frac{1}{e^x} - \varepsilon \geqslant \frac{1}{e} - \varepsilon = \frac{1-\varepsilon e}{e}, \tag{43}$$

on the interval [0, 1]. From (41), (42) and (43),

$$\left\| e^x - \frac{q(x)}{p(x)} \right\| \leqslant \frac{\varepsilon e^2}{1-\varepsilon e}. \tag{44}$$

Set $p(x) = \sum_{k=0}^{n} a_k x^k$, $a_k \geqslant 0$ $(k \geqslant 0)$; then, from (44) on [0, 1],

$$|e^x p(x) - q(x)| \leqslant \frac{\varepsilon e^2}{1-\varepsilon e}\, p(x) \leqslant \frac{\varepsilon e^2 p(1)}{1-\varepsilon e}. \tag{45}$$

Now by applying Lemma 2 to $e^x p(x)$, we obtain on [0, 1]

$$p(1)\frac{\varepsilon e^2}{1-\varepsilon e} \geqslant \|e^x p(x) - q(x)\| \geqslant \frac{\operatorname{Min}|(D+1)^{n+1}(p(x))|}{(n+1)!\, 4^n\, 2^{-1}}, \tag{46}$$

where as usual $D = d/dx$.

It is not hard to check that

$$\operatorname{Min}|(D+1)^{n+1} p(x)| \geqslant \sum_{k=0}^{n} a_k = p(1). \tag{47}$$

From (46) and (47),

$$\frac{\varepsilon e^2}{1-\varepsilon e} \geqslant \frac{2}{4^n (n+1)!}. \tag{48}$$

From (48), it follows easily that

$$\varepsilon \geqslant \{e + 2^{-1} e^2 4^n (n+1)!\}^{-1}.$$

Hence the result (39) is established.

THEOREM 8. *Let $p(x)$ and $q(x)$ be any polynomials of degrees at most $n-1$ where $n \geqslant 2$. Then we have*

$$\left\| e^{-x} - \frac{p(x)}{q(x)} \right\|_{L_\infty(N)} \geqslant \frac{(e-1)^n e^{-4n} 2^{-7n}}{n(3+2\sqrt{2})^{n-1}}. \tag{49}$$

Proof. Let us denote for any given $p(x)$ and $q(x)$ at $x = 0, 1, 2, 3, \ldots, n, n+1, \ldots$,

$$\left\| e^{-x} - \frac{p}{q} \right\| = \varepsilon. \tag{50}$$

Normalize $q(x)$, such that, for $k = 0, 1, 2, \ldots, n, \ldots, 2n$,

$$\operatorname{Max}|q(k)| = 1. \tag{51}$$

From (51) and Lemma 3, we obtain,

$$\max_{[0,\,2n]} |q(x)| \leqslant n4^{2n}. \tag{52}$$

From (52), we get by applying Lemma 4 that

$$\max_{[0,\,4n]} |q(x)| \leqslant n4^{2n}(3+2\sqrt{2})^{n-1}. \tag{53}$$

From (50) and (53), we have, for all $x = 0, 1, 2, \ldots, 4n$,

$$\|e^{-x}q(x)-p(x)\| \leqslant \varepsilon n4^{2n}(3+3\sqrt{2})^{n-1}. \tag{54}$$

Set

$$R(x) = e^{-x}q(x)-p(x).$$

Then we get by using Lemma 5 that

$$\Delta^n R(x) = \Delta^n(e^{-x}q(x)-p(x)) = \Delta^n(e^{-x}q(x)) = e^{-x}\left(\frac{\Delta+1-e}{e}\right)^n q(x). \tag{55}$$

On the other hand it is well known that

$$\Delta^n R(x) = \sum_{l=0}^{n} (-1)^{n-l}\binom{n}{l} R(x+l). \tag{56}$$

From (54) and (56), we get for $x = 0, 1, 2, \ldots, n, \ldots, 3n$,

$$|\Delta^n R(x)| \leqslant \sum_{l=0}^{n}\binom{n}{l}|R(x+l)| \leqslant 2^{3n}\varepsilon n4^n(3+2\sqrt{2})^{n-1}. \tag{57}$$

Now we have from (55) and (57), for $x = 0, 1, 2, \ldots, n, n+1, \ldots, 3n$,

$$|(\Delta+1-e)^n q(x)| \leqslant e^x e^n 2^{3n}\varepsilon n4^n(3+2\sqrt{2})^{n-1} \leqslant \varepsilon e^{4n}2^{5n}n(3+2\sqrt{2})^{n-1}. \tag{58}$$

Set

$$S(x) = (\Delta+1-e)^n q(x).$$

Then for $x = 0, 1, 2, \ldots, n, n+1, \ldots, 2n$, we get by using Lemma 6, that

$$|q(x)| = |(\Delta+1-e)^{-n}S(x)| = \left|(1-e)^{-n}\left(1-\frac{\Delta}{e-1}\right)^{-n}S(x)\right|$$

$$\leqslant \left|(1-e)^{-n}\sum_{i=0}^{n}\binom{n+i}{i}\left(\frac{\Delta}{e-1}\right)^i S(x)\right|$$

$$\leqslant (e-1)^{-n}\left|\sum_{i=0}^{n}\binom{n+i}{i}\Delta^i S(x)\right|$$

$$\leqslant (e-1)^{-n}\varepsilon e^{4n}2^{5n}n(3+2\sqrt{2})^{n-1}\sum_{i=0}^{n}\binom{n+i}{i}$$

$$\leqslant (e-1)^{-n}\varepsilon e^{4n}2^{7n}n(3+2\sqrt{2})^{n-1}. \tag{59}$$

From (59), we get for $x = 0, 1, 2, 3, \ldots, 2n$,

$$\operatorname{Max}|q(x)| \leqslant \varepsilon e^{4n}2^{7n}n(3+2\sqrt{2})^{n-1}(e-1)^{-n}. \tag{60}$$

From (51) and (60) we get

$$\varepsilon \geqslant (e-1)^n e^{-4n} 2^{-7n} n^{-1} (3+2\sqrt{2})^{-n+1}.$$

Hence (49) is established.

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References

1. S. N. Bernstein, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle* (Gauthier–Villars, Paris, 1926).
2. W. J. Cody, G. Meinardus and R. S. Varga, " Chebyshev rational approximation to e^{-x} in $[0, +\infty)$ and applications to heat conduction problems ", *J. Approximation Theory*, 2 (1969), 50–65.
3. P. Erdős and A. R. Reddy, " Rational approximation on the positive real axis ", *Proc. London Math. Soc.*, 31 (1975), 439–456.
4. P. Erdős and A. R. Reddy, " Rational approximation ", *Advances in Mathematics*, 21 (1976), 78–109.
5. G. Freud, D. J. Newman and A. R. Reddy, " Chebyshev rational approximation to $e^{-|x|}$ on the whole real line ", *Quart. J. Math. (Oxford)*, 28 (1977), 117–122.
6. C. Jordan, *Calculus of Finite Differences* (Chelsea Publishing Co., New York, 1947).
7. D. J. Newman, " Rational approximation to e^{-x} ", *J. Approximation Theory*, 10 (1974), 301–303.
8. D. J. Newman, " Rational approximation to e^x with negative zeros and poles ", *J. Approximation Theory*, 20 (1977) to appear.
9. D. J. Newman and A. R. Reddy, " Rational approximation to e^{-x} on the positive real axis ", *Pacific J. Math.*, 64 (1976), 227–232.
10. A. R. Reddy, " A contribution to rational approximation ", *J. London Math. Soc.*, 14 (1976), 441–444.
11. A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, (Macmillan, New York, 1963).

Hungarian Academy of Sciences,
Budapest, Hungary

and

Yeshiva University,
New York, New York 10033

and

Institute for Advanced Study,
Princeton, New Jersey 08540.