ADVANCES IN MATHEMATICS 25, 91–93 (1977)

Addendum to "Rational Approximation"

PAUL ERDÖS AND A. R. REDDY

Bell Laboratories, Murray Hill, New Jersey 07974 and
The Institute for Advanced Study, Princeton, New Jersey 08540

Recently we proved the following [1, Theorem 37]:

THEOREM I. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function. Denote* $M(r) = \operatorname{Max}_{|z|=r} |f(z)|$, *and assume that*

$$1 < \limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 < \infty$$

and

$$\lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = \begin{matrix}\alpha \\ \beta\end{matrix} \qquad (5 < 2\beta < 2\alpha < \infty).$$

Then for every sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, *of degree at most n,*

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty]} \right\}^{n^{-1-\Lambda^{-1}}} \geqslant \frac{1}{e}. \tag{2}$$

Now it is natural to ask, what conclusion one expects by replacing $2\beta > 5$ and $\beta < \alpha$ in (1) by $2\beta > 0$ and $\beta \leqslant \alpha$.

In this connection by adopting an entirely different and new approach we prove here the following more general

THEOREM II. *Let* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function, satisfying the assumption that* $0 < \Lambda < \infty$ *and* $0 < \beta \leqslant \alpha < \infty$. *Then for every polynomial* $P_n(x)$ *and* $Q_n(x)$ *of degrees at most n, we have*

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty]} \right\}^{n^{-1-\Lambda^{-1}}} \geqslant G \tag{3}$$

where

$$G = \exp\left\{ -\left(\frac{2}{\beta}\right)^{1/\Lambda} \left[\alpha - 1 + \left(\frac{2\alpha}{\beta}\right)^{1/(\Lambda+1)} \right] \right\}.$$

We need the following lemma for our purpose.

LEMMA [2, p. 534–35]. *Let* $P(x)$ *be any algebraic polynomial of degree at most n.*

ISSN 0001-8708

If this polynomial is bounded by M on an interval of total length l contained in $[-1, 1]$, *then in* $[-1, 1]$.

$$|P(x)| \leqslant M\,|T_n(4l^{-1}-1)|,$$

where $2T_n(x) = (x + (x^2-1)^{1/2})^n + (x - (x^2-1)^{1/2})^n$.

Proof of Theorem II. Let for a $P(x)$ and $Q(x)$ of degree at most n,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty]} \leqslant \delta. \tag{4}$$

Normalize $Q(x)$, such that

$$\max_{[0,A]} |Q(x)| = 1 \qquad \text{where} \quad (\log A)^A = 2n\beta^{-1}. \tag{5}$$

Now by applying our lemma to Eq. (5) over the interval $[0, 2DA]$, we get

$$\max_{[0,2DA]} |Q(x)| \leqslant (8D)^n \qquad \text{where} \quad 2\alpha\beta^{-1} = \left(\frac{\log DA}{\log A}\right)^{A+1}. \tag{6}$$

Then there must be a point $x_1 \in [0, A]$, for which

$$|Q(x_1)| = 1. \tag{7}$$

From Eqs. (4) and (7), we get

$$|P(x_1)| \geqslant \frac{1}{f(x_1)} - \delta. \tag{8}$$

For any given $\epsilon > 0$, by choosing A to be large, we get from Eqs. (1) and (8)

$$|P(x_1)| \geqslant \exp(-(\log A)^{A+1}\beta(1+\epsilon)) - \delta. \tag{9}$$

From Eqs. (4) and (6), we get for $x \in [DA, 2DA]$

$$|P(x)| \leqslant |Q(x)| \left[\frac{1}{f(x)} + \delta\right] \leqslant (8D)^n \left[\exp(-(\log DA)^{A+1}\beta(1-\epsilon)) + \delta\right]. \tag{10}$$

Now we apply again our lemma to Eq. (10) over the interval $[0, 2DA]$, and obtain

$$\max_{[0,2DA]} |P(x)| \leqslant (48D)^n \left[\exp(-(\log DA)^{A+1}\beta(1-\epsilon)) + \delta\right]. \tag{11}$$

From Eqs. (9) and (11), we get

$$\exp(-(\log A)^{A+1}\alpha(1+\epsilon)) - \delta \leqslant (48D)^n \exp(-(\log DA)^{A+1}\beta(1-\epsilon)) + \delta(48D)^n,$$

i.e.,

$$\exp(-(\log A)^{A+1}\alpha(1+\epsilon))\left[1-\frac{(48D)^n\exp((\log A)^{A+1}\alpha(1+\epsilon))}{\exp((\log DA)^{A+1}\beta(1-\epsilon))}\right] \leqslant \delta(1+(48D)^n). \tag{12}$$

From Eq. (12) we obtain for all large n,

$$\delta \geqslant 4^{-1}(48D)^{-n}\exp(-(\log A)^{A+1}\alpha(1+\epsilon)). \tag{13}$$

Now by substituting the values of D and A we get the required result, i.e.,

$$\liminf_{n\to\infty}\left\{\left\|\frac{1}{f(x)}-\frac{P_n(x)}{Q_n(x)}\right\|_{L_{\infty[0,\infty]}}\right\}^{n^{-1-A^{-1}}} \geqslant G.$$

where

$$G=\exp\left\{-\left(\frac{2}{\beta}\right)^{1/A}\left[\alpha-1+\left(\frac{2\alpha}{\beta}\right)^{1/(A+1)}\right]\right\}.$$

References

1. P. Erdös and A. R. Reddy, Rational approximation, *Adv. Math.* 21 (1976), 78–109.
2. P. Erdös and P. Turan, On interpolation III, interpolatory theory of polynomials, *Ann. Math.* 41 (9940), 510–553.