Ellis' article the alternative conditions (2') and (3') (misprinted as (3)) that are stated in the last paragraph. Concerning A. D. Wadha's *An interesting subseries of the harmonic series* (this MONTHLY, 82 (1975) 931–3) Robert Baillie writes that he has computed the sums $S_i = \Sigma\, 1/n$ (n has no digit equal to i) for $i = 0, 1, \ldots, 9$ to twenty decimal places. Further information can be obtained by writing to him at Computer-Based Educ. Res. Lab., University of Illinois, Urbana, IL 61801.

The article *Probabilities in Proofreading* by G. Polya (this MONTHLY, 83 (1976) 42) has stimulated a lot of reader response. V. N. Murty has informed us that the estimate obtained by Polya for the number of unnoticed misprints was obtained by Edward Deming and Chandra Sekhar (J. Amer. Stat. Assoc., 44 (1949) 101–15) and that demographers use it to estimate vital events. Ralph Winter writes that if the number C of misprints noticed by both proofreaders is 0, then Polya's estimate is undefined. He then adds that if most probable numbers (rather than expected numbers) are used, the estimate becomes $(A - C)(B - C)/(C + 1)$. L. Glickman has informed us that the problem Polya solves appears as Exercise 23 on page 170 of W. Feller's book *An Introduction to Probability Theory and its Applications*, vol. I, 3rd edition (J. Wiley, New York, 1968).

Richard I. Loebl has written that Chandler Davis has informed him that an example similar to that in his article *The non-commutative triangle inequality fails* (this MONTHLY, 83 (1976) 259–60) appears in Davis' article *Notions generalizing convexity for functions defined as spaces of matrices* (Amer. Math. Soc. Proc. Symp. Pure Math. VII, 1963).

Research problems. Due to the unusual amount of activity concerning the research problem posed by I. Cahit in *Are all complete binary trees graceful* (this MONTHLY, 83 (1976) 35–7) and the fact that the next updating article for the Research Problems section is scheduled for December, 1977, we are reporting here that the question posed by Cahit has been answered in the affirmative by several people. It has been discovered also that the solution is immediately implicit in the article *Labelling of balanced trees* by R. G. Stanton and C. R. Zarnke (Congressus Numerantium VIII, Proc. 4th S. E. Conf. on Combinatorics, Graph Theory and Computing, Boca Raton, 1973, 479–95). More details will be given in the December 1977 article by Richard K. Guy.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4. (From July 1976 to June 1977: Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, 16 Mill Lane, Cambridge CB2 1SB, England.)

THE POWERS THAT BE

R. B. EGGLETON, P. ERDÖS AND J. L. SELFRIDGE

If n is a fixed positive integer, a any integer greater than unity and m the unique integer such that

$$a^m \leqq n < a^{m+1},$$

then we shall call a^m a **maximal power** for n, a^{m+1} a **minimal power** above n, and m the **exponent** of a for n. (The exponent is just the characteristic of the base a logarithm of n.) In particular, if the prime p

has exponent m for n we call p^m a **maximal prime-power** for n and p^{m+1} a **minimal prime-power** above n, and m will be referred to as a **P-exponent** for n.

It is easily seen that the least common multiple L_n of the positive integers up to n is equal to the product of the maximal prime-powers for n. More generally, let A_k be the largest possible least common multiple of any set of k positive integers not exceeding n. Then the $\pi(n)$ maximal prime-powers for n are a suitable set for showing that $A_{\pi(n)} = L_n$, where $\pi(n)$ denotes the number of primes up to n. As no positive integer up to n is divisible by two different maximal prime-powers for n it follows that the sequence $(A_k)_{1 \leq k \leq \pi(n)}$ is strictly monotonic increasing. We have investigated **[1, 2]** a number of properties of this sequence. The questions we shall raise here have their origins in these investigations.

Let any two integers greater than unity be called **exponentially equivalent** for n if they have the same exponent for n. This is an equivalence relation on the set of integers greater than unity and, by restriction, also on the set P of primes. Let $E(n)$ denote the set of exponents for n, that is, the exponents corresponding to the various exponential equivalence classes for n; similarly let $E_P(n)$ denote the set of P-exponents for n. If m is an exponent for n, let a_m and b_m denote respectively the smallest and largest integer in the corresponding exponential equivalence class; if m is also a P-exponent for n, let p_m and q_m denote respectively the smallest and largest prime in the corresponding exponential equivalence class. (Note that b_0 and q_0 do not exist.) We illustrate these ideas in the following tabulation.

For $n = 90$:

m	a_m	b_m	p_m	q_m
0	91	–	97	–
1	10	90	11	89
2	5	9	5	7
3	4	4	–	–
4	3	3	3	3
5	–	–	–	–
6	2	2	2	2

Thus 90 has 6 exponential equivalence classes, $E(90) = \{0, 1, 2, 3, 4, 6\}$. Since 4 is alone in its equivalence class, this class does not contain a prime and $E_P(90) = \{0, 1, 2, 4, 6\}$.

There are several questions which now arise. We shall mention some of these, and follow up with such comments or partial solutions as we can offer.

Q1. When is $a_m = b_m$?

Q2. If $m, m' \in E(n)$ and $m < m'$, for which integers n do we have $a_{m'} = b_{m'}$ whenever $a_m = b_m$?

Q3. When is $p_m = q_m$?

Q4. When does a_m exist while p_m fails to exist?

Q5. Are there infinitely many n for which $p_m = a_m$ or for which $q_m = b_m$, whenever $m \in E_P(n)$?

Q6. How many exponential equivalence classes does n have, that is, what is the cardinality of $E(n)$?

Q7. What is the cardinality of $E_P(n)$?

An asymptotic answer to Question 1 is not difficult. It is clear that $a_m = b_m$ must hold if $n^{1/m} - n^{1/(m+1)} < 1$ and cannot hold if $n^{1/m} - n^{1/(m+1)} \geqq 2$. The reader may wish to verify that if n is sufficiently large, $a_m = b_m$ holds if $m \geqq \log n/(\log\log n - 2\log\log\log n)$ and fails if $m < \log n/(\log\log n - (2-\varepsilon)\log\log\log n)$ for any given $\varepsilon > 0$. The phenomenon in Question 2 occurs for small n, but for $15625 = 5^6 \leqq n < 16384 = 4^7$ we have $a_5 = b_5 = 6$ and $a_6 = 4$, $b_6 = 5$. For sufficiently large n, the number of integers m satisfying $1 < n^{1/m} - n^{1/(m+1)} < 2$ is large enough to ensure that there is an exponential equivalence class containing two integers with larger exponent than that of some other class which contains only one integer. Thus the set of integers n described in Question 2 is finite; what is its largest member?

For Question 3, it is probable that the asymptotic answer is the same as for Question 1, though in this case the distribution of the primes complicates the matter, and the same difficulty arises in Question 4. For Question 5, we note for example that $p_m = a_m$ whenever $m \in E_P(100)$, and $q_m = b_m$ whenever $m \in E_P(127)$. However, one should be able to prove that only finitely many integers n have either of these properties, but finding the largest is probably very difficult. The calculations for Question 1 are adequate to obtain an asymptotic answer to Question 6, since $m \in E(n)$ certainly holds as long as $n^{1/m} - n^{1/(m+1)} \geqq 1$. If m_0 is the largest exponent satisfying this inequality, the number of exponents not counted in this way is at most n^{1/m_0}, which is of smaller order than m_0. Hence $|E(n)| \sim \log n/\log\log n$. We conjecture the same asymptotic answer for Question 7, though we cannot prove this. It has long been known (cf. [6]) that for all sufficiently large x there is a prime between x and $x + x^\theta$, where $\theta < 1$ is a suitable constant; this implies that $|E_P(n)| > c \log n/\log\log n$ for some positive constant $c < 1$. We can show, however, that $E_P(n) = E(n)$ holds for only finitely many integers n. Indeed, $E_P(4095) = E(4095) = \{0, 1, 2, 3, 4, 5, 7, 11\}$ is the largest such example, because 4 is alone in its exponential equivalence class for $4096 = 4^6 \leqq n < 15625 = 5^6$ and for $16384 = 4^7 \leqq n$, and 6 is alone in its class for $15625 = 5^6 \leqq n < 16807 = 7^5$.

We shall now consider the numbers $\alpha_m = p_m^m$, $\beta_m = q_m^m$, $\gamma_m = p_m^{m+1}$ and $\delta_m = q_m^{m+1}$. Denote the sequences of these numbers by $\alpha(n) = \{\alpha_m\}$, $\beta(n) = \{\beta_m\}$, $\gamma(n) = \{\gamma_m\}$ and $\delta(n) = \{\delta_m\}$, where the subscripts run through $E_P(n)$ by increasing magnitude for α and γ, and likewise through $E_P(n)\backslash\{0\}$ for β and δ. (The corresponding sequences of powers of a_m and b_m are also interesting, but do not necessarily have all terms distinct, e.g., $a_3^3 = a_6^6 = 64 = b_3^3 = b_6^6$ when $n = 90$.) We now illustrate.

For $n = 90$:

m	α_m	β_m	γ_m	δ_m
0	1	—	97	—
1	11	89	121	7921
2	25	49	125	343
4	81	81	243	243
6	64	64	128	128

Since p_m is approximately $n^{1/(m+1)}$, the sequence $\alpha(n)$ increases monotonically, at least initially. Similarly q_m is approximately $n^{1/m}$, so $\delta(n)$ decreases monotonically, at least initially. There are no such compelling reasons for $\beta(n)$ or $\gamma(n)$ to be monotonic, even initially, though if it happens that $n^{1/m} - q_m$ does not decrease too rapidly as m increases then $\beta(n)$ will decrease initially; likewise if $p_m - n^{1/(m+1)}$ does not decrease too rapidly as m increases then $\gamma(n)$ will increase initially.

We are now in a position to raise several more questions, also to be followed by comments and answers as far as we know them.

Q8α. Are there infinitely many integers n for which $\alpha(n)$ increases monotonically throughout?

Q8β, Q8γ are the corresponding questions about $\beta(n)$ and $\gamma(n)$.

Q9δ. Are there infinitely many integers n for which $\delta(n)$ decreases monotonically throughout?

Q9β, Q9γ are the corresponding questions about $\beta(n)$ and $\gamma(n)$.

Q10. Are there infinitely many integers n for which $\alpha(n)$ increases throughout and simultaneously $\delta(n)$ decreases throughout?

Q11. To what point can we be sure that $\alpha(n)$ will increase monotonically?

Q12. To what point can we be sure that $\delta(n)$ will decrease monotonically?

For Question 8α it is easy to find integers n for which $\alpha(n)$ increases throughout. This is the case for $1 \leqq n < 9$, $16 \leqq n < 27$, $32 \leqq n < 81$, $256 \leqq n < 625$, $1024 \leqq n < 2187$, $8192 \leqq n < 15625$ and $16777216 = 2^{24} \leqq n < 19487171 = 11^7$.

For $n = 2^{24}$:

m	p_m	α_m	m	p_m	α_m
0	16777259	1	6	11	1771561
1	4099	4099	8	7	5764801
2	257	66049	10	5	9765625
3	67	300763	15	3	14348907
4	29	707281	24	2	16777216
5	17	1419857			

However, the question is whether there are infinitely many integers n for which $\alpha(n)$ increases throughout. We conjecture a negative answer, and possibly one can prove this. By computer we verified that outside the seven intervals listed, there are no cases with $n \leqq 10^{250000}$ in which $\alpha(n)$ increases throughout. (The corresponding question can be asked for such sequences as $\{a_m^m\}$, where m runs through $E(n)$. Perhaps $n = 15624$ is the last case in which this sequence increases throughout.) It would be interesting to decide if there are infinitely many n for which those terms of $\alpha(n)$ corresponding to exponents $m < (\log n)^{\frac{1}{2}}$ are increasing. For Question 8γ we note that $\gamma(n)$ can increase throughout only when $\alpha(n)$ does so. Thus $1 \leqq n < 3$ and $16 \leqq n < 23$ are the only intervals with $n \leqq 10^{250000}$ in which $\gamma(n)$ increases throughout; we conjecture that there are no later examples.

For Question 8β we have the examples $2 \leqq n < 5$ and $n = 8$. Moreover, in this case we can prove that there are only finitely many integers n for which $\beta(n)$ increases monotonically. A classical result of Ingham **[6]** states that for any $\varepsilon > 0$ and all sufficiently large x there is a prime between x and $x + x^{(5/8)+\varepsilon}$. Hence we deduce that between any two sufficiently large cubes there exists a prime. Ingham's result also ensures that 3 and 6 are P-exponents for any sufficiently large n. Thus if n is sufficiently large $\beta(n)$ contains the cubes β_3 and β_6, and between them there is a prime, so $\min\{\beta_3, \beta_6\} < \beta_1$, showing that $\beta(n)$ does not increase monotonically. Incidentally, it has never been proved that between every two consecutive cubes (or every two consecutive tetrahedral numbers) there is a prime: it would be nice to have such results available.

The parts of Question 9 have a similar status to the corresponding parts of Question 8. It may be easier to resolve Question 10 negatively than to resolve either Question 8α or Question 9δ alone. The interval $1331 \leqq n < 2048$ is the last interval with $n \leqq 10^{250000}$ for which $\alpha(n)$ and $\delta(n)$ are both monotonic throughout.

For Question 11 we note that a result of Huxley **[5]** improves Ingham's result used above by replacing the exponent $\frac{5}{8}$ by $\frac{7}{12}$. Thus for any given $\varepsilon > 0$, if $n^{1/m}$ is sufficiently large, then $p_{m-1} < n^{1/m} + n^{(7+\varepsilon)/12m}$. Also $n^{1/(m+1)} < p_m$, so a short calculation shows that if n is sufficiently large, $\alpha_{m-1} < \alpha_m$ holds so long as $m \leqq (\frac{5}{24} - \varepsilon)\log n/\log\log n$. How good is this estimate? (Probably the result is still true if the coefficient is replaced by $\frac{1}{4} - \varepsilon$, and perhaps even by $\frac{1}{2} - \varepsilon$.) Are there infinitely many n for which this monotonicity fails before the P-exponent reaches $\log n/\log\log n$? This discussion essentially carries over for Question 12 as well.

We shall now introduce two functions (from P to the integers) related to the monotonicity problems for α and δ. In the present context, let p_i denote the ith prime and let m_i denote the exponent of p_i for n. For a given k, suppose n is the smallest integer for which $2^{m_1} > 3^{m_2} > \cdots > p_k^{m_k}$. (Recall that $p_i^{m_i+1} > n$ for each $i \leqq k$.) This sequence is conveniently recorded by the function $f(p_k) = m_1$, since $n = 2^{m_1}$. Similarly suppose n' is the smallest integer for which

$$2^{m'_1+1} < 3^{m'_2+1} < \cdots < p_k^{m'_k+1},$$

where m'_i is the exponent of p_i relative to n'. The function $g(p_k) = m'_1$ records this sequence.

As part of our computations related to the monotonic behavior of α, we obtained the following data:

p_k	$f(p_k)$	p_k	$f(p_k)$	p_k	$f(p_k)$
2	0	11	10	23	5627
3	1	13	40	29	14501
5	2	17	40	31	330861
7	5	19	106	37	658110

Here are three further questions.

Q13. What are the corresponding values of g, and how do both sequences of values continue?

Q14. What is the asymptotic behavior of f and g?

Q15. Does f increase strictly for $p_k \geqq 17$? (Probably not.)

In closing, we note the following result related to the problems we have raised. Let p, q be primes such that $1 < p < q \leqq n$, and let p^r and q^s be the highest powers of p and q dividing $n!$; then $p^r > q^s$ if $r > s$. This result is the content of a problem originally proposed by Erdös [3], with published solution due to Harrington [4].

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CLASSROOM NOTES

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MOTIVATING EXISTENCE—UNIQUENESS THEORY FOR APPLICATIONS ORIENTED STUDENTS

ARTHUR DAVID SNIDER

When one is teaching a course in partial differential equations to an applications-oriented audience, a dilemma arises about halfway through the semester: namely, the available techniques for constructing solutions (method of characteristics, transforms, separation of variables, Green's