SOME EXTREMAL PROBLEMS IN GEOMETRY IV

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1. Introduction

We will be discussing some old and new problems and results in Combinatorial Geometry. We begin with some old problems.

Some time ago the senior author conjectured that a convex polygon in the plane always has a vertex which does not have three vertices equidistant from it. In [3] it is mentioned that Danzer disproved this. It is stated there that Danzer also considered the following general problem and settled it in the affirmative:

Given $k \geq 3$, does there exist a convex polygon of n_k vertices so that every vertex has k other vertices equidistant from it? However, Danzer now says he only has the result for $k=3$, hence the problem is still open for $k \geq 4$.

Some time ago the senior author posed the following problem: Given n distinct points in the plane, what is the maximum number $f(n)$ of lines that can have exactly k points on them if no $k+1$ points lie on a line? Karteszi showed that $f(n) \geq c_k n \log n$. B. Grünbaum has recently improved this to $f_k(n) \geq c_k n^{1+1/(k-2)}$. Here, as always, c's denote positive constants. S. Burr, B. Grünbaum and

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N.J.A. Sloane in [1] obtain a result of the form $f_3(n) \geq \frac{n^2}{6} - cn$, and clearly $f_3(n) \leq \frac{n^2}{6}$.

For k=4, $f_4(n) \leq \frac{n^2}{12}$ trivially and we conjecture $f_4(n) = o(n^2)$, but we can't even show $f_4(n) \leq (1-\varepsilon)\frac{n^2}{12}$, for any positive ε.

2. Some problems involving points in General Position

Let $f_2^c(n)$ denote the maximum number of pairwise congruent triangles that can occur among all the triples formed from n distinct points in the plane. In a previous paper [6] we showed that $f_2^c(n) = o(n^{3/2})$. We now add to this the lower bound $f_2^c(n) \geq cn \log n$. We can actually prove this lower bound under the restriction that the points are in general position--no three on a line.

We have

Theorem 1 Let f(n) denote the maximum number of pairwise congruent triangles that can occur in the plane among n points if no three points lie on a line. Then

$$f(n) \geq \frac{n \log \frac{n}{3}}{9 \log 3}.$$

We postpone the proof until later.

Many of the problems considered in 6 and 2 can be looked at with the new restriction that we have no three points on a line, and we discuss some of these in this section.

Theorem 2 Let f(n) denote the maximum number of times that unit distance can occur among n points in the plane if no three points lie on a line. Then

$$f(n) \geq \frac{2n \log \frac{n}{6}}{3 \log 3}.$$

<u>Remark</u> Without the restriction of no three points on a line, it is shown in [2] that

$$f(n) \geq n^{1+ c/\log \log n}.$$

<u>Proof</u> Clearly $f(2) = 1$. We start by showing

(1) $$f(2n) \geq 2f(n) + n.$$

We represent the points by complex numbers. Let $S = \{z_1,\ldots,z_n\}$ be a set of n points with unit distance occurring $f(n)$ times. For any a of unit modulus the number of unit distances occurring in $S \cup (S + a) = \{z_1,\ldots,z_n, z_1+a,\ldots,z_n+a\}$ is at least $2f(n)+n$, since there are $f(n)$ occurring in each of S and $S+a$, and $|z_i+a-z_i| = |a| = 1$ for all i. We shall show that a can be chosen so that no three points will lie on a line.

Let z_i and z_j be fixed, and let ℓ be the line through them. For each point z_k, the locus of points z_k+a such that a has modulus one is a circle intersecting ℓ in at most two points. Hence for each of the $\binom{n}{2}$ pairs of points in S there are at most $n\binom{n}{2}$ choices for a that must be avoided to prevent a point of $S+a$ being collinear with two points of S. Similarly there are only a finite number of choices for a to be avoided to prevent a point of S from being collinear with two points of $S+a$. Since there are infinitely many choices for a there will be one avoiding collinearity, and (1) follows. We next show

(2) $\quad f(3n) \geq 3f(n) + 3n$.

Let S be a set of complex numbers $\{z_1,\ldots,z_n\}$ with $f(n)$ unit distances. If a is a complex number of unit length, and $\omega = \frac{1}{2} + \frac{i\sqrt{3}}{2}$(a primitive cube root of unity), then 0, a and ωa form an equilateral triangle of side one. We seek an a such that $S \cup (S+a) \cup (S+\omega a)$ has no three collinear points, from which (2) clearly follows.

The collinearity of two points in one set with one point of another set can be avoided by excluding only finitely many choices for a, as in the proof of (1).

Suppose three points from different sets are collinear: z_i, z_j+a and $z_k+\omega a$. Then $z_i = \lambda(z_j+a) + (1-\lambda)(z_k+\omega a)$, where λ is real. Solving for a, we get $a = \frac{c+\lambda d}{\lambda+(1-\lambda)\omega}$, where $c = z_i-z_k$ and $d = z_k-z_j$. Since $|a| = 1$, we have
$(c+\lambda d)(\bar{c}+\lambda\bar{d}) = \{\lambda+(1-\lambda)\omega\}\{\lambda+(1-\lambda)\bar{\omega}\}$. Hence λ satisfies a quadratic equation $A\lambda^2 + B\lambda + C = 0$, where
$A = d\bar{d} - 1 - \omega\bar{\omega} + \bar{\omega} + \omega = |z_k - z_j|^2 + 1 \neq 0$.
There are at most two such λ, and once λ is fixed, a is determined. As we range over all triples of points, we see that only finitely many choices of a have to be avoided, and (2) is proved.

It follows immediately from (2) that

$$f(3^k) \geq \frac{3^k \log 3^k}{\log 3} .$$

Let $3^m \leq n < 3^{m+1}$, $m \geq 1$. If $3^m \leq n < \frac{4}{3} 3^m$, then $f(n) \geq f(3^m) \geq \frac{3^m \log 3^m}{\log 3} \geq \frac{3n \log \frac{3}{4} n}{4 \log 3}$. If $\frac{4}{3} 3^m \leq n < 2.3^m$ then, by (1),

$$f(n) \geq f(4.3^{m-1}) \geq 4 g(3^{m-1}) \geq \frac{4.3^{m-1} \log 3^{m-1}}{\log 3} \geq \frac{2n \log(\frac{n}{6})}{3 \log 3} .$$

Finally, if $2.3^m \leq n < 3^{m+1}$, then, by (1), $f(n) \geq f(2.3^m) \geq 2f(3^m) \geq \frac{2.3^m \log 3^m}{\log 3} > \frac{2n \log(\frac{n}{3})}{3 \log 3}$, and the theorem is proved. At the moment it is not clear if the restriction of no three points on a line really decreases the number of unit distances. We may also ask how many unit distances you get if there are no four points on a circle.

Proof of Theorem 1

We now sketch the proof of theorem 1. Let $g(n)$ be as in theorem 1, except for equilateral triangles of side one. Clearly $g(3) = 1$. We shall show that

(3) $\qquad g(3n) \geq 3g(n) + n$

and then theorem 1 will follow. Let S be a set of n points $\{z_1, \ldots, z_n\}$ in the complex plane with $g(n)$ equilateral triangles. If a is a complex number of unit modulus and $\omega = \frac{1}{2} + \frac{i\sqrt{3}}{2}$, then the set $S \cup (S+a) \cup (S+\omega a)$ contains the n equilateral triangles z_i, z_i+a, $z_i+\omega a$ of side one, and $3g(n)$ others. By the same argument as in the proof of theorem 2 we can avoid three points on a line, and (3) follows. Hence $g(3^k) \geq \frac{n \log n}{3 \log 3}$ and it follows that $g(n) \geq \frac{n \log(\frac{n}{3})}{9 \log 3}$.

How many isosceles triangles can occur among n points in E_k no three on a line? Let the maximum number be $g_k^i(n)$.

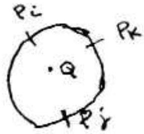
Theorem 3

$(n-2)(n-4) \leq g_2^i(n) \leq n(n-1)$. Further, if n is even and not of the form $3k+1$, then $g_2^i(n) \geq (n-1)(n-2)$.

Proof

We first prove the lower bound. Let n be even and let $P_1, \ldots, P_{n-1}$ be a regular $(n-1)$gon inscribed in a unit circle with center Q. No three of the points $Q, P_1, \ldots, P_{n-1}$ are collinear. If P_i and P_j are distinct points, then the triangle QP_iP_j is isosceles, and $\binom{n-1}{2}$ triangles are obtained in this way.

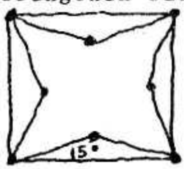
If P_i and P_j are distinct points, then since $n-1$ is odd, there is a point P_k equidistant from P_i and P_j so that $P_iP_jP_k$ is isosceles. Equilateral triangles get counted three times in this way. If $n = 3k+1$, then there are k equilateral triangles, and the number of distinct isosceles triangles $P_iP_jP_k$ is $\binom{n-1}{2} - \frac{2(n-1)}{3}$, and the total number of isosceles triangles is therefore $2\binom{n-1}{2} - \frac{2(n-1)}{3} = \frac{3(n-1)(n-2)-2(n-1)}{3} = \frac{n-1}{3}(3n-6-2) = \frac{1}{3}(n-1)(3n-8)$. If n is not of the form $3k+1$, then $g_2^i(n) \geq 2\binom{n-1}{2} = (n-1)(n-2)$. If n is odd, then $g_2^i(n) \geq g_2^i(n-1) \geq \frac{1}{3}(n-2)(3n-1) \geq (n-2)(n-4)$. Thus we have obtained the lower bounds claimed.



To obtain the upper bound, let $x_1, \ldots, x_n$ be n points in the plane. For fixed x_i and x_j the points x_k forming isosceles triangles with x_i and x_j lie on the perpendicular bisector. Since at most two points are on a line, this gives $g_2^i(n) \le 2\binom{n}{2} = n(n-1)$, and the theorem is proved.

We can also consider the same problem with the additional restriction that no four points lie on a circle. However, we can't even show that the number of triangles would be $o(n^2)$.

Theorem 3 is related to a problem of L. M. Kelly: Is it possible to find n points such that every perpendicular bisector of two points has two points on it. It is possible for eight points, but Kelly conjectures that it is not possible for any other number. A proof of this conjecture with quantitative results would strengthen our upper bound on $g_2^i(n)$.



Kelly's figure actually contains fewer than $g_2^i(8)$ isosceles triangles, since it contains eight equilateral triangles and therefore $56-16 = 40$ isosceles triangles compared with 42 for the example of our lower bound.

3. Triangles of different areas

In [5] we discussed the following problem: Given n points in the plane, what is the maximum number $f(n)$ of triangles of the same area that can occur, and we showed

$$c_1 n \log \log n \leq f(n) \leq c_2 n^{5/2}.$$

We now discuss a related problem.

Theorem 4 Let $g(n)$ be the minimum number of triangles of different areas which must occur among n points in the plane, not all on a line.

Then $c_1 n^{3/4} \leq g(n) \leq c_2 n$.

Proof The upper bound comes from considering the points (i,j) for $1 \leq i,j \leq \sqrt{n}$ and observing that every area is half an integer and bounded by $\frac{n}{2}$.

We now prove the lower bound. It follows from theorem 4.1 [8] of L. M. Kelly and Leo Moser that if you have n points, with no $n-\sqrt{n}$ on a line, and you form the lines through pairs of points, then there are at least $cn^{3/2}$ different lines.

Let ℓ be a line with points $P_o,\ldots,P_m$ in order, and let Q be a point not on ℓ. Then the areas of triangles QP_oP_i, $i = 1,2,\ldots,m$ form an increasing sequence. We may therefore certainly assume that less than $n-\sqrt{n}$ points lie on any line, so that we have at least $cn^{3/2}$ different lines.

Suppose that no direction has more than $\epsilon n^{3/4}$ parallel lines. Then there are lines in $\frac{cn^{3/4}}{\epsilon}$ different directions.

Let ℓ be a line determined by A and B and consider the lines parallel to it. Assume that p points are covered by these lines. Three uncovered points P_1, P_2 and P_3 cannot give rise to triangles P_iAB of equal area, since this would imply that two of the P_i were on a line parallel to ℓ. Hence the uncovered points give at least $\frac{n-p}{2}$ different areas.

For the covered points, let h_i be the number of points on the i^{th} line parallel to ℓ, $1 \le i \le r$. Then the number of pairs determining this direction is $\sum\binom{k_i}{2}$. Since $\sum k_i = p$, we have $\sum\binom{k_i}{2} \geq \frac{p^2}{2r} > \frac{\frac{1}{2}(n-2cn^{3/4})^2}{\epsilon n^{3/4}} > \frac{n^{5/4}}{3\epsilon}$ for ϵ small enough. The number of directions is at least $\frac{cn^{3/4}}{\epsilon}$. Hence the total number of pairs is at least $\frac{1}{3\epsilon}n^{5/4}\frac{cn^{3/4}}{\epsilon} > \binom{n}{2}$ for ϵ sufficiently small, which is absurd. Hence some direction has more than $\epsilon n^{3/4}$ parallel lines. Choose one line ℓ through A and B and one point from each of the other lines and you get at least $\frac{\epsilon n^{3/4}-1}{2}$ different areas for some $\epsilon > 0$. Hence the theorem follows.

Remark If the following old conjecture is true, then by a proof similar to the above $g(n) \geq c_3 n$ and the order of magnitude of $g(n)$ is known:

Given n points in the plane with no $(1-\epsilon)n$ on a line, where $\epsilon > 0$, there exist positive c and N such that there are more than cn^2 lines if $n > N$.

4. Covering lattice points by circles and lines

Theorem 5 Let $f(n)$ be the minimum number k such that there exist k points in the n by n lattice L_n so that the lines through any two of them cover all of the points of L_n. Then

$$f(n) \geq cn^{2/3}.$$

Proof Let k points be given which satisfy the above hypothesis, let x_o be one of these points, and consider the points of L_n covered by the lines through x_o and the other points $x_1, \ldots, x_{k-1}$. By moving the origin, we may suppose that $x_o = (0,0)$ and $x_i = (x_i, y_i)$, $1 \leq i < k$. Let ℓ_i be the line through x_o and x_i. We shall show that the number of points covered by all the lines ℓ_i is at most $cn\sqrt{k}$.

We may suppose that the (x_i, y_i) are distinct and primitive. The distance between consecutive points on ℓ_i is $\sqrt{x_i^2 + y_i^2}$, the diameter of L_n is less than $\sqrt{2}n$, and so ℓ_i covers at most $1 + \frac{\sqrt{2}n}{\sqrt{x_i^2+y_i^2}}$ points of L_n. The number of points excluding x_o covered by all of the ℓ_i cannot excede

$$\sum_{i=1}^{k-1} \frac{\sqrt{2}n}{\sqrt{x_i^2+y_i^2}} .$$

We shall show that this sum is bounded above by $cn\sqrt{k}$ even without the restriction to primitive points.

The maximum is clearly obtained when the points all lie within a circle of radius $r = \sqrt{k} + O(1)$. Hence

$$\sqrt{2}n \sum_{i=1}^{k-1} \frac{1}{\sqrt{x_i^2+y_i^2}} \leq 2\sqrt{2}n \sum_{\substack{0 \leq u_i \leq v_i \leq r \\ (u_i,v_i) \neq (0,0)}} \frac{1}{\sqrt{u_i^2+v_i^2}}$$

$$\leq 2\sqrt{2}n \sum_{1 \leq v_i \leq r} \frac{1}{v_i} + 2\sqrt{2}n \sum_{1 \leq u_i \leq v_i \leq r} \frac{1}{\sqrt{u_i^2+v_i^2}} \leq 4\sqrt{2}nr \leq cn\sqrt{k}.$$

Hence, as claimed, the total number of points on lines through the fixed point x_o is at most $cn\sqrt{k}$, and so all of the lines cover at most $cnk^{3/2}$ points. Since the n^2 points of L_n are covered, we must have $c'k^{3/2} \geq n$, or $k \geq c\ n^{2/3}$, as stated in the theorem.

We are unable to prove $f(n) = o(n)$ but we conjecture this.

Theorem 6 Let $f(n)$ be the minimum number of circles needed to cover all of the points of the n by n lattice. (The points are covered by a circle if they lie on its perimeter.) Then

$$f(n) \leq \frac{8n^2 \log n}{n^{c/\log \log n}}$$

Proof As usual let the n by n lattice L_n be the set of points (i,j) such that $1 \leq i,j \leq n$. It follows from theorems in number theory--see [2] theorem 2 for details and references--that there is an absolute positive constant c so that some circle contained entirely in L_n, centered on a lattice point, has at least $t = n^{c/\log \log n}$ lattice points on it.

Let our first covering circle be that circle and let r denote its radius. Any circle of radius r

centered on a point of L_n will contain at least $\frac{t}{4}$ points of L_n. We shall choose circles succesively as follows: Suppose that $k \geq 1$ and that the first k circles have been chosen. Let N_k be the number of points of L_n not covered by them. We shall show that the next circle may be chosen so that

(4) $$N_{k+1} \leq N_k(1 - \frac{t}{4n^2}).$$

To see this, consider the circles c_i of radius r centered on the N_k uncovered points. Each c_i contains at least $\frac{t}{4}$ points of L_n. Hence some point P of L_n must belong to at least $\frac{tN_k}{4n^2}$ circles c_i. For our next covering circle we choose the circle with center P and radius r. This circle covers at least $\frac{tN_k}{4n^2}$ new points, and so

$$N_{k+1} \leq N_k - \frac{tN_k}{4n^2} = N_k(1 - \frac{t}{4n^2}).$$

Hence (4) is proved, and by induction $N_k \leq n^2(1 - \frac{t}{4n^2})^k$. Hence $N_k < 1$ when $k > \frac{-2 \log n}{\log(1-\frac{t}{4n^2})} < \frac{8n^2 \log n}{t}$, and so

$$f(n) \leq \frac{8n^2 \log n}{n^{c/\log \log n}}.$$

5. Congruent subsets of a set

Theorem 7 Let $f(n)$ be the maximum number of congruent subsets of a set of n points that can occur in the plane. Then

$$f(n) = o(n^{3/2}).$$

Proof Let $\{x_1, \ldots, x_n\}$ be a set of n points in the plane, and let $\{A_1, \ldots, A_m\}$ be a fixed subset. By a theorem of E. Pannwitz [9] the maximum distance d

occurring among $A_1,\dots,A_m$ can occur at most m times. By a theorem of S. Józsa and E. Szemerédi [7] the number of pairs of points x_ix_j at distance d is $o(n^{3/2})$. For each line segment $\overline{A_iA_j}$ and $\overline{x_rx_s}$ there are at most four ways of placing $\{A_1,\dots,A_m\}$ onto $\{x_1,\dots,x_m\}$ so that these line segments coincide. Hence the number of ways of placing $\{A_1,\dots,A_m\}$ congruently in $\{x_1,\dots,x_n\}$ is $o(4mn^{3/2}) = o(n^{5/2})$.

<u>Corollary</u> Given n points in the plane there are more than $\frac{\psi(n)2^n}{n^{5/2}}$ incongruent subsets, where $\psi(n)$ tends to infinity.

<u>Theorem 8</u> Let $f(n)$ be the maximum number of congruent subsets of a set of n points that can occur in E_k. Then

$$f(n) \leq cn^{2k+2}.$$

<u>Proof</u> Let $\{x_1,\dots,x_n\}$ be a set of n points in E_k, and let $A = \{A_1,\dots,A_m\}$ be a fixed subset. Let ℓ be the dimension of A and assume without loss of generality that $A_1,\dots,A_{\ell+1}$ span the subspace generated by A. For each of the less than $\binom{m}{\ell+1}$ subsets of $\{B_1,\dots,B_{\ell+1}\}$ congruent to $\{A_1,\dots,A_{\ell+1}\}$ and for each of the less than $\binom{n}{\ell+1}$ subsets $\{y_1,\dots,y_{\ell+1}\}$ of $\{x_1,\dots,x_n\}$ congruent to $\{A_1,\dots,A_{\ell+1}\}$ there are at most $(\ell+1)!$ ways of making the simplices $B_1B_2\cdots B_{\ell+1}$ and $y_1\cdots y_{\ell+1}$ coincide. Hence $f(n) \leq \binom{m}{\ell+1}\binom{n}{\ell+1}(\ell+1)! \leq cn^{2k+2}$. We suspect that in fact $f(n) \leq cn^{k/2}$ for even k.

<u>Corollary</u> Given n points in E_k there are at least

$\frac{c2^n}{n^{2k+2}}$ incongruent subsets.

In Hilbert space it is possible to have a countable set of points such that only countably many incongruent subsets occur. Simply take the sequence $\{E_i\}$, where E_i is the i^{th} coordinate vector. Subsets of the same finite cardinality are congruent, and the countable subsets are all congruent.

However if $m > \aleph_o$ and S is a subset of Hilbert space of power m, then there are always 2^m incongruent subsets of S. To see this, observe that there are at most c sets congruent to any subset S_1 of Hilbert space. Thus if $2^m > c$ our statement immediately follows. If $2^m = c$ then S contains a convergent subsequence $\{X_n\}$ and it is easy to see that the sequence contains c incongruent subsets.

For some further problems in Hilbert space see [4]p.541, where the following problem is given: In Hilbert space, does every set of c points have a subset of c points without any right triangles? In E^k the answer is affirmative.

We are indebted to Ernst Straus for stimulating discussion of some of the problems of this paper.

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