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Rational Approximation

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1. INTRODUCTION

Recently several authors have investigated the question of approximating certain functions by reciprocals of polynomials under the uniform norm on the positive real axis. Perhaps these results have some applications in industry and elsewhere (cf. [1, 36]). Our present motivation is to give a detailed list of all the known results with simplified proofs in some cases and many new results and finally many open problems. Gonchar's article [14] may be of great help to people interested in finite intervals.

Long ago Chebyshev has shown "for any function $f(x)$ continuous on the whole real axis and having the finite limit $\lim_{x\to\pm\infty} f(x) = C$, there exists a sequence of continuous rational functions of the form $R_n(x) = P_n(x)/Q_n(x)$ (where $P_n(x)$ and $Q_n(x)$ are polynomials of degree n) such that $\lim \| f(x) - R_n(x)\|_{L_\infty(-\infty,\infty)} \to 0$." But Chebyshev never discussed the rate of convergence of the error to zero. This kind of result has been obtained by Freud and Szabodas [13] in 1968.

In 1955, Hastings has shown [15] by computation functions such as

$$e^{-x}, \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

can be approximated under the uniform norm very closely by reciprocals of polynomials on $[0, \infty)$. In 1969, Cody, Meinardus and Varga have shown [4] that e^{-x} can be uniformly approximated on $[0, \infty)$ by reciprocals of polynomials of degree n with an error $(2.298)^{-n}$. In 1973, Schönhage has shown [33] that e^{-x} can be approximated uniformly by reciprocals of polynomials of degree n on $[0, \infty)$ with an error 3^{-n} but not much better. In 1974, D. J. Newman has proved [18] that e^{-x} cannot be approximated on $[0, \infty)$ under the uniform norm by general rational functions of degree n with an error better than $(1280)^{-n-1}$. Recently Freud, Newman, and Reddy [12] have shown that $e^{-|x|}$ can be approximated by reciprocals of polynomials of degree n on $(-\infty, +\infty)$ with an error $C_1(\log n)/n$ but not like a C_2/n. Further, Freud, Newman, and Reddy have shown that $e^{-|x|}$ can be approximated on $(-\infty, +\infty)$ by general rational functions of degree n with an error like a $C_3e^{-C_4\sqrt{n}}$ but not like a $C_5e^{-C_6\sqrt{n}}$. In 1970, Meinardus and Varga [16] have extended the results of [4] to reciprocals of certain entire functions of perfectly regular growth. In 1974, Reddy [20] has extended the results of [16]. In 1972, Meinardus, Reddy, Taylor and Varga [17] have obtained some direct and converse results. Subsequently in a series of papers by developing certain new techniques, Erdös, Newman, and Reddy [5], Erdös and Reddy [6–11], Newman and Reddy [37–39], and Reddy [19–28] have obtained many results.

We present results in this article not according to the chronological order but according to certain pattern, perhaps convenient to the readers to follow. At the end we mention a few results for certain unbounded domains of the complex plane.

2. Definitions and Notations

Let $f(Z)$ be a nonconstant entire function. As usual write $M_f(r) = M(r) = \max_{|z|=r} |f(z)|$; then the order ρ and the lower order β of $f(Z)$ are defined thus

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} = \begin{matrix}\rho\\ \beta\end{matrix} \qquad (0 \leqslant \beta \leqslant \rho \leqslant \infty). \tag{2.1}$$

If $0 < \rho < \infty$, then the type τ and the lower type ω of f are:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^\rho} = \begin{matrix}\tau\\ \omega\end{matrix} \qquad (0 \leqslant \omega \leqslant \tau \leqslant \infty). \tag{2.2}$$

If $\rho = 0$, then we define the logarithmic order $\rho_l = \Lambda + 1$ and the lower logarithmic order β_l of f as:

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r)}{\log\log r} = \begin{matrix}\rho_l\\ \beta_l\end{matrix} = \Lambda + 1 \qquad (1 \leqslant \beta_l \leqslant \rho_l \leqslant \infty). \tag{2.3}$$

If $\rho = 0$, $0 < \Lambda < \infty$, then we define the logarithmic types τ_l and ω_l of f as:

$$\lim_{r\to\infty} \frac{\sup}{\inf} \frac{\log M(r)}{(\log r)^{\Lambda+1}} = \frac{\tau_l}{\omega_l} \qquad (0 \leqslant \omega_l \leqslant \tau_l \leqslant \infty). \tag{2.4}$$

An entire function $f(Z)$ is of perfectly regular growth (ρ, τ) [35, p. 45] if and only if there exist two (finite) positive constants ρ and τ such that

$$\lim_{r\to\infty} \frac{\log M(r)}{r^\rho} = \tau.$$

Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$ be an entire function with nonnegative real $a_k (a_0 > 0)$. Then set $S_n(Z) = \sum_{k=0}^{n} a_k Z^k$ and

$$\lambda_{0,n} = \lambda_{0,n}\left(\frac{1}{f}\right) = \inf_{p\in\pi_n} \left\| \frac{1}{f(x)} - \frac{1}{P(x)} \right\|_{L_\infty[0,\infty)} \tag{2.5}$$

where π_n denotes the class of all ordinary polynomials of degree at most n.

For given $s > 1$ and $r > 0$, let $\delta(r, s)$ denote the unique open ellipse in the complex plane with foci at $x = 0$ and $x = r$ and semimajor and semiminor axes a and b such that $b/a = (s^2 - 1)(s^2 + 1)^{-1}$.

Denote $\tilde{M}_F(r, s) = \sup\{|F(z)| : Z \in \delta(r, s)\}$.

Let $h(x)$ be a real nonnegative continuous function on $[0, +\infty)$ such that, for all x large, $h(x) > 0$, and $h'(x)$ exists, is nonnegative, and satisfies

$$\lim_{x\to+\infty} h'(x) = 0. \tag{2.6}$$

Defining generically the set H_S, $0 \leqslant S \leqslant 1$, in the complex plane by

$$H_S = \{Z = x + iy : x \geqslant 0 \text{ and } |y| \leqslant Sh(x)\}. \tag{2.7}$$

$$\bar{H}_D = \{Z = x + iy : 0 \leqslant x \leqslant C_1 d_n \cos(n^{-(1/2+\epsilon)}) \text{ and } |y| \leqslant D\bar{h}(x)\}. \tag{2.8}$$

Set for a real $q > 1$

$$0 < D < \frac{\sqrt{q} - 1}{\sqrt{q} + 1} < 1. \tag{2.9}$$

Let $S(\theta)$ denote generically the infinite sector

$$S(\theta) = \{Z : |\arg Z| < \theta\}. \tag{2.10}$$

$C_1, C_2, C_3, C_4, \ldots, C_k$ are suitable positive constants may be different on different occasions. θ and δ also have different meanings in different theorems.

3. Theorems

THEOREM 1 (Chebyshev [34, p. 19]). *Let $f(x)$ be continuous on $(-\infty, +\infty)$ and*

$$\lim_{x\to+\infty} f(x) = \lim_{x\to-\infty} f(x) = C$$

is finite, then

$$\lim_{n\to\infty} \|f(x) - R_n(x)\|_{L_\infty(-\infty,\infty)} = 0. \tag{3.1}$$

THEOREM 2 (Freud and Szabados [13, p. 201]). *If $f(x)$ satisfies the assumptions of the above theorem, then*

$$\|f(x) - R_n(x)\|_{L_\infty(-\infty,\infty)} \leqslant 48\omega(1/n), \qquad n = 1, 2, 3, \ldots,$$

where $\omega(\delta)$ is the module of continuity of the function $f(\tan t/2)$ on the interval $[-\pi, \pi]$.

THEOREM 3. *Let $f(x)$ be continuous maintains sign and tends to zero on the positive real axis. Then there exist a sequence of polynomials $P_{n_k}(x) = \sum_{i=0}^{k} a_i x^{n_i}$, where $\{n_p\}$ is a sequence of natural numbers satisfying the assumptions that $0 = n_0 < n_1 < n_2 < \cdots < n_k$ and $\sum_1^\infty 1/n_k = \infty$, for which*

$$\lim_{k\to\infty} \left\| f(x) - \frac{1}{P_{n_k}(x)} \right\|_{L_\infty[0,\infty)} = 0. \tag{3.2}$$

Remark. There exist functions which can be approximated by general rational functions on $(-\infty, +\infty)$ but not by reciprocal of polynomials on $[0, \infty)$. One such example is $f(x) = 1 + e^{-|x|}$.

Proof. It is known [3, p. 391] that $f(x)$ satisfying the assumptions of Theorem 3 can be approximated uniformly on any finite interval $[0, 2b]$ as close as we like by reciprocals of polynomials $\{P_{n_k}\}$ of the form,

$$P_{n_k}(x) = \sum_{i=0}^{k} a_i x^{n_i},$$

where $\{n_j\}$ satisfies the above conditions. In other words

$$\lim_{k\to\infty} \left\| f(x) - \frac{1}{P_{n_k}(x)} \right\|_{L_\infty[0,2b]} \to 0, \tag{3.3}$$

for every finite interval $[0, 2b]$.

Now we choose $\epsilon > 0$, $b > 0$ and sufficiently large and a $n_q(q > k)$ such that

$$\lim_{k\to\infty} \left\| f(x) - \frac{1}{P_{n_k}(x) + \epsilon(x/b)^{n_q}} \right\|_{L_\infty[0,b]} \to 0. \tag{3.4}$$

This is certainly possible since $\epsilon(x/b)^{n_q}$ tends to zero very fast for $b > x > 0$, if $b = x$, ϵ being very small (3.4) is certainly valid.

Now we divide for convenience $[0, \infty)$ into $[0, b]$ and $[b, \infty)$, where is sufficiently big finite interval. For all $x > b, f(x)$ will be very small and

$$P_{n_k}(x) + \epsilon(x/b)^{n_q}$$

grow very fast for all large k, hence

$$\lim_{k\to\infty} \left\| f(x) - \frac{1}{P_{n_k}(x) + \epsilon(x/b)^{n_q}} \right\|_{L_\infty[b,\infty)} \to 0. \tag{3.5}$$

We get the result (3.2) from (3.4) and (3.5).

Remarks. If $1/f(x)$ is not entire, then the following theorems indicate that, it is not possible to approximate $f(x)$ very closely by reciprocals of polynomials.

THEOREM 3A (Erdös–Newman–Reddy [5]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ $(0 < \rho < \infty)$ *type* τ *and lower type* ω $(0 < \omega \leqslant \tau < \infty)$. *Then for all large n,*

$$\lambda_{0,n}\left(\frac{x}{f(x)}\right) \geqslant \frac{(\log n)^{1/\rho}}{10n^2(2\tau)^{1/\rho}} \left(f\left[\left(\frac{\log n}{2\tau}\right)^{1/\rho} n^{-2}\right]\right)^{-1}.$$

THEOREM 3B (Erdös–Newman–Reddy [5]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* $\rho(0 < \rho < \infty)$ *type* τ *and lower type* $\omega(0 < \omega \leqslant \tau < \infty)$. *Then*

$$\lambda_{0,n}\left(\frac{x}{f(x)}\right) \leqslant C_0(\log n)^{1/\rho}\, n^{-2}.$$

THEOREM 3C (Erdös–Newman–Reddy [5]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* $\rho(1 \leqslant \rho < \infty)$ *type* τ *and lower type* $\omega(0 < \omega \leqslant \tau < \infty)$. *Then there is a polynomial* $P_n(x)$ *of degree n for which*

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[1,\infty)} \leqslant \exp(-Cn^{1/2}).$$

THEOREM 3D (Erdös–Newman–Reddy [5]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* $\rho(0 < \rho < 1)$ *type* τ *and lower type* $\omega(0 < \omega \leqslant \tau < \infty)$. *Then there is a polynomial* $P(x)$ *of degree n for which*

$$\left\| \frac{x}{f(x)} - \frac{1}{P(x)} \right\|_{L_\infty[1,\infty)} \leqslant \exp\left(-C\left(\frac{n}{\log n}\right)^{\rho}\right).$$

THEOREM 3E (Erdös–Newman–Reddy [5]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of infinite order. Then there is a polynomial* $P_n(x)$ *of degree n for which for infinitely many n*

$$\left\| \frac{x}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant C(\log n)^2 n^{-2} \mid a_n \mid^{-1/n}.$$

THEOREM 4 (Erdös and Reddy [11, Theorem 1]). *Let* $f(x)$ *be continuous non-vanishing and tends to* $+\infty$ *on* $[0, \infty)$. *Then there exist polynomials* $P_{n_k}(x)$ *satisfying the assumptions of the above theorem for which,*

$$\lim_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} = 0.$$

THEOREM 5 (Erdös and Reddy [11, Theorem 2]). *Let* $f(x)(\neq 0)$ *be a continuous function defined on* $[0, \infty)$. *If there exist a sequence of polynomials* $P_{n_k}(x) = \sum_{i=0}^{k} a_{n_i} x^{n_i}$ *for which*

$$\lim_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{P_{n_k}(x)} \right\|_{L_\infty[0,\infty)} = 0,$$

where $0 = n_0 < n_1 < n_2 < \cdots < n_k$ *and* $\sum_{k=1}^{\infty} 1/n_k < \infty$. *Then* $f(x)$ *is the restriction to* $[0, \infty)$ *of an entire function* $F(Z)$.

THEOREM 6 (Reddy and Shisha [30, Theorem 1]). *Let $f(x)$ be a continuous function ($\neq 0$) defined on $[0, \infty)$. If there exist a sequence of polynomials $\{P_n(x)\}_{n=0}^{\infty}$, with nonnegative coefficients such that*

$$\lim_{n\to\infty}\left\|\frac{1}{f(x)}-\frac{1}{P_n(x)}\right\|_{L_\infty[0,\infty)}=0.$$

Then $f(x)$ is the restriction to $[0, \infty)$ of an entire function $F(Z)$.

THEOREM 7 (Erdös and Reddy [7, Theorem 1]). *Let $f(Z)=\sum_{k=0}^{\infty}a_kZ^k$ $a_0>0$, and $a_k\geqslant 0$ $(k\geqslant 1)$ be any entire function. Then for every $\epsilon>0$, there exist infinitely many n for which*

$$\lambda_{0,n}\leqslant\exp\left(\frac{-n}{(\log n)^{1+\epsilon}}\right).$$

Remarks. For functions which grow regularly, the above conclusion, is valid for all large n. For a slightly general result see (Erdös and Reddy [9, Theorem 1]).

THEOREM 8 (Erdös and Reddy [7, Theorem 2]). *Let $f(Z)$ be an entire function of infinite order with non-negative coefficients. Then for each $\epsilon>0$, there exist infinitely many n for which*

$$\lambda_{0,n}\geqslant e^{-\epsilon n}.$$

THEOREM 9. *Let $f(Z)=\sum_{k=0}^{\infty}a_kZ^k$, $a_0>0$, $a_k\geqslant 0$ $(k\geqslant 1)$ be a transcendental entire function of finite order $\rho(0\leqslant\rho<\infty)$. Then for every constant $c>0$, for all large n*

$$\lambda_{0,n}\leqslant 1/n^c. \tag{3.6}$$

Proof. By definitions for $0\leqslant x\leqslant r=(n/2)^{1/\rho+\epsilon}e^{-1}$, $\epsilon>0$.

$$\begin{aligned}0\leqslant\frac{1}{S_n(x)}-\frac{1}{f(x)}&\leqslant\left(\sum_{k=n+1}^{\infty}a_kr^k\right)C_1\leqslant C_1e^{-n}\sum_{k=n+1}^{\infty}a_k(\mathrm{re})^k\\&\leqslant C_1e^{-n}M(\mathrm{re})\leqslant C_1\exp((re)^{\rho+\epsilon}-n)\leqslant C_1e^{-n/2}.\end{aligned}\tag{3.7}$$

On the other hand for $x\geqslant r$.

$$\begin{aligned}0\leqslant\frac{1}{S_n(x)}-\frac{1}{f(x)}&\leqslant\frac{1}{S_n(x)}\leqslant\frac{1}{S_n(r)}\leqslant\frac{1}{f(r)-e^{-n/2}}\\&\leqslant\frac{C_2}{f(r)}\leqslant\frac{C_2}{r^{C_3}}\leqslant n^{-c}.\end{aligned}\tag{3.8}$$

(3.6) follows from (3.7) and (3.8) by properly choosing C_1, C_2, and C_3.

THEOREM 10 (Meinardus, Reddy, Taylor and Varga [17, Theorem 3]). *Let $f(x)$ be a real continuous function ($\neq 0$) on $[0,\infty)$ and assume that there exist a sequence of real polynomials $\{P_n(x)\}_{n=0}^{\infty}$, with $P_n \in \pi_n$ for each $n \geqslant 0$, and a real number $q > 1$ such that*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \leqslant \frac{1}{q} < 1. \tag{3.9}$$

Then, there exists an entire function $F(z)$ with $F(x) = f(x)$ for all $x \geqslant 0$, and $F(Z)$ is of finite order ρ, i.e.,

$$\limsup_{r\to\infty} \frac{\log\log M_F(r)}{\log r} \equiv \rho < \infty.$$

In addition, for every $S > 1$, there exist constants $K = K(S, q) > 0$, $\theta = \theta(S, q) > 1$ and $r_0 = r_0(S, q) > 0$ such that

$$\tilde{M}_F(r, S) \leqslant (K \| f \|_{L_\infty[0,r]})^\theta \qquad \text{for all} \quad r \geqslant r_0 .$$

If, for each $S > 1$, $\bar{\theta}(S)$ is defined by

$$\limsup_{r\to\infty} \left\{ \frac{\log \tilde{M}_F(r, S)}{\log \| f \|_{L_\infty[0,r]}} \right\} = \bar{\theta}(S)$$

when $\| f \|_{L_\infty[0,r]}$ is unbounded as $r \to \infty$, and $\bar{\theta}(S) \equiv 1$ otherwise, then the order ρ of F satisfies

$$\rho \leqslant \inf_{S>1} \left\{ \frac{\log \bar{\theta}(S)}{\log \left[\frac{1}{2} + \frac{1}{4}\left(S + \frac{1}{S}\right)\right]} \right\}$$

and this upper bound for the order ρ is in general best possible.

Remarks. It is very likely that $F(Z)$ may satisfy that

$$\lim_{r\to\infty} \frac{\log\log M_F(r)}{\log r} = \rho, \qquad (0 < \rho < \infty).$$

It is easy to give examples of entire functions of zero order for which (3.9) fails.

THEOREM 11 (Meinardus, Reddy, Taylor and Varga [17, Theorem 5]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$ be an entire function with $a_0 > 0$ and $a_k \geqslant 0$ for* *all $k \geqslant 1$. If there exist real numbers $A > 0$, $S > 1$, $\theta > 0$, and $r_0 > 0$ such that*

$$\tilde{M}_F(r, S) \leqslant A(\|f\|_{L_\infty[0,r]})^\theta \quad \text{for all} \quad r \geqslant r_0,$$

then there exist a sequence of real polynomials $\{P_n(x)\}_{n=0}^\infty$ with $P_n \in \pi_n$ for each $n \geqslant 0$, and a real number $q \geqslant S^{1/(1+\theta)} > 1$ such that

$$\limsup_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)}\right\}^{1/n} = \frac{1}{q} < 1.$$

Remarks. Quite recently the assumption $a_0 > 0$ and $a_k \geqslant 0$ $(k \geqslant 1)$ has been weakend by Blatt [2] and Roulier and Taylor [31].

THEOREM 12 (Meinardus, Reddy, Taylor and Varga [17, Theorem 6]). *Let $f(Z) = \sum_{k=0}^\infty a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order $\rho(0 < \rho < \infty)$ type τ and lower type $\omega(0 < \omega \leqslant \tau < \infty)$. Then there exist a sequence of real polynomials $\{P_n(x)\}_{n=0}^\infty$ for which*

$$\limsup_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)}\right\}^{1/n} < 1.$$

THEOREM 13 (Reddy [23]). *Let $f(Z) = \sum_{k=0}^\infty a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order $\rho(0 < \rho < \infty)$ type τ and lower type $\omega(0 < \omega \leqslant \tau < \infty)$. Then*

$$\limsup_{n\to\infty} \left\{\left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^n a_k x^k} \right\|_{L_\infty[0,\infty)}\right\}^{1/n} \leqslant \exp\left(\frac{-\omega}{\rho\tau e + \rho\omega}\right).$$

Remark. There exist functions which fail to satisfy the assumptions of the above theorem but for which the conclusion is valid in a slightly different form. One such example is

$$f(Z) = 1 + \sum_{k=2}^\infty Z^k \left(\frac{\log k}{k}\right)^k.$$

THEOREM 14 (Reddy [20, Theorem C]). *Let $f(Z) = \sum_{k=0}^\infty a_k Z^k$, $a_k \geqslant 0$ $(k \geqslant 0)$ be an entire function of order $\rho(0 < \rho < \infty)$, type τ, and lower type ω, $(0 < \omega \leqslant \tau < \infty)$ and $\tau < 2\omega$, then*

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \left(\frac{\tau}{2\omega}\right)^{x_2/\alpha x_1},$$

where x_1 is the largest and x_2 the smallest root of the equation,

$$x \log(x/e) + \omega/\tau = 0.$$

THEOREM 15 (Reddy and Shisha [29, Theorem 14]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0 (k \geqslant 1)$ be an entire function and suppose there exist constants $\delta > 1$, $C > 1$, $\epsilon > 0$ and $0 < C_1 < C_2 < 1$ for which, for all large r,*

$$M(r\delta) \geqslant \{M(r)\}^{\theta}, \qquad \textit{where} \quad \theta = \frac{C_2}{C_1} + \frac{\log(4\delta - 2)}{C_1 \log C} + \epsilon.$$

Then for every sequence $\{P_n(x)\}_{n=0}^{\infty}$, where each $P_n(x)$ is a real polynomial of degree $\leqslant n$, positive throughout $[0, \infty)$, we have

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \geqslant C^{-\theta} > 0.$$

THEOREM 16 (Erdös and Reddy [9, Theorem 3]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_k \geqslant 0 (k \geqslant 0)$ be an entire function of order $\rho (0 < \rho < \infty)$ type τ, and lower type $\omega (0 < \omega \leqslant \tau < \infty)$. Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant \left(\frac{e\omega^2}{e^{2\omega/[\tau(e+1)]} \tau^2 (e+1) 4^{\rho}} \right)^{x_1/\rho x_2}.$$

THEOREM 17 (Reddy [22]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_k \geqslant 0 (k \geqslant 0)$ be an entire function of order $\rho (0 < \rho < \infty)$ type τ and lower type $\omega (0 < \omega \leqslant \tau < \infty)$. Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant \left(2^{2+1/\rho} \tau^{1/\rho} \left(\frac{\tau}{\omega} \right)^{1/\rho} - 1 \right)^{-2}.$$

THEOREM 18 (Meinardus and Varga [16], Reddy [20]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_k \geqslant 0 (k \geqslant 0)$ be an entire function of perfectly regular growth (ρ, τ). Then*

$$\frac{1}{2^{2+1/\rho}} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{2^{1/\rho}}.$$

THEOREM 19 (Reddy [20]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_k \geqslant 0 (k \geqslant 0)$ be an entire function of perfectly regular growth (ρ, τ). Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant \frac{1}{2^{2+1/\rho}}.$$

THEOREM 20 (Cody, Meinardus and Varga [4]). *Let $f(Z) = e^Z$. Then*

$$\frac{1}{6} \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{1/n} \leqslant \frac{1}{2.298}.$$

THEOREM 21 (Schönhage [33]). *Let $f(Z) = e^Z$. Then*

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = \tfrac{1}{3}.$$

THEOREM 22 (Newman [18]). *Let $P(x)$ and $Q(x)$ be any real polynomials of degree less than n. Then*

$$\liminf_{n\to\infty} \left\{ \left\| e^{-x} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \geqslant \frac{1}{(1280)}.$$

THEOREM 23. *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0 (k \geqslant 1)$ be an entire function of perfectly growth (ρ, τ). Then for every non-vanishing polynomials $P(x)$ and $Q(x)$ of degree at most n, there is a constant $C > 4^{1+1/\rho}$ for which*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \geqslant C^{-2}. \tag{3.10}$$

Remarks. (a) If $P(x)$ is a constant then (3.10) is known in a better form (cf. [22, Theorem]). (b) The proof adopted here is different from the one used by D. J. Newman.

We need the following lemma for our purpose.

LEMMA ([34], [9, p. 68]). *Let $P(x)$ be any algebraic polynomial of degree at most n. If this polynomial is bounded by M on an interval $[a, b]$ then at any point outside the interval we have*

$$|P(x)| \leqslant M T_n \left| \left(\frac{2x - a - b}{b - a} \right) \right|$$

where

$$2T_n(x) = (x + \sqrt{x^2 - 1})^n + (x - \sqrt{x^2 - 1})^n.$$

Proof of Theorem 23. Let $M(r) = \max_{|z| \leqslant r} |f(z)|$. Then by assumption

$$\lim_{r\to\infty} \frac{\log M(r)}{r^\rho} = \tau, \quad \begin{pmatrix} 0 < \rho < \infty \\ 0 < \tau < \infty \end{pmatrix}.$$

Hence for each $\epsilon > 0$ and $\delta > 1$, there is an $r_0 = r_0(\epsilon)$, such that for all $r \geqslant r_0(\epsilon)$,

$$M(r\delta) \geqslant \{M(r)\}^{\delta^{\rho}(1-\epsilon)/(1+\epsilon)}. \tag{3.11}$$

Let us assume on the contrary the theorem is false. Then for all large n,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} < C^{-2n}. \tag{3.12}$$

Since $\lim_{x\to\infty} | P(x)| = \infty$, there exist arbitrary large r for which

$$| P(r)| \leqslant | P(t)|, \qquad \text{for all} \quad t \geqslant r. \tag{3.13}$$

For each of these values of r for which (3.13) is valid, we can find sufficiently large n and a constant $C > 4^{1+1/\rho}$ such that

$$f(r) = (C - \tfrac{1}{4})^n. \tag{3.14}$$

Then at this point $x = r$, we get

$$\left| \frac{Q(r)}{P(r)} \right| < C^n. \tag{3.15}$$

If (3.15) is not valid, then

$$\left| \frac{Q(r)}{P(r)} \right| \geqslant C^n. \tag{3.16}$$

From (3.14) and (3.16), we obtain

$$C^{-2n} < (C - \tfrac{1}{4})^{-n} - C^{-n} \leqslant \frac{1}{f(r)} - \frac{P(x)}{Q(x)}. \tag{3.17}$$

(3.17) clearly contradicts (3.12). Hence (3.15) is valid.

At $x = r\delta = r(C/4)$, we have from (3.11) and (3.14)

$$f(r\delta) \geqslant \{f(r)\}^{\delta^{\rho}(1-\epsilon)/(1+\epsilon)} \geqslant (C - \tfrac{1}{4})^{2(1-\epsilon)n/(1+\epsilon)}. \tag{3.18}$$

Since $P(x) \neq 0$, we get from (3.15).

$$|Q(r)| < | P(r)| \, C^n. \tag{3.15$'$}$$

Now by applying lemma to $| Q(r)|$ over the interval $[0, r\delta]$, we get

$$|Q(r\delta)| < | P(r)| \{2(2\delta - 1)C\}^n. \tag{3.19}$$

From (3.13) and (3.19), we get by choosing $t = r\delta$,

$$\left| \frac{Q(r\delta)}{P(r\delta)} \right| \leqslant \{2(2\delta - 1)C\}^n. \tag{3.20}$$

Clearly (3.18) and (3.20) contradicts (3.12) for all those values of n for which (3.14) is valid. Since ϵ being arbitrary

$$C^{-2n} < [(4\delta - 2)C]^{-n} - (C - \tfrac{1}{4})^{-2n} < \frac{P(r\delta)}{Q(r\delta)} - \frac{1}{f(r\delta)}.$$

Hence the theorem is proved.

THEOREM 24 (Reddy [21]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order $\rho (0 < \rho < \infty)$, type τ and lower type $\omega (0 < \omega \leqslant \tau < \infty)$. Then one cannot find for $n = 0, 1, 2,\ldots$, polynomials $P_n(x)$ and $Q_n(x)$ with nonnegative coefficients and of degree at most n for which*

$$\liminf_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} < (5.2)^{-\tau/\rho\omega}.$$

THEOREM 25. *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0 (k \geqslant 1)$ be an entire function of order ρ and maximal type or of (finite) order $\rho + \epsilon$. If $P_n(x)$ and $Q_n(x)$ are, for $n = 0, 1, 2,\ldots$ polynomials with nonnegative real coefficients of degree at most n, then*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{\rho/n} > (2.75)^{-1}. \tag{3.21}$$

Proof. If (3.21) is not valid, then for all large n,

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \leqslant (2.75)^{-n/\rho}. \tag{3.22}$$

By our assumption $f(Z)$ is either an entire function of order $\rho(0 < \rho < \infty)$ and maximal type or of order $\rho + \epsilon(\epsilon > 0)$, then it is known ([3], p. 8) that

$$\limsup_{n\to\infty} \frac{\log M(r)}{r^\rho} = \infty, \tag{3.23}$$

where $M(r) = \operatorname{Max}_{|Z|=r} |f(Z)|$. From (3.23), it follows that there exists arbitrarily large values of r for which

$$\frac{\log M(r)}{r^\rho} \geqslant \frac{\log M(t)}{t^\rho} \qquad 0 < t < r. \tag{3.24}$$

From (3.24) we obtain for all those values of r, with

$$t = C_1^{1/\rho} r, \quad 0 < C_1 < 1,$$

$$M(r) \geqslant [M(t)]^{(r/t)^\rho} = [M(t)]^{C_1^{-1}}. \tag{3.25}$$

For sufficiently large r, we can find an n such that

$$M(C^{1/\rho} r) = (2.75)^{(n/\rho)C_1}. \tag{3.26}$$

At this point, that is at $x = C_1^{1/\rho} r$,

$$\frac{Q(x)}{P(x)} < (2.75)^{(n/\rho)C_2}, \qquad \text{where} \quad 0 < C_1 < C_2 < 1. \tag{3.27}$$

(3.27) follows easily from (3.22) and (3.26). But at $x = r$, we get by (3.25) and (3.26),

$$M(r) \geqslant [M(t)]^{C_1^{-1}} = (2.75)^{n/\rho}. \tag{3.28}$$

Because of the assumption that the coefficients of $P(x)$ and $Q(x)$ are nonnegative, we get along with (3.27),

$$\frac{Q(r)}{P(r)} \leqslant \left(\sum_{k=0}^{n} b_k C_1^{k/\rho} C_1^{-k/\rho} r^k \Big/ P(rC_1^{1/\rho}) \right) \leqslant C_1^{-n/\rho} \frac{Q(rC_1^{1/\rho})}{P(rC_1^{1/\rho})} \leqslant (2.75)^{(n/\rho)C_2} C_1^{-n/\rho}. \tag{3.29}$$

From (3.28) and (3.29) we get at $x = r$, with $C_1 = 0.945$, $C_2 = 0.95$.

$$(2.75)^{-n/\rho} < (2.75)^{(-n/\rho)C_2} C_1^{-n/\rho} - (2.75)^{-n/\rho} < \frac{P(x)}{Q(x)} - \frac{1}{f(x)}. \tag{3.30}$$

This flatly contradicts (3.22), hence the result is proved.

Theorem 26 (Erdös and Reddy [8, Theorem 1]). *Let $g(n)$ be any sequence tending to infinity arbitrarily fast, then there is an entire function of infinite order such that for infinitely many $n = n_p$*

$$\lambda_{0,n_p} \leqslant \frac{1}{g(n_p)}.$$

Theorem 27. *Let $h(n)$ be any sequence tending to $+\infty$ very slowly. Then there is an entire function such that for infinitely many n*

$$\lambda_{0,n} \geqslant \frac{1}{h(n)}.$$

The proof of this is similar to Theorem 2 of [7], hence the details are omitted.

THEOREM 28 (Erdös and Reddy [10, Theorem 1]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of finite order* $\rho (0 \leqslant \rho < \infty)$. *Then for every* $\epsilon > 0$, *there exist infinitely many* n *for which*

$$\lambda_{0,n} \leqslant \frac{1}{(1.4)^{n/\rho+\epsilon}}.$$

THEOREM 29 (Erdös and Reddy [11A, Theorem 2]). *Let* $f(Z) = \sum_{k=0}^{\infty} b_k Z^k$, $b_0 > 0$, $b_k \geqslant 0$, $b_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ *and lower order* β $(0 \leqslant \beta < \rho < \infty)$. *Then there exist an entire function* $h(Z) = \sum_{p=0}^{\infty} a_{n_p} Z^{n_p}$ *(for convenience we let* $a_{n_0} = C$, $n_0 = 0$*) formed from the series* $f(Z)$ *for which*

$$\liminf_{n\to\infty} \left[\lambda_{0,n}\left(\frac{1}{h(x)}\right)\right]^{1/n\log n} \leqslant \exp\left(1 - \frac{\rho}{\beta}\right).$$

THEOREM 30 (Erdös and Reddy [11A, Theorem 3]). *There is an entire function of positive lower order for which*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n\log\log n} = 0.$$

THEOREM 31 (Erdös and Reddy [11A, Theorem 1]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of lower order* β *and order* $\rho (0 < \beta \leqslant \rho < \infty)$. *Then for every* $\epsilon > 0$, *there is an* $n_0 = n_0(\epsilon)$, *such that for all* $n \geqslant n_0(\epsilon)$,

$$\lambda_{0,n} \leqslant \exp(-n^{\beta(1-\epsilon)/\rho(1+\epsilon)}).$$

THEOREM 32. *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* ρ *and lower order* $\beta (0 < \beta \leqslant \rho < \infty)$. *Then for all large* $n \geqslant n_0(\epsilon)$,

$$\lambda_{0,n} \geqslant \exp(-n^{\rho(1+\epsilon)/\beta(1-\epsilon)}). \tag{3.31}$$

Proof. Let us assume (3.31) is false. Then there exist infinitely many n for which

$$\left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} < \exp(-n^{-\rho(1+\epsilon)/\beta(1-\epsilon)}). \tag{3.32}$$

By assumption $f(x)$ is increasing, hence for every large n and any $\epsilon > 0$ there is an $r > 0$ such that

$$f(r) = \exp\left(\frac{n^{\rho(1+\epsilon)/\beta(1-\epsilon)}}{4}\right). \tag{3.33}$$

By assumption for each $\epsilon > 0$, there is an $r_0 = r_0(\epsilon)$ such that for all $r \geqslant r_0(\epsilon)$.

$$\exp(r^{\beta(1-\epsilon)}) \leqslant f(r) \leqslant \exp(r^{\rho(1+\epsilon)}). \tag{3.34}$$

In (3.33) we choose n so large such (3.33) and (3.34) valid simultaneously. From (3.32) and (3.33) it is easy to see that

$$P_n(r) < \exp\left(\frac{n^{\rho(1+\epsilon)/\beta(1-\epsilon)}}{3}\right). \tag{3.35}$$

From (3.33) and (3.34) we obtain

$$r \geqslant n^{1/\beta(1-\epsilon)} 4^{-1/\rho(1+\epsilon)}. \tag{3.36}$$

Now at $x = r\delta$, where δ satisfies the assumption that $n\delta^{\beta(1-\epsilon)} = 2n^{\rho(1+\epsilon)/\beta(1-\epsilon)}\, 4^{\beta(1-\epsilon)/\rho(1+\epsilon)}$, we have from (3.36) along with the definition of lower order,

$$f(r\delta) \geqslant \exp[(r\delta)^{\beta(1-\epsilon)}] \geqslant \exp\left[n\left(\frac{\delta^{\rho(1+\epsilon)}}{4}\right)^{\beta(1-\epsilon)/\rho(1+\epsilon)}\right] \geqslant \exp[2n^{\rho(1+\epsilon)/\beta(1-\epsilon)}]. \tag{3.37}$$

But by using lemma of Theorem 23 we get

$$P_n(r\delta) < (4\delta)^n P_n(r) < (4\delta)^n \exp\left(\frac{n^{\rho(1+\epsilon)/\beta(1-\epsilon)}}{3}\right) < \exp(n^{\rho(1+\epsilon)/\beta(1-\epsilon)}). \tag{3.38}$$

(3.37) and (3.38) clearly contradicts (3.32), hence the result is proved.

THEOREM 33 (Erdös and Reddy [11, Theorem 4]). *Let* $f(Z) = \sum_{k=0}^{\infty} b_k Z^k$, $b_0 > 0$, $b_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of order* $\rho (0 < \rho < \infty)$ *type* τ *and lower type* ω, *satisfying the assumption that* $0 \leqslant \omega < \delta < \tau \leqslant \infty$. *Then there exist an entire function* $h(Z) = \sum_{m=0}^{\infty} a_m Z^m$ *formed from the series of* $f(Z)$ *for which we get*

$$\liminf_{n\to\infty} \left(\lambda_{0,n}\left(\frac{1}{h(x)}\right)\right)^{1/n} \leqslant \frac{\omega}{\delta}.$$

THEOREM 34 (Meinardus, Reddy, Taylor and Varga [17, Theorem 7]). *Let $f(Z)$ be an entire function of logarithmic order $\rho_1 = \Lambda + 1$ $(0 < \Lambda < \infty)$ and logarithmic types τ_l and ω_l $(0 < \omega_l \leqslant \tau_l < \infty)$. Then*

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = 0.$$

Remark. $0 < \omega_l \leqslant \tau_l < \infty$, guarantees the following

$$0 < \lim_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 < \infty.$$

On the other hand

$$0 < \lim_{r\to\infty} \frac{\log\log M(r)}{\log\log r} = \Lambda + 1 < \infty$$

may not imply that $0 < \omega_l \leqslant \tau_l < \infty$.

EXAMPLES

$$f_1(Z) = 1 + \sum_{n=2}^{\infty} \frac{Z^n}{\exp[(n \log n)^2]},$$

$$f_2(Z) = 1 + \sum_{n=2}^{\infty} \frac{Z^n}{\exp[(n/\log n)^2]}.$$

It is easy to verify that for

$$f_1(Z), \quad \Lambda = 1 = \beta_l - 1, \quad \tau_l = 0.$$
$$f_2(Z), \quad \Lambda = 1 = \beta_l - 1, \quad \omega_l = \infty.$$

THEOREM 35 (Reddy [19] and [22]). *Let $f(Z)$ satisfy the assumptions of Theorem 34. Then*

$$\exp\left(\frac{-\Lambda}{(\Lambda+1)[(\Lambda+1)\,\tau_l]^{1/\Lambda}}\right) \leqslant \limsup_{n\to\infty} (\lambda_{0,n})^{n^{-[1+\Lambda^{-1}]}} < 1.$$

THEOREM 36 (Reddy [24]). *Let $f(Z)$ be an entire function satisfying the assumptions that*

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log\log r} < \liminf_{r\to\infty} \frac{\log\log M(r)}{\log\log r} + 1.$$

Then

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = 0.$$

THEOREM 37. *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ *be an entire function satisfying the assumptions that* $0 < \Lambda < \infty$, $5 < 2\omega_l < 2\tau_l < \infty$. *Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{n^{-(1+1/\Lambda)}} \geqslant \frac{1}{e}.$$

The proof of this theorem is very similar to the proof of Theorem 32, with the only difference we use here

$$f(r) = \exp\left\{\frac{n^{\Lambda+1/\Lambda}}{6}\right\},$$

and

$$\log \delta = \frac{(2.4)^{1/\Lambda+1}\, n^{1/\Lambda}}{[\omega_l(1-\epsilon)]^{1/\Lambda+1}} - \frac{n^{1/\Lambda}}{[6\tau_l(1+\epsilon)]^{1/(\Lambda+1)}}.$$

We omit all the details to the reader.

THEOREM 38 (Erdös and Reddy [9, Theorem 6]). *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of logarithemic order* $\rho_l = \Lambda + 1 < \infty$. *Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{n^{-(1+1/\Lambda+\epsilon)}} < 1.$$

THEOREM 39. *Let* $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ *be an entire function of logarithemic order* $\rho_l = \Lambda + 1 < \infty$. *Then for each* $\epsilon > 0$,

$$\limsup_{n\to\infty} (\lambda_{0,n})^{n^{-1/\Lambda+\epsilon}} < 1.$$

The proof of this theorem is very similar to the proof of Theorem 9, with the only difference we use here

$$\Lambda = \limsup_{n\to\infty} \frac{\log n}{\log\left\{\frac{1}{n}\log\left|\frac{1}{a_n}\right|\right\}},$$

instead of

$$\limsup_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho = \limsup_{n\to\infty} \frac{n\log n}{\log\left|\frac{1}{a_n}\right|},$$

hence we omit the details to the reader.

THEOREM 40 (Erdös and Reddy [9, Theorem 4]). *Let* $f(Z) = 1 + \sum_{k=1}^{\infty} Z^k/(d_1 d_2 \cdots d_k)$ *with* $d_{k+1} > d_k > 0$, $k = 1, 2, 3,\ldots$, *be an entire function of finite order* ρ. *Then for any* $\epsilon > 0$, *we have for all large* n,

$$\frac{d_1 d_2 \cdots d_n}{2^{4n} d_n^{2(\rho+\epsilon)} d_{n+1} d_{n+2} \cdots d_{2n}} \leqslant \lambda_{0,2n-1} \leqslant \frac{d_1 d_2 \cdots d_n}{d_{n+1} d_{n+2} \cdots d_{2n}} \left(\frac{d_{2n+1}}{d_{2n+1} - d_{2n}}\right).$$

EXAMPLES.

(1) $$f(Z) = 1 + \sum_{k=1}^{\infty} \frac{Z^k}{2^{\log 2} 3^{\log 3} \cdots k^{\log k}}.$$

For this function, $\Lambda = \infty$. But

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n\log n} = \frac{1}{2}.$$

(2) $$f(Z) = 1 + \sum_{k=1}^{\infty} Z^k \delta^{-2^k}, \quad (1 < \delta < \infty).$$

For this function $\Lambda = 0$. But

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/2^{(n+1)}} = \frac{1}{\delta}.$$

(3) $$f(Z) = 1 + \sum_{k=1}^{\infty} \frac{Z^k}{2^2 3^3 4^4 \cdots k^k}.$$

For this function $\Lambda = 1$, $\tau_l = 0$. But

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n^2\log n} = e^{-1/4}.$$

THEOREM 41 (Erdös and Reddy [9, Theorem 7]). *Let* $f(Z) = 1 + \sum_{k=1}^{\infty} a_{n_k} Z^{n_k}$, $\liminf_{k\to\infty} n_{k+1}/n_k \geqslant \theta > 1$ *be an entire function of order* ρ $(0 \leqslant \rho < \infty)$. *Then*

$$\liminf_{n\to\infty} (\lambda_{0,n})^{(\rho+\epsilon)/n} \leqslant \frac{1}{\sqrt{\theta}}. \tag{3.39}$$

Remarks. We stated this result in [9] without proof. Now we present a proof.

Proof. By assumption for each $\epsilon > 0$,

$$\lim_{k\to\infty} n_k^{1/\rho+\epsilon} a_{n_k}^{1/n_k} \to 0. \tag{3.40}$$

From (3.40) we get for a sequence of values of k and all $i \geqslant k$,

$$n_i^{1/\rho+\epsilon} a_{n_i}^{1/n_i} \leqslant a_{n_k}^{1/n_k} (n_k)^{1/\rho+\epsilon}. \tag{3.41}$$

Now as usual by definitions for $0 \leqslant w \leqslant r = \theta^{1/2(\rho+\epsilon)} a_{n_k}^{-1/n_k}$ along with (3.41) we get

$$0 \leqslant \frac{1}{S_{n_k}(x)} - \frac{1}{f(x)} \leqslant \sum_{i=k+1}^{\infty} a_{n_i} r^{n_i} \leqslant \sum_{i=n+1}^{\infty} \theta^{-(n_i/2(\rho+\epsilon))} \leqslant \theta^{-n_k/2(\rho+\epsilon)} C_5 . \tag{3.42}$$

On the other hand for $x \geqslant r = \theta^{1/2(\rho+\epsilon)} a_{n_k}^{-1/n_k}$,

$$0 \leqslant \frac{1}{S_{n_k}(x)} - \frac{1}{f(x)} \leqslant \frac{1}{S_{n_k}(r)} \leqslant \frac{1}{a_{n_k} r^{n_k}} \leqslant \theta^{-n_k/2(\rho+\epsilon)} , \tag{3.43}$$

(3.39) follows from (3.42) and (3.43).

THEOREM 42 (Erdös and Reddy [10]). *For all large $n \geqslant n_0(c)$, we have*

$$\left\| \frac{1}{\sum_{k=0}^{\infty} x^k} - \frac{1}{\sum_{k=0}^{n} x^k} \right\|_{L_\infty[0,\infty)} \leqslant \frac{c \log n}{n} .$$

THEOREM 43 (Erdös and Reddy [10]). *There is a polynomial $P_n(x)$ of degree at most n for which*

$$\left\| \frac{1}{(x+1)^{n+1}} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant 2^{-n}.$$

THEOREM 44 (Erdös and Reddy [10]). *For every polynomial $P_n(x)$ of degree at most n, we have*

$$\left\| \frac{1}{(x+1)^{n+1}} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \geqslant (16)^{-n}.$$

THEOREM 45 (Reddy [26, Theorem 4]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order $\rho = 2$, type τ and lower type $\omega(\frac{1}{25} \leqslant \omega \leqslant \tau < \infty)$ or order $\rho(2 < \rho < \infty)$, type τ and lower type $\omega(0 < \omega \leqslant \tau < \infty)$. Then it is not possible to find exponential polynomials of the form $\sum_{i=0}^{n} b_k e^{kx}$ $(b_k \geqslant 0)$ for which*

$$\liminf_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k e^{kx}} \right\|_{L_\infty[0,\infty)}^{\rho\omega/n^2\tau} \leqslant e^{-1}$$

EXAMPLES. Let $f(Z) = 1 + \sum_{k=1}^{\infty} e^{Zk}/1^1 2^2 3^3 \cdots k^k$. This is an entire function of order $\rho = 2$ and type $\tau = 0$. This function fails to satisfy the assumption of Theorem 45. But for this function it is easy to show that

$$\limsup_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} a_k e^{kx}} \right\|_{L_\infty[0,\infty)}^{1/n^2 \log n} \leqslant \frac{1}{e}.$$

(2) The following example suggests the assumption $\rho = 2$, $\tau > 0$ is not sufficient for the conclusion of

THEOREM 45. *Let*

$$f(Z) = \sum_{k=0}^{\infty} \frac{e^{Zp_k}}{e^{p_k^2}}.$$

$$0 = p_0 < p_1 < p_2 < p_3 < \cdots < p_k < \cdots,$$

$$\lim_{k\to\infty} \frac{p_{k+1}}{p_k} = \infty.$$

This is an entire function of order $\rho = 2$ and type $\tau > 0$. For this function we can show easily

$$\limsup_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum\limits_{k=0}^{n} \dfrac{e^{xp_k}}{e^{p_k^2}}} \right\|_{L_\infty[0,\infty)}^{1/p_n^2} = 0.$$

THEOREM 46 (Reddy [26, Theorem 5]). *Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$, $a_k > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be any entire function of order ρ $(1 \leqslant \rho < \infty)$ type τ and lower type $\omega (0 < \omega \leqslant \tau < \infty)$. Let $\phi(Z)$ be any transcendental entire function with non-negative coefficients satisfying the assumption that*

$$0 < \lim_{r\to\infty} \frac{\log M_\phi(r)}{(\log r)^2} = \theta < 1.$$

Then for every $g_n(x) = \sum_{k=0}^{\infty} b_k\{\phi(x)\}^k$, with $b_k \geqslant 0$, we have

$$\liminf_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} b_k\{\phi(x)\}^k} \right\|_{L_\infty[0,\infty)}^{\rho/n(\log n)(\log\log n)} > e^{-\tau/\omega}.$$

Remark. There exist entire functions of infinite order, whose reciprocals can be approximated by reciprocals of $\sum_{k=0}^{n} \{\phi(x)\}^k (k!)^{-1}$ or $[0, \infty)$, with an error C^n $(0 < C < 1)$. For example let

$$f(Z) = \sum_{k=0}^{\infty} \frac{\{\phi(Z)\}^k}{k!},$$

where

$$\phi(Z) = 1 + \sum_{i=1}^{\infty} \frac{Z^i}{1^1 2^2 3^3 \cdots i^i},$$

clearly $f(Z)$ is an entire function of infinite order. For this function we can show easily

$$\limsup_{n\to\infty} \left\| \frac{1}{f(x)} - \frac{1}{\sum_{k=0}^{n} \frac{\{\phi(x)\}^k}{k!}} \right\|_{L_\infty[0,\infty)}^{1/n} < 1.$$

THEOREM 47 (Saff and Varga [32]). *Assume that g is a continuous function ($\neq 0$) on $[0, \infty)$, and assume that there exist a sequence of polynomials $\{P_n(x)\}_{n=1}^{\infty}$, with $P_n \in \pi_n$ for each $n \geqslant 1$, and a real number $q > 1$ such that*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{g(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \leqslant \frac{1}{q} < 1.$$

Then as is known [17, *Theorem* 3], *there exists an entire function $G(Z)$ of finite order with $G(x) = g(x)$ for all $x \geqslant 0$. Next, assume that h is a continuous function on $[0, +\infty)$ with $h(x) > 0$ for all $x > 0$, and such that $h'(x)$ exists, is nonnegative for all x large, and satisfies $\lim_{x\to\infty} h'(x) = 0$. Assume further that no zeros of P_n lie in the interior of H_1 (defined in* (2.7)) *for all n sufficiently large. If D satisfies* (2.9) *and if G is nonzero on the vertical segment $\{Z = iy: |y| \leqslant Dh(0)\}$, then*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{G(Z)} - \frac{1}{P_n(Z)} \right\|_{L_\infty(H_D)} \right\}^{1/n} \leqslant \frac{1}{q} \left(\frac{1+D}{1-D} \right)^2 < 1.$$

THEOREM 48. *Let $g(Z) = \sum_{k=0}^{\infty} Z^k a^{-k^2}$, where $a \geqslant 2$, and let $S_n(Z) = \sum_{k=0}^{n} Z^k a^{-k^2}$. Then, on every closed sector $\bar{S}(\theta)$ (defined in* (2.10)) *with $0 < \theta < \pi$*

$$\lim_{n\to\infty} \left\{ \left\| \frac{1}{g} - \frac{1}{S_n} \right\|_{L_\infty(\bar{S}(\theta))} \right\}^{1/n^2} = \frac{1}{\sqrt{a}}.$$

THEOREM 49. *Let* $f(Z) = 1 + \sum_{k=1}^{\infty} Z^k/d_1 d_2 \cdots d_k$, *with* $d_{k+1} > d_k \geqslant 1$ $(k \geqslant 1)$ *be an entire function of order* $\rho < 2$. *If there exist a sequence of polynomials* $\{P_n(x)\}_{n=1}^{\infty}$ *with* $P_n \in \pi_n$ *for each* $n \geqslant 1$ *and a real constant* $q > 1$ *such that*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(x)} - \frac{1}{P_n(x)} \right\|_{L_\infty[0,\infty)} \right\}^{1/n} \leqslant \frac{1}{q} < 1.$$

If D satisfies (2.9), *then*

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{F(Z)} - \frac{1}{P_n(Z)} \right\|_{L_\infty(B(D))} \right\}^{1/n} \leqslant \frac{1}{q} \left(\frac{1+D}{1-D} \right)^2 < 1.$$

We need the following lemma for our purpose

LEMMA 2. *Let* $f(Z) = 1 + \sum_{k=1}^{\infty} Z^k/d_1 d_2 \cdots d_k$ *be an entire function satisfying the assumption that*

$$\left(1 + \frac{C_1}{n}\right) \leqslant \frac{d_{n+1}}{d_n} \leqslant \left(1 + \frac{C_2}{n}\right).$$

Then $S_l(Z) = 1 + \sum_{k=1}^{l} Z^k/d_1 d_2 \cdots d_k$, $l \geqslant n$ *is zero free in a region bounded by* $r = C_3 d_n$ *and* $\theta = \pm n^{-((1/2)+\epsilon)}$, *where* $Z = re^{i\theta}$, *and as usual* $C_1, C_2, C_3, \ldots$ *are suitable constants.*

Proof. Let $Z = re^{i\theta}$, $t_n = a_n Z^n$, $a_n = (d_1 d_2 \cdots d_n)^{-1}$, and assume $d_n \leqslant r < d_{n+1}$, then clearly nth term of $f(Z)$ becomes the maximum term. To be very precise, let $r = d_n$. Then

$$\begin{aligned} S_l(Z) &= t_l + t_{l-1} + \cdots + t_n + t_{n-1} + \cdots + t_{n-k} + \cdots + t_1 + t_0 \\ &= t_n \left(\frac{t_0}{t_1} + \frac{t_1}{t_n} + \cdots + \frac{t_{n-k-1}}{t_n} \right) + t_{n-k} \left(1 + \frac{t_{n-k+1}}{t_{n-k}} + \cdots + \frac{t_{n-1}}{t_{n-k}} \right) \\ &\quad + t_n + t_n \left(\frac{t_{n+1}}{t_n} + \frac{t_{n+2}}{t_n} + \cdots + \frac{t_{n+k}}{t_n} \right) + t_n \left(\frac{t_{n+k+1}}{t_n} + \cdots + \frac{t_l}{t_n} \right). \end{aligned}$$

We show here for all $k \geqslant n^{(1/2)+\epsilon}$

$$t_{n-k} = o(t_n). \tag{3.44}$$

By definition

$$\left| \frac{t_{n-k}}{t_n} \right| = \left| \frac{d_{n-k+1} d_{n-k+2} \cdots d_n}{Z^k} \right| \leqslant \left| \frac{d_{n-C_4 k}}{d_n} \right|^k.$$

Now by using the fact that

$$1 + \frac{C_1}{n} \leqslant \frac{d_{n+1}}{d_n} \leqslant 1 + \frac{C_2}{n}.$$

We get

$$\left| \frac{t_{n-k}}{t_n} \right| \leqslant \left(\frac{d_{n-C_8k}}{d_n} \right)^k \leqslant \left(1 + \frac{C_8 k}{n} \right)^{-k} \leqslant e^{-C_8k^2/n}.$$

If $k \geqslant n^{(1/2)+\epsilon}$, then clearly $e^{-C_8k^2/n} \to 0$. Similarly we can show if $k \geqslant n^{(1/2)+\epsilon}$

$$t_{n+k} = o(t_n). \tag{3.45}$$

On the other hand

$$\begin{aligned}
&\left| 1 + \frac{t_{n-k+1}}{t_{n-k}} + \frac{t_{n-k+2}}{t_{n-k}} + \cdots + \frac{t_{n-1}}{t_{n-k}} \right| \\
&\quad \leqslant \left| 1 + \frac{Z}{d_{n-k+1}} + \frac{Z^2}{d_{n-k+1}d_{n-k+2}} + \frac{Z^3}{d_{n-k}d_{n-k+3}} + \cdots + \frac{Z^k}{d_{n-k+1} \cdots d_{n-k}} \right| \\
&\quad \leqslant \left| 1 + \left(\frac{d_n}{d_{n-k+1}} \right) + \left(\frac{d_n}{d_{n-k+1}} \right)^2 + \cdots + \left(\frac{d_n}{d_{n-k+1}} \right)^k \right| \\
&\quad \leqslant \left| 1 + C_9 \left(\frac{d_{n-k}}{d_{n-k+1}} \right) + C_9 \left(\frac{d_{n-k}}{d_{n-k+1}} \right)^2 + \cdots + C_9 \left(\frac{d_{n-k}}{d_{n-k+1}} \right)^k \right| \\
&\quad \leqslant \frac{1}{1 - \left(\frac{d_{n-k}}{d_{n-k+1}} \right)} \leqslant C_{10}(n-k).
\end{aligned} \tag{3.46}$$

Similarly

$$\left| \frac{t_{n+1}}{t_n} + \frac{t_{n+2}}{t_n} + \cdots + \frac{t_{n+k}}{t_n} \right| \leqslant C_{11} n. \tag{3.47}$$

Hence, we get from (3.44)–(3.47).

$$\begin{aligned}
S_l(Z) &= t_n + t_n C_{12} n + C t_{n-k}(n-k)C + t_n C_{13} \\
&= t_n(1 + C_{12} n + C_{13}) + C(n-k)\, t_{n-k} \\
&= C_{14} n t_n + C_{15}(n-k)\, t_{n-k}.
\end{aligned}$$

$$S_l(Z) = 0, \quad \text{if} \quad C_{14} n t_n = -C_{15}(n-k)\, t_{n-k}$$

i.e.,

$$\frac{C_{16} Z^n}{d_1 d_2 \cdots d_n} = - \frac{Z^{n-k}(n-k)\, C_{16}}{(n)}.$$

Then clearly

$$r^k = C_{17} d_{n-k+1} d_{n-k+2} \cdots d_n \frac{(n-k)}{n}$$
$$= C_{18}(d_n)^k (1 - k/n)$$

$$e^{i\theta_1 k} = e^{(2j+1)i\pi}, \qquad j = 0, \pm 1, \pm 2, \pm 3, \pm 4 \pm \cdots$$

$$\theta_1 = \frac{(2j+1)}{k}.$$

Hence $S_n(Z)$ is zero free in a region bounded by

$$r = C_3 d_n, \qquad \theta = \pm C_{20} n^{-((1/2)+\epsilon)}.$$

Proof of the Theorem. From the above lemma it follows that $S_n(Z)$ is zero free in $\bar{H}_D$. Now by adopting the reasoning of Theorem 47 the result follows.

Remark. For functions of order $\rho \geqslant 2$, clearly y becomes zero, hence the theorem is proved for $\rho < 2$.

THEOREM 50. *Let $f(Z) = 1 + \sum_{k=1}^{\infty} Z^k (d_1 d_2 \cdots d_k)^{-1}$, be an entire function with $d_{k-1} + d_k < d_{(k+\alpha)}$ $(0 < \alpha < 1, k \geqslant 2)$, where d_k is positive and continuous for all positive values of $k \geqslant 1$ and increases to $+\infty$ with k. If there exist a $\psi(n) > n$ and a ξ $(0 < \xi < 1)$ for which*

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/\psi(n)} = \xi.$$

Then on every closed sector $\bar{S}(\theta)$, with $0 < \theta < \pi$,

$$\lim_{n\to\infty} (\tilde{\lambda}_{0,n})^{1/\psi(n)} = \xi,$$

where

$$\tilde{\lambda}_{0,n} = \inf_{P_n(Z)\in\pi_n^*} \left\| \frac{1}{f(Z)} - \frac{1}{P_n(Z)} \right\|_{L_\infty(S(\theta))}.$$

π_n^* denote the set of all complex polynomials of degree at most n in the variable Z.

We need the following lemma to prove the above theorem.

LEMMA 3. *Let $f(Z) = 1 + \sum_{k=1}^{\infty} Z^k (d_1 d_2 \cdots d_k)^{-1}$, $d_{k-1} + d_k < d_{(k+\alpha)}$ $(0 < \alpha < 1)$, where d_k is positive and continuous for all positive values of*

$k \geqslant 1$ and increases steadily to $+\infty$ with k. Then all the partial sums of $f(Z)$ have zeros on the negative real axis only.

Proof. Let $Z = re^{i\theta}$, $t_n = a_n Z^n$, $a_n = (d_1 d_2 \cdots d_n)^{-1}$ $T_n = | t_n |$ and assume $d_n \leqslant r < d_{n+1}$, then clearly nth term of $f(Z)$ becomes the maximum term. To be very precise let $r = d_{n+\alpha}$ $(0 < \alpha < 1)$, then

$$S_n(Z) = \sum_{k=0}^{n} Z^k (d_1 d_2 \cdots d_k)^{-1} = t_n + t_{n-1}\Big(1 + \frac{t_{n-2}}{t_{n-1}} + \frac{t_{n-3}}{t_{n-2}} + \cdots + \frac{t_0}{t_1}\Big)$$

$$= t_n + t_{n-1}\Big(1 + \frac{d_{n-1}}{Z} + \frac{d_{n-2}d_{n-1}}{Z^2} + \cdots + \frac{d_1 d_2 \cdots d_{n-1}}{Z^{n-1}}\Big).$$

It is easy to verify that

$$\frac{d_{n-1}}{| Z |} = \Big(\frac{d_{n-1}}{d_{n+\alpha}}\Big), \qquad \Big| \frac{d_{n-2}d_{n-1}}{Z^2} \Big| \leqslant \Big(\frac{d_{n-1}}{d_{n+2}}\Big)^2.$$

Therefore

$$\Big| \frac{d_{n-1}}{Z} + \frac{d_{n-2}d_{n-1}}{Z^2} + \cdots + \frac{d_1 d_2 \cdots d_{n-1}}{Z^{n-1}} \Big|$$

$$\leqslant \frac{d_{n-1}}{d_{n+\alpha}} + \Big(\frac{d_{n-1}}{d_{n+\alpha}}\Big)^2 + \cdots + \Big(\frac{d_{n-1}}{d_{n+\alpha}}\Big)^n$$

$$= \frac{1}{1 - \Big(\dfrac{d_{n-1}}{d_{n+\alpha}}\Big)} - 1$$

$$= \frac{d_{n-1}}{d_{n+\alpha} - d_{n-1}}.$$

Hence $S_n(Z) = t_n(1 + \beta)$, where

$$| \beta | \leqslant \frac{| t_{n-1} |}{| t_n |}\Big(1 + \frac{d_{n-1}}{d_{n+\alpha} - d_{n-1}}\Big) = \Big(\frac{d_n}{d_{n+\alpha}}\Big)\Big(\frac{d_{n+\alpha}}{d_{n+\alpha} - d_{n-1}}\Big)$$

$$= \Big(\frac{d_n}{d_{n+\alpha} - d_{n-1}}\Big).$$

But by our assumption $d_n/(d_{n+\alpha} - d_{n-1}) < 1$, therefore $S_n(Z)$ has "n" zeros in the circle $r = d_{n+\alpha}$, and $(n - 1)$ zeros in the circle $r = d_{n-1+\alpha}$ and therefore one zero between the circle $d_{n-1+\alpha}$ and $d_{n+\alpha}$. Let $Z = -d_{n+\alpha}$, then $S_n(Z)$ has the sign of t_n, i.e. of $(-1)^n$ and when

$Z = -d_{n-1+s}$ it has the sign of $(-1)^{n-1}$. Hence $S_n(Z)$ has a real negative zero lying between the circles $r = d_{n+s}$ and d_{n-1+s}. We noted earlier that only one zero lies between two circles (successive circles) hence all the zeros are real and negative.

Proof of the Theorem. From the statement of the theorem it is clear that $f(Z)$ is an entire function of zero order. For example $d_k = \delta^{2^{k-1}}$ satisfies the assumptions of the above theorem with $A = 0$. For this d_k it is known (Theorem 40, example 2) that $\psi(n) = 2^{n+1}$, $\xi = (1/\delta)$ $(1 < \delta < \infty)$. Now by adopting the reasoning of Theorem 48, we get

$$\limsup_{n\to\infty} \left\{ \left\| \frac{1}{f(Z)} - \frac{1}{S_n(Z)} \right\|_{S(\psi)} \right\}^{1/\psi(n)} \leqslant \xi. \tag{3.48}$$

On the other hand we get easily from Theorem 40, that

$$\xi \leqslant \liminf_{n\to\infty} (\lambda_{0,n})^{1/\psi(n)} \leqslant \liminf_{n\to\infty} (\bar{\lambda}_{0,n})^{1/\psi(n)}. \tag{3.49}$$

The result follows from (3.48) and (3.49).

Open Problems

Problem 1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order ρ $(0 < \rho < \infty)$, with the further assumption that

$$\limsup_{n\to\infty} (\lambda_{0,n})^{1/n} < 1.$$

Then there is a c $(0 < c < 1)$ such that

$$\liminf_{n\to\infty} (\lambda_{0,n})^{1/n} \geqslant c.$$

Problem 2. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ $(k \geqslant 0)$ be an entire function of order ρ $(0 < \rho < \infty)$ with the further assumption that $0 < \omega = \tau < \infty$. Then there is a $\delta > 1$, for which

$$\lim_{n\to\infty} (\lambda_{0,n})^{1/n} = \delta^{-1}.$$

Remark. For $f(x) = e^x$, $\delta = 3$ (cf. [33]).

PROBLEM 3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ (>0 on $[0, \infty)$) be an entire function with the additional property that it grows on $[0, \infty)$ as fast as anywhere else in the complex plane, then for each $\epsilon > 0$, there exist infinitely many n for which

$$\lambda_{0,n} \leqslant \exp\left(\frac{-n}{(\log n)^{1+\epsilon}}\right).$$

PROBLEM 4. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $k \geqslant 1)$ be an entire function of order $\rho(0 < \rho < \infty)$ type τ and lower type ω $(0 < \omega \leqslant \tau < \infty)$. Then for any polynomials $P(x)$ and $Q(x)$ of degree less than n, there is a $c_1 > 1$ for which

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geqslant c_1^{-n}.$$

Remark. For $f(x) = e^x$, $c_1 = 1280$ (cf. [18]).

PROBLEM 5. Let $f(x)$ be any nonvanishing infinitely differentiable and monotonic function tending to $+\infty$. Then for infinitely many n

$$\lambda_{0,n} \leqslant \frac{1}{(\log n)}.$$

PROBLEM 6. Let $f(x)$ be any non-vanishing infinitely differentiable and monotonic function tending to $+\infty$. Then, there exist polynomials of the form

$$Q(x) = \sum_{i=0}^{k} a_i x^{n_i}$$

with $n_0 = 0$, $n_0 < n, < n_2 < n_3 < \cdots$, $\sum_{i=1}^{\infty} 1/n_i = \infty$, for which for infinitely many k

$$\left\| \frac{1}{f(x)} - \frac{1}{Q(x)} \right\|_{L_\infty[0,\infty)} \leqslant \left(\frac{1}{\log\log n_k}\right).$$

PROBLEM 7. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function. Then for infinitely many k and any $c > 1$,

$$\left\| \frac{1}{f(x)} - \frac{1}{Q(x)} \right\|_{L_\infty[0,\infty)} \leqslant \frac{1}{(\log n_k)^c}.$$

PROBLEM 8. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function of order $\rho(0 < \rho < \infty)$ type τ and lower type $\omega(0 < \omega \leqslant \tau < \infty)$. Then there exist polynomials of the form

$$Q(x) = \sum_{i=0}^{k} a_i x^{n_i},$$

where

$$0 = n_0 < n_1 < n_2 < \cdots < n_i, \qquad \sum_{i=1}^{\infty} \frac{1}{n_i} = \infty,$$

for which for infinitely many k

$$\left\| \frac{1}{f(x)} - \frac{1}{Q(x)} \right\|_{L_\infty[0,\infty)} \leqslant \frac{1}{n_k}.$$

PROBLEM 9. Let $f(x)$ be any entire function satisfying the assumption that $\lim_{x\to\infty} f(x)$ is finite. Then there exist rational functional of the form $(P_n(x)/Q_n(x))$ of the degree at most n for which for each $\epsilon > 0$ there exist infinitely many n, such that,

$$\left\| \frac{1}{f(x)} - \frac{P_n(x)}{Q_n(x)} \right\|_{L_\infty[0,\infty)} \leqslant \exp\left(\frac{-n}{(\log n)^{1+\epsilon}} \right).$$

PROBLEM 10. Let $f(z)$ and $g(z)$ be entire functions of perfectly regular growth (ρ, τ), $(\rho + \epsilon, \tau)$ respectively for any $\epsilon > 0$. Then for all large n

$$\lambda_{0,n}(1/f) \leqslant \lambda_{0,n}(1/g).$$

CONCLUDING REMARKS

It is clear from Theorem 3A continuous functions which maintain sign and satisfy $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to 0} f(x)$ cannot be approximated well. The method used to obtain a lower bound in Example 1 can be applied to any function which vanishes at the origin and tends to zero at infinity. As far as we know no other method is known to attack this kind of problem. This method was first used in [5] which is slight variation of the technique we used in [9].

The method used in Theorem 32 can be applied very successfully to find lower bounds to $\lambda_{0,n}(1/f)$, where f is an arbitrary entire function. Unfortunately for entire functions of perfectly regular growth this method does not yield sharper bounds. For functions of this category the method used in [16] and [20] is very successful. Unfortunately the method used in [16] and [20] is useless for entire functions where lower order is less than order.

The method used to prove Theorem 13 is very elementary and can be applied to all those entire functions for which we know the upper bound of $M(r)$ for all large r.

It is interesting to know, what connection exists between the structural properties of f and the rate of convergence of

$$\limsup_{n\to\infty} \left\| \frac{1}{f} - \frac{P_n}{Q_n} \right\|_{L_\infty[0,\infty)}^{1/n} < 1.$$

It is likely that $f(x)$ is quasianalytic in the sense of S. N. Bernstein.

Let $f(Z) = \sum_{k=0}^{\infty} a_k Z^k$, $a_0 > 0$ and $a_k \geqslant 0$ $(k \geqslant 1)$ be an entire function satisfying the further assumption that

$$0 < \lim_{r\to\infty} \frac{\log\log M(r)}{\log r} = \rho < \infty, \tag{*}$$

then it is known ([35, p. 43]) that there exist a subsequence $\{n_p\}$ of natural numbers satisfying the assumptions that

$$\lim_{p\to\infty} \frac{\log n_{p+1}}{\log n_p} = 1,$$

and

$$\lim_{p\to\infty} \frac{n_p \log n_p}{\log \left| \dfrac{1}{a_{n_p}} \right|} = \rho. \tag{A}$$

If $f(Z)$ satisfies the assumption (*) then it follows from Theorems (31) and (32) that

$$\lim_{n\to\infty} \frac{\log\log \dfrac{1}{\lambda_{0,n}}}{\log n} = 1 \tag{B}$$

From (A) and (B) we get for the above sequence $\{n_p\}$

$$n_p \log\log\left(\frac{1}{\lambda_{0,n_p}}\right) \sim \rho \log \left| \frac{1}{a_{n_p}} \right|.$$

It is interesting to observe certain class of continuous function which vanish at the origin and tends to zero at infinity can be approximated much better by rational functions than by reciprocals of polynomials. One such example is $f(x) = xe^{-x}$, this can be approximated by P_n/Q_n roughly with an error of 2^{-n}, this cannot be approximated by reciprocals of polynomials even like $c \log n/n^2$.

Just recently the Problem 4 has been solved completely in [28].

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