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PROBLEMS AND RESULTS IN RATIONAL APPROXIMATION

by

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Several mathematicians have discussed (cf. [3]–[9]) the problem of approximating reciprocals of certain entire functions by reciprocals of polynomials under the uniform norm on the positive real axis. The question of approximating reciprocals of non-entire functions by reciprocals of polynomials is left open. In this note we obtain, a few results for non-entire functions (we approximate functions which are continuous and tend to zero by reciprocals of polynomials, proofs of these results are very simple) and also a few results for entire functions. Finally, we present some open problems.

NOTATIONS. Let π_n denote the class of all algebraic polynomials of degree at most n, π_n^* denote the class of all algebraic integervalued polynomials (polynomials which assume integervalues at integers) of degree at most n.

Let $f(x)$ be any non-vanishing continuous function on $[0, \infty)$. Denote

$$(1) \qquad \lambda_{0,n} = \inf_{P(x)\in\pi_n} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{L^\infty[0,\infty)},$$

$$(2) \qquad \bar{\lambda}_{0,n} = \inf_{P(x)\in\pi_n^*} \left\| \frac{1}{f(x)} - \frac{1}{p(x)} \right\|_{L^\infty[0,\infty)}.$$

$S_n(z)$ denotes the n-th partial sum of $f(z) = \sum_{k=0}^{\infty} a_k z^k$. $c_1, c_2, c_3, \ldots, c_k$ are suitable constants.

THEOREM 1. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k \geq 0$ be an entire function of finite order ϱ. Then for each $\varepsilon > 0$,*

$$(3) \qquad \liminf_{n\to\infty} (\lambda_{0,n})^{\frac{\varrho+\varepsilon}{n}} \leq (1.4)^{-1}.$$

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PROOF. From the notation it is clear that

$$0 \leq \frac{1}{S_{2n-1}(x)} - \frac{1}{f(x)} \leq \frac{\sum_{k=2n}^{\infty} a_k x^k}{a_n^2 x^{2n}}. \tag{4}$$

Since $f(z)$ is an entire function of finite order ϱ, we get for any $\varepsilon > 0$,

$$\lim_{n\to\infty} n^{\frac{1}{\varrho+\varepsilon}} |a_n|^{\frac{1}{n}} = 0 \qquad ([2], \text{p. } 8).$$

Then there exist infinitely many n for which

$$(n+l)^{\frac{1}{\varrho+\varepsilon}} |a_{n+l}|^{\frac{1}{n+l}} \leq n^{\frac{1}{\varrho+\varepsilon}} |a_n|^{\frac{1}{n}}, \quad l = 0, 1, 2, 3, \ldots . \tag{5}$$

From (5), we obtain by taking $l = n + j,\ j = 0, 1, 2, \ldots,$

$$|a_{2n+j}| \leq \left(\frac{n}{2n+j}\right)^{\frac{2n+j}{\varrho+\varepsilon}} |a_n|^{\frac{2n+j}{n}}. \tag{6}$$

Set

$$|a_n|^{\frac{1}{n}} x < (1.96)^{\frac{1}{\varrho+\varepsilon}}. \tag{7}$$

Then from (4), (6) and (7) we get

$$\frac{\sum_{k=2n}^{\infty} a_k x^k}{a_n^2 x^{2n}} \leq \frac{\sum_{j=0}^{\infty} a_{2n+j} x^j}{a_n^2} \leq \sum_{j=0}^{\infty} |a_n|^{\frac{j}{n}} 2^{-\left(\frac{2n+j}{\varrho+\varepsilon}\right)} x^j \leq \tag{8}$$

$$\leq \sum_{j=0}^{\infty} \left(\frac{1.96}{2}\right)^{\frac{j}{\varrho+\varepsilon}} 2^{-\left(\frac{2n}{\varrho+\varepsilon}\right)} \leq c_1 2^{-\frac{2n}{\varrho+\varepsilon}}.$$

On the other hand, let

$$|a_n|^{\frac{1}{n}} x \geq (1.96)^{\frac{1}{\varrho+\varepsilon}}.$$

Then

$$0 \leq \frac{1}{S_{2n-1}(x)} - \frac{1}{f(x)} \leq \frac{1}{S_{2n-1}(x)} \leq \frac{1}{a_n x^n} \leq \frac{1}{(1.96)^{\frac{n}{\varrho+\varepsilon}}}. \tag{9}$$

From (8) and (9) we get the result (3).

REMARK. This result improves a recent result of ERDŐS and REDDY ([3], Theorem 3).

THEOREM 2A. *Let* $g(z) = (z+1)^{n+1}$. *Then*

$$\bar{\lambda}_{0,n} \leq 2^{-n}, \qquad n = 1, 2, 3, \ldots .$$

Proof. It is easy to see that for all $x \geq 0$

$$0 \leq \frac{1}{\sum_{k=0}^{n} \binom{n+1}{k} x^k} - \frac{1}{(x+1)^{n+1}} \leq \left(\frac{x}{x+1}\right)^{n+1}. \tag{10}$$

Set

$$0 \leq x < 1.$$

Then

$$\left(\frac{x}{x+1}\right)^{n+1} \leq 2^{-(n+1)}. \tag{11}$$

On the other hand for $x \geq 1$

$$0 \leq \frac{1}{\sum_{k=0}^{n} \binom{n+1}{k} x^k} - \frac{1}{(x+1)^{n+1}} \leq \frac{1}{\sum_{k=0}^{n} \binom{n+1}{k} x^k} \leq 2^{-n}. \tag{12}$$

Therefore we have from (10), (11) and (12) the required result.

Remarks. It is natural to ask whether one can approximate $(x+1)^{-(n+1)}$ by reciprocals of unrestricted polynomials under the uniform norm with an error much better than a $\delta^n (0 < \delta < 1)$. We show in the next theorem that one cannot.

Theorem 2B. *Let* $f(z) = (z+1)^{n+1}$. *Then*

$$\liminf_{n \to \infty} (\lambda_{0,n})^{\frac{1}{n}} \geq \frac{1}{16}.$$

Proof. For each $\varepsilon > 0$, $0 \leq x \leq 1 - \varepsilon$ and all large n,

$$0 \leq f(x) \leq f(1-\varepsilon) \leq (2-\varepsilon)^{n+1} < 2^n \leq \frac{1}{\lambda_{0,n}}, \tag{13}$$

(cf. Theorem 2A). Now consider $q_n(x) \in \pi_n$ which gives best approximation in sense of uniform norm, that is

$$\lambda_{0,n} \equiv \max \left| \frac{1}{f(x)} - \frac{1}{q_n(x)} \right|_{[0,\infty)}. \tag{14}$$

From (14) we get, with a simple calculation for $0 \leq x \leq 1 - \varepsilon$,

$$-\frac{f^2(x)}{f(x) + (\lambda_{0,n})^{-1}} \leq q_n(x) - f(x) \leq \frac{f^2(x)}{(\lambda_{0,n})^{-1} - f(x)}. \tag{15}$$

Since the right-hand side of (15) is monotonic increasing with $f(x)$, we can write from (13),

$$|q_n(x) - f(x)| \leq \frac{(2-\varepsilon)^{2(n+1)}}{(\lambda_{0,n})^{-1} - (2-\varepsilon)^{n+1}} \qquad 0 \leq x \leq 1 - \varepsilon. \tag{16}$$

Now let

$$E_n[(x+1)^{n+1}] = \inf_{p_n \in \pi_n} \sup |p_n(x) - (x+1)^{n+1}|_{[0,1-\varepsilon]}. \tag{17}$$

From (16) and (17) we get

$$E_n \leq \frac{(2-\varepsilon)^{2(n+1)}}{(\lambda_{0,n})^{-1} - (2-\varepsilon)^{n+1}}. \tag{18}$$

To obtain a lower bound for E_n, we use a result of BERNSTEIN [1, p. 10] which gives for the interval $[0, 1-\varepsilon]$

$$E_n \geq \frac{(1-\varepsilon)^{n+1}}{2^{2n+1}}. \tag{19}$$

Now by (18) and (19) we have

$$\frac{(1-\varepsilon)^{n+1}}{2^{2n+1}} \leq \frac{(2-\varepsilon)^{2(n+1)}}{(\lambda_{0,n})^{-1} - (2-\varepsilon)^{n+1}} \text{ for all large } n. \tag{20}$$

From (20) we get with a simple calculation

$$\lambda_{0,n} \geq \frac{c_2(1-\varepsilon)^n}{[2(2-\varepsilon)]^{2n}}. \tag{21}$$

ε being arbitrary, (21) gives us the required result.

REMARK. It is very likely the limit may exist in Theorem 2B.[1]

THEOREM 3. *Let $f(z) = \sum_{k=0}^{\infty} (k+1)z^k$. Then for all large n,*

$$\lambda_{0,n} \leq \frac{c_3}{n}. \tag{22}$$

PROOF. As earlier we get for $0 < x < 1$

$$0 \leq \frac{1}{\sum_{k=0}^{n} (k+1)x^k} - \frac{1}{f(x)} \leq \frac{\sum_{k=n+1}^{\infty} (k+1)x^k}{f(x) \sum_{k=0}^{n} (k+1)x^k} \leq (n+2)\, x^{n+1}. \tag{23}$$

Set

$$x < 1 - \frac{2\log n}{n}.$$

Then

$$(n+2)x^{n+1} \leq \frac{c_3}{n}. \tag{24}$$

[1] Added in proof: Recently D. J. NEWMAN has shown the limit to be $\left(\frac{4}{27}\right)$.

On the other hand we get for

$$x \geq 1 - \frac{2 \log n}{n}$$

(25) $$0 \leq \frac{1}{\sum_{k=0}^{n} (k+1)x^k} - \frac{1}{f(x)} \leq \frac{1}{\sum_{k=0}^{n} (k+1)x^k} \leq \frac{1}{\sum_{k=0}^{n} (k+1)\left(1 - \frac{\log n}{n}\right)^k} \sim$$

$$\sim \left(\frac{2 \log n}{n}\right)^2 < c_4 \left(\frac{\log n}{n}\right)^2.$$

We obtain the required result from (24) and (25).

THEOREM 4. *Let* $f(z) = \sum_{k=0}^{\infty} z^k$. *Then for all large* n

(26) $$\lambda_{0,n} \leq \frac{c_5 \log n}{n}.$$

PROOF. As usual

(27) $$0 \leq \frac{1}{\sum_{k=0}^{n} x^k} - \frac{1}{\sum_{k=0}^{\infty} x^k} \leq \frac{\sum_{k=n+1}^{\infty} x^k}{\sum_{k=0}^{\infty} x^k} = x^{n+1}.$$

Set

$$x < 1 - \frac{\log n}{n}.$$

Then

(28) $$x^{n+1} \leq \left(1 - \frac{\log n}{n}\right)^{n+1} \leq n^{-\frac{n+1}{n}}.$$

On the other hand for $x \geq \left(1 - \frac{\log n}{n}\right)$

(29) $$0 \leq \frac{1}{\sum_{k=0}^{n} x^k} - \frac{1}{\sum_{k=0}^{\infty} x^k} \leq \frac{1}{\sum_{k=0}^{n} x^k} \leq \frac{1}{\sum_{k=0}^{n} \left(1 - \frac{\log n}{n}\right)^k} \leq \frac{c_5 \log n}{n},$$

because for all large n

$$\sum_{k=0}^{n} x^k \geq \sum_{k=0}^{n} \left(1 - \frac{\log n}{n}\right)^k \sim \frac{c_6(n-1)}{\log n}.$$

The result (26) follows from (28) and (29).

Theorem 5. *Let* $f(z) = 1 + \sum_{k=1}^{\infty} \frac{z^k}{k}$. *Then for all sufficiently large* n

$$\lambda_{0,n} \leq \frac{c_7}{\log n}. \tag{30}$$

Proof. As usual it is easy to see for $0 \leq x < 1 - \frac{\log n}{n}$

$$0 \leq \frac{1}{1 + \sum_{k=1}^{n} \frac{x^k}{k}} - \frac{1}{1 + \sum_{k=0}^{\infty} \frac{x^k}{k}} \leq \frac{\sum_{k=n+1}^{\infty} \frac{x^k}{k}}{1 + \sum_{k=0}^{\infty} \frac{x^k}{k}} \leq x^{n+1} \leq \frac{c_8}{n}. \tag{31}$$

On the other hand for

$$x \geq 1 - \frac{\log n}{n}$$

(32)

$$0 \leq \frac{1}{1 + \sum_{k=1}^{n} \frac{x^k}{k}} - \frac{1}{1 + \sum_{k=1}^{\infty} \frac{x^k}{k}} \leq \frac{1}{1 + \sum_{k=1}^{n} \frac{x^k}{k}} \leq \frac{1}{1 + \sum_{k=1}^{n} \frac{1}{k}\left(1 - \frac{\log n}{n}\right)^k} \leq \frac{c_9}{\log n}.$$

Hence the result.

Concluding remarks. By observing Theorems 3,4 and 5, it is easy to note that the error ($\lambda_{0,n}$) becomes less when the function $f(x)$ grows faster. This fact is quite contrary to the known case of entire functions (cf. [4]–[7]), where the error (for all large n) becomes smaller for functions which grow regularly and of small growth in comparison with those which grow regularly and of fast growth. Whether or not this is only an isolated case or there is a general result for non-entire functions is not clear.

Open problems

Problem 1. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_0 > 0$, $a_k \geq 0$ $(k \geq 1)$ be an entire function of order ϱ $(0 < \varrho < \infty)$, with the further assumption that

$$\limsup_{n \to \infty} (\lambda_{0,n})^{\frac{1}{n}} < 1.$$

Then there is a c $(0 < c < 1)$ such that

$$\liminf_{n \to \infty} (\lambda_{0,n})^{\frac{1}{n}} \geq c.$$

Problem 2. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $a_k > 0$ $(k \geq 0)$ be an entire function of order ϱ $(0 < \varrho < \infty)$ with the further assumption that $0 < \omega = \tau < \infty$. Then there is a $\delta > 1$, for which

$$\lim_{n\to\infty} (\lambda_{0,n})^{\frac{1}{n}} = \delta^{-1}.$$

Remark. For $f(x) = e^x$, $\delta = 3$ (cf. [9]).

Problem 3. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ (>0 on $[0, \infty)$) be an entire function with the additional property that it grows on $[0, \infty)$ as fast as anywhere else in the complex plane, then for each $\varepsilon > 0$, there exist infinitely many n for which

$$\lambda_{0,n} \leq \exp\left(\frac{-n}{(\log n)^{1+\varepsilon}}\right).$$

Problem 4. Let $f(z) = \sum_{k=0}^{n} a_k z^k$, $a_0 > 0$, $a_k \geq 0$ $(k \geq 1)$ be an entire function of order ϱ $(0 < \varrho < \infty)$ type τ and lower type ω $(0 < \omega \leq \tau < \infty)$. Then for any polynomials $P(x)$ and $Q(x)$ of degree less than n, there is a $c_1 > 1$ for which

$$\left\| \frac{1}{f(x)} - \frac{P(x)}{Q(x)} \right\|_{L_\infty[0,\infty)} \geq c_1^{-n}.$$

Remark. For $f(x) = e^x$, $c_1 = 1280$ (cf. [8]).[2]

Problem 5. Let $f(x)$ be any non-vanishing infinitely differentiable and monotonic function tending to $+\infty$. Then for infinitely many n

$$\lambda_{0,n} \leq \frac{1}{\log n}.$$

Problem 6. Let $f(x)$ be any non-vanishing infinitely differentiable and monotonic function tending to $+\infty$. Then, there exist polynomials of the form

$$Q(x) = \sum_{l=0}^{k} a_l x^{n_l}$$

with $n_0 = 0, n_0 < n_1 < n_2 < n_3 < \ldots, \sum_{i=1}^{\infty} \frac{1}{n_i} = \infty$, for which for infinitely many k

$$\left\| \frac{1}{f(x)} - \frac{1}{Q(x)} \right\|_{L_\infty[0,\infty)} \leq \frac{1}{\log\log n_k}.$$

[2] Added in proof: A. R. Reddy has settled this problem.

Problem 7. Let $f(z)=\sum_{k=0}^{\infty} a_k z^k,\ a_0>0,\ a_k\geq 0\ (k\geq 1)$ be an entire function. Then for infinitely many k and any $c>1$,

$$\left\|\frac{1}{f(x)}-\frac{1}{Q(x)}\right\|_{L_{\infty[0,\infty)}}\leq\frac{1}{(\log n_k)^c}.$$

Problem 8. Let $f(z)=\sum_{k=0}^{\infty} a_k z^k,\ a_0>0,\ a_k\geq 0\ (k\geq 1)$ be an entire function of order ϱ $(0<\varrho<\infty)$ type τ and lower type ω $(0<\omega\leq\tau<\infty)$. Then there exist polynomials of the form

$$Q(x)=\sum_{i=0}^{k} a_i x^{n_i},$$

where

$$0=n_0<n_1<n_2<\ldots<n_i,\qquad \sum_{i=1}^{\infty}\frac{1}{n_i}=\infty,$$

for which for infinitely many k

$$\left\|\frac{1}{f(x)}-\frac{1}{Q(x)}\right\|_{L_{\infty[0,\infty)}}\leq\frac{1}{n_k}.$$

Problem 9. Let $f(x)$ be any entire function satisfying the assumption that $\lim_{x\to\infty} f(x)$ is finite. Then there exist rational functions of the form $\frac{P_n(x)}{Q_n(x)}$ of the degree at most n for which for each $\varepsilon>0$ there exist infinitely many n, such that,

$$\left\|\frac{1}{f(x)}-\frac{P_n(x)}{Q_n(x)}\right\|_{L_{\infty[0,\infty)}}\leq\exp\left(\frac{-n}{(\log n)^{1+\varepsilon}}\right).$$

Problem 10. Let $f(z)$ and $g(z)$ be entire functions of perfectly regular growth (ϱ,τ), $(\varrho+\varepsilon,\tau)$, respectively for any $\varepsilon>0$. Then for all large n

$$\lambda_{0,n}\left(\frac{1}{f}\right)\leq\lambda_{0,n}\left(\frac{1}{g}\right).$$

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