PROBLEMS AND RESULTS IN GRAPH THEORY AND COMBINATORIAL ANALYSIS

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I have published during my long life many papers on this subject. In this paper I will restrict myself almost entirely to finite problems and will concentrate on new problems, older questions will be mentioned only if they seem to me to be exceptionally attractive or if they have been (in my opinion) undeservably neglected. It is perhaps unnecessary to state that I do not claim that the problems discussed here are the most important ones, my choice of them is much more subjective - I include them because I am interested in them and have given them some thought - as stated in a previous paper this method of choice has the advantage that I am likely to know more about them than the reader.

First of all I give references to some of my previous papers on solved and unsolved combinatorial problems.

1. Problems and results on finite and infinite graphs, Recent advances in graph theory, Proc. Symp. Prague 1974, Academia Praha 1975, Editor M. Fiedler, 183-190.

2. Problems and results of combinatorial analysis, Symposium held in Rome September 1973 will appear soon.

3. Some unsolved problems in graph theory and combinatorial analysis, Combinatorial Math. and its Applications, Oxford Conference 1969, Acad. Press, London 1971, 97-109.

4. Problems and results in chromatic graph theory, Proof techniques in graph theory, New York Acad. Press 1969, 27-35.

PROC. 5TH BRITISH COMBINATORIAL
CONF. 1975, pp. 169-192.

5. Problems and results in combinatorial analysis, Proc. Symp. Pure Math. Vol. XIX, Combinatorics Amer. Math. Soc., 77-89.

6. (with D. Kleitman) Extremal problems among subsets of a set, Proc. second Chapel Hill Conference, Univ. of North Carolina, August 1970, 146-170, see also Discrete Math.

7. Extremal problems in graph theory, Theory of graphs and its applications, (M. Fiedler, editor) Proc. Symp. Smolenice 1963, Acad. Press, New York 1964, 29-36.

8. On some new inequalities concerning extremal properties of graphs, Theory of graphs, Proc. Coll. Tihany 1966, Acad. Press and Akadémiai Kiadó 1968, 77-81.

9. Some recent results on extremal problems in graph theory, Theory of graphs, International Symposium Rome 1966, 117-130.

10. Topics in combinatorial analysis, Proc. second Louisiana Conference on Combinatorics, Graph Theory and Computing, (R.C. Mullin et al editor) Louisiana State Univ., Baton Rouge 1971, 2-20.

11. Extremal problems on graphs and hypergraphs, Hypergraph Seminar held at Columbus, Ohio 1972, Lecture Notes in Mathematics 411, Springer Verlag, 75-83.

Finally, two papers in the Boca-raton Conference on combinatorial analysis February 1974 and March 1975 published I believe by Utilitas Math.

I will refer to these papers by their number in this list.

1. Lovász and I investigated the following question: Let f(n) be the smallest integer for which there is a family of sets $\{A_k\}$, $1 \le k \le f(n)$, $|A_k| = n$, $|A_{k_1} \cap A_{k_2}| \ge 1$ and the family can not be represented by n-1 elements. In other words: If $|\mathcal{S}| = n-1$ there always is a k so that $\mathcal{S} \cap A_k = \emptyset$.

We proved

(1) $$\frac{8n}{3} \le f(n) < n^{3/2+\epsilon}.$$

The upper bound is (1) can probably be improved to $f(n) < c\, n\log n$, but we have no idea if $f(n) < cn$ is true. I offer 50 pounds for a proof or disproof.

P. Erdös and L. Lovász, Problems and results on 3-chromatic hypergraphs and some related questions, Infinite and finite sets, Coll. Math. Soc. J. Bólyai 10 (Bólyai-North Holland, 1975).

2. Faber, Lovász and I conjectured that if $|A_k| = n$, $1 \le k \le n$ and $|A_{k_1} \cap A_{k_2}| \le 1$, $1 \le k_1 < k_2 \le n$ then one can colour the elements of the union $\bigcup_{k=1}^{n} A_k$ by n colours so that every set has elements of all the colours. It is very surprising that no progress has been made with this problem and I offer 50 pounds for a proof or disproof.

3. A family of sets $\{A_k\}$ is called a strong Δ system if the intersection of any two of them is the same i.e. the intersection of any two of them equals the intersection of all of them. It is called a weak Δ system if the intersection of any two of them has the same size. Rado and I and Milner, Rado and I settled all the problems on the existence of strong and weak Δ systems for infinite sets, but very challenging finite problems remain. Denote by $F(n,r)$ the smallest integer so that if $\{A_k\}$, $|A_k| = n$, $1 \le k \le F(n,r)$ is any family of sets then there always are r of them $A_{k_1},\dots,A_{k_r}$ which form a strong Δ system. Rado and I conjectured if C_r is a constant which only depends on r then

(1) $$F(n,r) < C_r^n.$$

I find (1) one of the most challenging unsolved problems and offer 300 dollars for a proof or disproof. (1) would have many applications in number theory and combinatorial analysis. Instead of (1) Rado and I proved

(2) $$F(n,r) < C_r'^{\,n} n!$$

Our value of C_r' has been improved by Abbott, Hanson and others but it is not yet known whether for every A and $n > n_0(A,r)$.

(3) $$F(n,r) < A^{-n}n!$$

(1) and (3) are unsolved even for $r = 3$. Abbott and Hanson also proved $F(n,3) > 10^{n/2}n^{-c}$. Abbott and Gardner further proved $F(3,3) = 21$.

Denote by $f(n,r)$ the smallest integer so that if $\{A_k\}$, $|A_k| = n$, $1 \le k \le f(n,r)$ then there are always r of them which form a weak Δ system. In our paper with Milner and Rado we conjectured

$$f(n,r) < c_r^n$$

but cannot even prove $f(n,r) < n!^{1-\varepsilon}$. Trivially $f(2,3) = 6$ and Hanson proved $f(3,3) = 11$, $f(4,3) \ge 26$. It is easy to see that $f(a+b,3)-1 \ge (f(a,3)-1)(f(b,3)-1)$ thus $\lim_{n=\infty} f(n,3)^{1/n}$ exists but we do not know whether it is finite or infinite.

Denote by $g(n,r)$ (respectively $\mathcal{g}(n,r)$) the smallest integer so that if $|\mathcal{S}| = n$ and $\{A_k\}$, $1 \le k \le g(n,r)$ (respectively $\mathcal{g}(n,r)$) is a family of subsets of $\mathcal{S}$ then our family has a subfamily of r sets forming a weak (respectively strong) Δ system. Abbott called attention to the fact that it is not obvious that

(4) $$\lim_{n=\infty} g(n,3)/n = \infty.$$

Szemerédi proved a much stronger result: for every t

$$\lim_{n=\infty} g(n,3)/n^t = \infty.$$

It is easy to see that

$$\lim_{n=\infty} g(n,3)^{1/n} = c \text{ and } \lim_{n=\infty} \mathcal{g}(n,3)^{1/n} = C$$

exists and the probability method easily gives $C > 1$. Abbott just informs me that he and Hanson gave a constructive proof for $C > 1$. We could not prove $c < 2$. $C < 2$ would follow from 1.

P. Erdös and R. Rado, Intersection theorems for systems of sets I and II, J. London Math. Soc. 35 (1960), 85-90 and 44 (1969), 467-479.

P. Erdös, E. Milner and R. Rado, Intersection theorems for systems of sets III, J. Australian Math. Soc. 18 (1974), 22-40.

J.L. Abbott and B. Gardner, On a combinatorial theorem of Erdös and Rado, in W.T. Tutte, ed., Recent progress in combinatorics, Acad. Press, New York (1969), 211-215.

H.L. Abbott, D. Hanson and N. Sauer, Intersection theorems for systems of sets, J. Combinatorial Theory (1972).

4. An old problem of Hajnal and myself states: Is there a function f(k) so that every graph of chromatic number f(k) contains a subgraph which has no triangle and has chromatic number k. As far as I know no progress has been made with this interesting conjecture. I offer 50 dollars for a proof or disproof. More generally we conjectured that there is a $f_\ell(k)$ for which every graph of chromatic number $f_\ell(k)$ contains a subgraph of girth ℓ and chromatic number k.

For infinite graphs we conjectured that every graph of chromatic number m (m is an infinite cardinal) contains a subgraph of chromatic number m which contains no triangle, or more generally whose smallest odd circuit has size $\geq 2\ell+1$. (By one of our theorems if $m > \aleph_0$ $\mathcal{G}$ must contain all even circuits and in fact a $K(n,\aleph_1)$ for every integer n.)

P. Erdös and A. Hajnal, On the chromatic number of graphs and set systems, Acta Math. Acad. Sci. Hungar. 16 (1966), 61-99.

5. Let $\mathcal{G}$ be a graph of 5n vertices which contains no triangle. Is it true that $\mathcal{G}$ can be made bipartite by the omission of (at most) n^2 edges? Is it true that $\mathcal{G}$ can contain at most n^5 pentagons? It is easy to see that if true then these are best possible. Both of these conjectures of mine are old and as far as I know no progress

has been made with them. Clearly many related more general problems can be raised.

6. Edwards and I proved that every graph of m edges contains a bipartite subgraph of $\frac{m}{2} + cm^{\frac{1}{2}}$ edges and Edwards determined the best possible value of c. Lovász and I proved that if $\mathcal{G}$ has m edges and contains no triangle then it contains a bipartite graph of at least $\frac{m}{2} + Cm^{2/3}$ edges for every C if $m > m_0(C)$. The above result no longer holds if $\frac{2}{3}$ is replaced by $1 - \varepsilon$. Our results are not yet published.

C.S. Edwards, An improved lower bound for the number of edges in a largest bipartite subgraph, Recent advances in graph theory, Proc. Symp. Prague 1974, Academia Praha 1975, Editor M. Fiedler, 167-181, see also Some extremal properties of bipartite subgraphs, Can. J. Math. 25 (1973), 475-485.

7. Dirac calls the k-chromatic graph critical if the omission of any of its edges decreases its chromatic number. Denote by $f_k(n)$ the largest integer for which there is a $\mathcal{G}(n;f_k(n))$ which is k-chromatic and critical ($\mathcal{G}(n;\ell)$ denotes a graph of n vertices and ℓ edges). Nearly thirty years ago I asked whether for $k \geq 4$ $f_k(n) > c_k n^2$. Dirac proved $f_6(4n+2) \geq 4n^2 + 8n + 3$ and Toft proved $f_4(n) > \frac{n^2}{16}$. Simonovits and Toft proved $f_4(n) < \frac{n^2}{4} + cn$. I conjectured

(1) $\lim_{n=\infty} f_{3k}(n)/n^2 = \lim_{n=\infty} f_{3k+1}(n)/n^2 = f_{3k+2}(n)/n^2 = \frac{1}{2}(1-\frac{1}{k})$.

Toft disproved (1). In fact he showed

$$\lim_{n=\infty} f_{3k+1}(n)/n^2 > \tfrac{1}{2}(1-\tfrac{1}{k}) \text{ for } k \geq 2.$$

It would be very interesting to determine $\lim_{n=\infty} f_k(n)/n^2 = c_k$. For $k \geq 4$ it is not even known whether the limit exists.

G. Dirac, A property of 4-chromatic graphs and some remarks on critical graphs, J. London Math. Soc. 27 (1952), 85-92.

B. Toft, On the maximal number of edges of critical k-chromatic graphs, Studia Sci. Math. Hungar. 5 (1970), 461-470.

8. Is it true that every graph of girth greater than four can be directed in such a way that it contains no directed circuit and if one reverses the direction of any of its edges the resulting new digraph should also not contain a directed circuit? I asked this question several years ago (p.99 of 3) but as far as I know there are no results.

9. M. Rosenfeld told me a few weeks ago the following very pretty conjecture. The weak Rosenfeld conjecture states as follows: Every finite graph $\mathcal{G}$ which contains no triangle can be imbedded in the following graph $\mathcal{G}_H$. The vertices of $\mathcal{G}_H$ are the points of the unit sphere, two of these points are joined if their distance is greater than $\sqrt{3}$. The strong Rosenfeld conjecture states that this imbedding can be made to be faithful. In other words if $\mathcal{G}$ has n vertices $x_1,\dots,x_n$ we can find n points $y_1,\dots,y_n$ on the unit sphere so that the distance from y_i to y_j is greater than $\sqrt{3}$ if and only if x_i is joined to x_j. Clearly if these conjectures hold Hilbert space can be replaced by the unit sphere of n dimensional space and it might be interesting to determine which graph $\mathcal{G}(n)$ needs the unit sphere of highest dimension - perhaps K_n?

10. Let $\mathcal{G}^{(r)}(n;\ell)$ be an r-graph of n vertices, ℓ edges and chromatic number k. Is it true that for $k > k_0(r)$ $\ell \geq \binom{(k-1)(r-1)+1}{r}$. Equality only for the complete r-graph $K^{(r)}((k-1)(r-1)+1)$. For $r = 2$ it is easy to see that the conjecture always holds i.e. $k > k_0(r)$ can be omitted. For $r = 3$ we already run into difficulties since the conjecture certainly fails for $r = 3$, $k = 3$. The smallest three chromatic 3-graph is given by the seven triples of the Fano plane and not by the 10 triplets of $K^{(3)}(5)$.

11. The following interesting conjecture is due to Jean-Claude Meyer: Let $\{A_i\}$, $1 \le i \le n$, $|A_i| = h$ be a family of sets satisfying $A_i \cap A_j \ne \emptyset$, $1 \le i < j \le n$. Further the family is maximal with respect to these properties i.e. if A is any set of h elements there always is an i, $1 \le i \le n$ for which $A \cap A_i = \emptyset$. Conjecture: Let $h = p^\alpha + 1$, p prime. Then $n \ge h^2 - h + 1$. We clearly have equality for the lines of a finite projective plane.

12. The following surprising conjecture is due to Chvatal: Let F be a family of subsets of a finite set $\mathcal{S}$ such that $X \in F$, $Y \subset X$ implies $Y \in F$. Then there is a $t \in \mathcal{S}$ such that every intersecting subfamily G of F satisfies

$$|G| \le |\{X \in F: t \in X\}|.$$

A family is called intersecting if any two of its members have a non empty intersection.

Problems 10,11 and 12 are not new they appeared in the Hypergraph Seminar Lecture Notes of Math. 411 Springer Verlag. I restated them here because I find them particularly attractive.

13. In a recent paper of Chvatal the following question is posed: Let $|\mathcal{S}| = n$. A family $\{A_t\}$ of distinct subsets of $\mathcal{S}$ is called m intersecting if any m of the A_t have a non empty intersection. Assume now that all the A_t have size k and denote by $f(n,k,m)$ the largest m intersecting family of subsets of $\mathcal{S}$ of size k. Chvatal conjectured

$$(1) \qquad f(n,k,m) = \binom{n-1}{k-1}, \text{ for } 1 \le m < k \text{ and } n \ge \frac{m+1}{m}k.$$

For $m = 1$ this is the well known theorem of Ko, Rado and myself. For $m = 2$ I conjectured (1) in 9 of the introduction. $f(n,2,2)$ is simply Turán's theorem that every graph of n vertices which has no triangle has at most $\left[\frac{n^2}{4}\right]$ edges. Chvatal proved my conjecture for $k = 3$.

V. Chvatal, An extremal set-intersection theorem, J. London Math. Soc. (second series) 9 (1974), 355-359.

C. Ko, P. Erdős and R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford 12 (1961), 313-320.

Now I discuss a few extremal problems in graph theory. $\mathcal{G}^{(r)}(n;m)$ denotes an r-graph of n vertices and m edges (i.e. r-tuples). $f(n;\mathcal{G}^{(r)}(k,\ell))$ is the smallest integer for which every $\mathcal{G}^{(r)}(n;f(n;\mathcal{G}^{(r)}(k;\ell)))$ contains a $\mathcal{G}^{(r)}(k;\ell)$ as a subgraph. New and interesting complications arise if we also prescribe the structure of $\mathcal{G}^{(r)}(k;\ell)$. $K^{(r)}(t)$ denotes the complete r graph of t vertices ($K^{(r)}(t)$ is of course identical with $\mathcal{G}^{(r)}(t;\binom{t}{r})$). Turán's well known old problem states: Determine $f(n;K^{(r)}(t)$ for every $t > r$ and also determine the structure of the extremal graphs. Turán completely solved this problem for $r = 2$ and every $t > r$ but for $r > 2$ nothing is known, though Turán has some plausible conjectures. Put

(1) $$\lim f(n;K^{(r)}(t))\binom{n}{r}^{-1} = \alpha(t,r)$$

Turán proved $\alpha(t,2) = 1 - \frac{1}{t-1}$, but for $t > r > 2$ none of the $\alpha(t,r)$ are known.

I will now state some new problems and will try to restrict myself to recent and unpublished ones, but first I give some literature which is heavily biased in favour of papers of my collaborators and myself.

1. M. Simonovits, A method for solving extremal problems in graph theory, stability problems, Theory of graphs, Proc. Coll. held at Tihany, Hungary 1966, Acad. Press, 279-319.

2. M. Simonovits, Extremal graph problems with conditions, Combinatorial Theory and its Applications, Coll. Math. Soc. J. Bolyai 1970, Vol. III, 999-1012 (North Holland).

3. P. Erdős and M. Simonivits, A limit theorem in graph theory, Studia Sci. Math. Hungar. 1 (1966), 51-57.

4. P. Erdős and M. Simonovits, Some extremal problems in graph theory, Coll. Math. Soc. J. Bolyai 1970, Vol. I 378-392.

5. P. Erdös, On extremal problems of graphs and generalised graphs, Israel J. Math. 2 (1965), 183-190.

6. W.G. Brown, P. Erdös and V. T. Sós, Some extremal problems on r-graphs, New directions in the theory of graphs, Proc. Third Ann Arbor Conference on Graph Theory, (Ed. F. Harary), Acad. Press 1973, 53-63, also On the existence of triangulated spheres in 3-graphs.

B. Bollobás, Three graphs without two triples whose symmetric difference is contained in a third, J. London Math. Soc.

I will refer to these papers by their number and to avoid confusion if I refer to a paper of the introduction by number I will state that I refer to the list in the introduction.

14. Sauer and I investigated the following problem: Denote by $f(k;n)$ the smallest integer so that every $\mathcal{G}(n;f(k;n))$ contains a regular subgraph of valency (or degree) k. Trivially $f(2;n) = n$. It seems likely that $f(k;n) < n^{1+\varepsilon}$ for every k and every $\varepsilon > 0$ if $n > n_0(k,\varepsilon)$, but we cannot even prove this for $k = 3$. The best upper bound we have is $f(3,n) < cn^{8/5}$. Chvatal observed $f(3;n) > \frac{8n}{3} - c$. As far as we know $f(k;n) < c_k n$ has never been disproved. It is not known whether

$$\lim_{n=\infty} \frac{1}{n} f(k;n) = C_k$$

exists.

Sauer and Berge conjectured that every regular graph of valency four contains a regular subgraph of valency three. Chvatal conjectured that to every k there is an α_k so that if $n > n_0(k)$ then every $\mathcal{G}(n)$ each vertex of which has valency $\geq \alpha_k$ contains a regular subgraph of valency k. This beautiful conjecture if true would of course imply $f(k;n) < c_k n$.

Szemerédi asked: Denote by $F(k;n)$ the smallest integer so that every $\mathcal{G}(n;F(k;n))$ contains a spanned regular subgraph of valency k. Determine or estimate $F(k;n)$. I proved $F(3,n) < cn^{5/3}$. In fact I showed that every $\mathcal{G}(n;[cn^{5/3}])$ contains $K^{(4)}$ or a spanned $K(3,3)$. Unfotunately we could not prove

$$\lim_{n=\infty}\frac{1}{n}F(3;n) = \infty.$$

15. Let $\mathcal{G}$ be a bipartite graph. Simonovits and I conjectured that there always is a rational α, $1 \le \alpha < 2$ for which

(1) $$\lim_{n=\infty} f(n;\mathcal{G})/n^{\alpha} = c_{\alpha}(\mathcal{G}).$$

Conversely we conjectured that for every rational $1 \le \alpha < 2$ there is a graph $\mathcal{G}$ which satisfies (1). We are very far from being able to prove any of these conjectures. We have at present no guess about the possible values of the constants $c_{\alpha}(\mathcal{G})$. We proved (see 4) that α does not have to be of the form $1 + \frac{1}{k}$ or $2 - \frac{1}{k}$.

The situation is certainly much more complicated for hypergraphs. Szemerédi recently proved that

(2) $$f(n,\mathcal{G}^{(3)}(6;3)) = o(n^2)$$

and Ruzsa proved that for every $\varepsilon > 0$ and $n > n_0(\varepsilon)$

(3) $$f(n;\mathcal{G}^{(3)}(6;3)) > n^{2-\varepsilon}.$$

The joint paper of Ruzsa and Szemerédi will be published soon. (2) and (3) implies that (1) certainly does not hold for hypergraphs. I hope and believe that for every hypergraph $\mathcal{G}^{(r)}$

$$\lim_{n=\infty} \log f(n;\mathcal{G}^{(r)})/\log n$$

exists and is rational.

16. Let $\mathcal{G}$ be a graph of chromatic number $K(\mathcal{G})$. $\beta(\mathcal{G})$ is the smallest non negative number for which there is a sequence of graphs $\mathcal{G}_t$ with $e(\mathcal{G}_t) \to \infty$ ($e(\mathcal{G})$ is the number of edges of $\mathcal{G}$) so that any subgraph of $\mathcal{G}_t$ having more than $(\beta+\varepsilon)e(\mathcal{G}_t)$ edges contains $\mathcal{G}$ as a subgraph.

For ordinary graphs ($r = 2$) this concept does not seem to be fruitful. A well known theorem of Stone, Simonovits and myself (see 3) asserts that

$$(1) \qquad \lim_{n = \infty} \frac{f(n;\mathcal{G})}{\binom{n}{2}} = 1 - \frac{1}{K-1}.$$

Clearly for every $\mathcal{G}$, $\beta(\mathcal{G})$ is not less than the limit in (1) and it is easy to see that in fact $\beta(\mathcal{G}) = 1 - \frac{1}{k-1}$. In other words $\beta(\mathcal{G})$ is assumed if $\mathcal{G}_t$ is the complete graph.

On the other hand it is not impossible that for hypergraphs new and intersecting situations will arise. Define $\mathcal{G}^{(3)}(2n;(n-1)n^2)$ as the hypergraph of vertices $x_1,\ldots,x_n$; $y_1,\ldots,y_n$ and edges (x_i,x_j,y_ℓ); (y_i,y_j,x_ℓ), $1 \le i < j \le n$, $1 \le \ell \le n$. Perhaps the following result holds: For every $\varepsilon > 0$ and $n > n_0(\varepsilon)$ every subgraph of our $\mathcal{G}^{(3)}(2n;(n-1)n^2)$ having more than $(\frac{1}{2}+\varepsilon)n^3$ edges contains a $K^3(4)$. If true then $\beta(K^{(3)}(4)) \ge \frac{1}{2}$. On the other hand Turán observed in 1940 that $\alpha(4,3) \ge \frac{5}{9}$ in other words if my conjecture is true the value of β is not given by the complete graph.

I expect that interesting new phenomena will occur for ordinary bipartite graphs $\mathcal{G}$ if the definition of $\beta(\mathcal{G})$ is modified. Let $\beta(\mathcal{G})$ be the smallest number for which for every $\varepsilon > 0$ there is a graph $\mathcal{G}_1$ with $e(\mathcal{G}_1)$ arbitrarily large every subgraph of which having

$$(2) \qquad (e(\mathcal{G}_1))^{\beta(\mathcal{G})+\varepsilon}$$

edges contains $\mathcal{G}$ as a subgraph. For C_4 (C_k is a circuit of k edges) it follows from results of Folkman and Szemerédi that $\beta(C_4) = \frac{2}{3}$ (see 3 of the introduction p.97). The complete graph would give the exponent $\frac{3}{4}$.

By the way perhaps (2) can be replaced by the simpler expression $(c(\mathcal{G}) + o(1))(e(\mathcal{G}_1))^{\beta}$ and it could very well be true that β must be rational. Some of these conjectures are formulated while I am writing the paper - I hope not too many of them turn out to be nonsensical.

F.R.K. Chung and R.L. Graham, On multicolor Ramsey numbers for complete bipartite graphs, J. Combinatorial Theory, Ser. B 18 (1975), 164-169.

17. V.T. Sós and I last week investigated the following question: Denote by $S^{(1)}(n)$ the smallest integer so that every $\mathcal{G}^{(3)}(n;S^{(1)}(n))$ contains at least one Steiner triple system. $S_{\ell}^{(1)}(n)$ is defined as the smallest integer so that every $\mathcal{G}^{(3)}(n;S_{\ell}^{(1)}(n))$ contains at least one Steiner system belonging to a set of size ℓ. It will no doubt be extremely difficult to determine

(1) $$\lim_{n=\infty} S^{(1)}(n)/\binom{n}{3} \text{ and } \lim_{n=\infty} S_{\ell}^{(1)}(n)/\binom{n}{3}.$$

It is easy to see that the limits in (1) exist. We expect that it will not be difficult to prove that for $n > n_0$ $S^{(1)}(n) = S_7^{(1)}(n)$ though we did not carry out all the details. In fact equality probably holds for all n.

In view of the difficulty in handling these questions we introduced two new functions. $S^{(2)}(n)$ is the smallest integer for which for every $\mathcal{G}^{(3)}(n;S^{(2)}(n))$ there is a subset A, $|A| > 3$ so that to every $x \in A$, $y \in A$ there is a $z \in A$ so that (x,y,z) is one of the edges of our $\mathcal{G}^{(3)}(n;S^{(2)}(n))$. $S^{(3)}(n)$ is defined similarly except that the condition $z \in A$ is dropped. $S_{\ell}^{(i)}(n)$, $i = 2,3$ can clearly be defined similarly and all these problems can be asked for r-tuples instead of triples. There is no shortage of definitions and problems but a regrettable shortage of theorems. $S^{(1)}(n) \geq S^{(2)}(n) \geq S^{(3)}(n)$ is of course obvious. It seems that $S^{(2)}(n)$ is nearly as hard to handle as $S^{(1)}(n)$, perhaps $S^{(2)}(n) = S_4^{(2)}(n) = f(n;\mathcal{G}^{(3)}(4,3))$ but we had no time to look into this. $S^{(3)}(3n) = n^3 + 1$ is easy to prove and it will not be difficult to determine $S_r^{(3)}(n)$.

Bollobás and I discussed the following question: Let $\mathcal{G}$ be a graph, $|\mathcal{G}| = n$, denote by $A(\mathcal{G};n)$ the smallest number of triples of $\mathcal{G}$ so that the graph spanned by the edges of these triples should contain $\mathcal{G}$ as a subgraph. We observed that it easily follows from known results that $A(C_4;n) = (1+o(1))n^{3/2}/6$. $A(C_k;n) > n^{1+\varepsilon_k}$ is easy to prove but we have no asymptotic formula say for $A(C_5,n)$ and in fact do not even know the best possible value for ε_5. It might be of some interest to investigate $A(K(r,r,r);n)$.

18. Denote by $C_\ell(\mathcal{G})$ the number of K_ℓ's contained in $\mathcal{G}$. Determine or estimate

$$\min(C_\ell(\mathcal{G}(n;f(n;K_\ell)+u)) = B(n;\ell,u)$$

as a function of n and u where the minimum is to be taken over all graphs of n vertices and $f(n;K_\ell) + u$ edges. Important work on this problem has recently been done on this subject by Bollobás and Lovász-Simonovits which in fact was reported at this conference by the authors. Many unsolved problems remain. In particular let $u = cn^2$ determine

$$\lim_{n=\infty} B(n;\ell,cn^2)n^{-\ell} = g_\ell(c).$$

As far as I know this is unsolved even for $\ell = 3$.

Denote by $\bar{\mathcal{G}}(n)$ the complementary graph of $\mathcal{G}(n)$ (i.e. the edges of $\bar{\mathcal{G}}(n)$ are the non-edges of $\mathcal{G}(n)$). A. Goodman determined $\min(C_3(\mathcal{G}(n)) + C_3(\bar{\mathcal{G}}(n)))$. I proved

$$\min(C_\ell(\mathcal{G}(n)) + C_\ell(\bar{\mathcal{G}}(n))) \le 2\binom{n}{\ell}2^{-\binom{\ell}{2}}$$

and conjectured

$$\lim_{n=\infty} \min(C_\ell(\mathcal{G}(n)) + C_\ell(\bar{\mathcal{G}}(n)))\binom{n}{\ell}^{-1} = 2^{1-\binom{\ell}{2}}.$$

P. Erdős, On a theorem of Rademacher-Turán, Illinois J. Math. 9 (1962), 59-60.

P. Erdős, On the number of complete subgraphs and circuits contained in graphs, Ĉasopis Pest. Mat. 94 (1969), 290-296.

P. Erdős, On the number of complete subgraphs contained in certain graphs, Publ. Math. Inst. Hung. Acad. Sci. 7 (1962), 459-464 - see also "The Art of Counting" 145-150.

The paper of Bollobás will appear very soon in Proc. Cambridge Phil. Soc. A part of the paper of Lovász and Siminovits will appear in the Proceedings of our conference.

19. Simonovits and I a few days ago considered the following problems. Two bipartite graphs $\mathcal{G}_1(k_1;\ell_1)$ and $\mathcal{G}_2(k_2;\ell_2)$ are said to be equivalent if for every $n > n_0$

(1) $$f(n;\mathcal{G}_1(k_1;\ell_1) = f(n;\mathcal{G}_2(k_2;\ell_2)).$$

Several problems can be posed which seem interesting but are perhaps very difficult. Is it true that (1) implies that there is a $\mathcal{G}(k;\ell)$ which is a subgraph of both $\mathcal{G}_1(k_1;\ell_1)$ and $\mathcal{G}_2(k_2;\ell_2)$ and for which

(2) $$f(n;\mathcal{G}(k;\ell) = f(n;\mathcal{G}_1(k_1;\ell_1)) = f(n;\mathcal{G}_2(k_2;\ell_2))?$$

Assume next that (1) holds and that $\mathcal{G}_1(k_1;\ell_1)$ is a subgraph of $\mathcal{G}_2(k_2;\ell_2)$ and that $k_1 = k_2$. In other words $\mathcal{G}_2(k_2;\ell_2)$ is obtained from $\mathcal{G}_1(k_1;\ell_1)$ by adding some edges. Estimate $\ell_2 - \ell_1$ from above and below. Perhaps $\ell_2 - \ell_1 < c_1 k$ always holds and $\ell_2 - \ell_1 > c_2 k$ is possible for suitable graphs $\mathcal{G}_1(k_1;\ell_1)$ and $\mathcal{G}_2(k_2;\ell_2)$.

It is not possible that if we assume that (1) holds for a sequence n_k tending to infinity then it will hold for all sufficiently large n.

Two graphs are called weakly equivalent if for $n \to \infty$

(3) $$f(n;\mathcal{G}_1(k_1;\ell_1)) = (1+o(1))f(n;\mathcal{G}_2(k_2;\ell_2))$$

and very weakly equivalent if

(4) $$c_1 f(n;\mathcal{G}_1(k_1;\ell_1)) < f(n;\mathcal{G}_2(k_2;\ell_2) < c_2 f(n;\mathcal{G}_1(k_1;\ell_1)).$$

It seems certain that (3) and (4) will hold if it is assumed to hold for a sequence n_k tending to infinity. The analogous questions for weakly or very weakly equivalent graphs are probably easier than for the equivalent ones.

For non-bipartite graphs this concept of equivalence is less illuminating since all the odd cycles are equivalent.

20. Now I discuss a few problems on Ramsey numbers. For the "older" literature I refer to the excellent survey paper of S. Burr. Several papers on these problems are published in the Proc. of the Colloquium on Finite and Infinite Sets at Kenthely 1973 held in the memory of the poor old author of this paper, and Harary has a paper on Ramsey theory in the Proceedings of our Colloquium. $r_k(\mathcal{G}_1,\dots,\mathcal{G}_k) = \mathcal{N}$ is the smallest integer with the property that if we colour the edges of $K(\mathcal{N})$ by k colours then for some i, $1 \le i \le k$ the i-th colour contains $\mathcal{G}_i$ as a subgraph. Usually it is very difficult to obtain exact results for $r_k(\mathcal{G}_1,\dots,\mathcal{G}_k)$. In a paper of Burr, Spencer and myself which will appear very soon in Trans. Amer. Math. Soc. we obtain very accurate (and in fact often exact) estimates for $r_k(\mathcal{G}_1,\dots,\mathcal{G}_k)$ if k tends to infinity and all the $\mathcal{G}$'s are identical.

The graph $\mathcal{G}$ is said to have edge density c if c is the smallest real number so that for every subgraph $\mathcal{G}'$ of $\mathcal{G}$ we have $e(\mathcal{G}') \le cv(\mathcal{G}')$ where $e(\mathcal{G})$ and $v(\mathcal{G})$ denotes the number of edges and vertices of $\mathcal{G}$ respectively. Burr and I conjectured that for graphs of bounded edge density the Ramsey function has linear growth. More precisely: There is a function f(c) so that if $\mathcal{G}$ has edge density $\le c$ then

(1) $$r(\mathcal{G},\mathcal{G}) \le f(c)v(\mathcal{G}).$$

We proved (1) for many special cases but are very far from being able to prove (1) in full generality. (1) may hold for a sequence of graphs of unbounded edge density too. One of our most challenging problems with Burr states: Let $\mathcal{G}^{(n)}$ be the skeleton of the n-dimensional cube. Is it true that

(2) $$r(\mathcal{G}^{(n)},\mathcal{G}^{(n)}) < c_1 2^n.$$

(2) if true is in some sense best possible. The probability method easily gives that if $\mathcal{G}_n$ is an infinite sequence of graphs satisfying $r(\mathcal{G}_n,\mathcal{G}_n) < cv(\mathcal{G}_n)$ then $e(\mathcal{G}_n) < c'v(\mathcal{G}_n) \log v(\mathcal{G}_n)$ for some $c' = c'(c)$ and we have

$v(\mathcal{G}^{(n)}) = 2^n$, $e(\mathcal{G}^{(n)}) = (n-1)2^{n-1}$.

For further unsolved problems see also my papers with S. Burr and R.L. Graham, Colloquium on Finite and Infinite sets, Kesthely 1973, North Holland 1974. A paper of Harary in our conference states eight challenging unsolved problems in this subject. (S.A. Burr, Graphs and Combinatorics, Springer, Berlin (1974) 52-75.

21. V.T. Sós and I considered a few weeks ago the following problem. Denote by $g_1^{(3)}(n)$ the largest integer so that if we colour the triples of $|\mathcal{S}| = n$ by two colours there always is a monochromatic Steiner system of size $g_1^{(3)}(n)$. It is probably very difficult to estimate $g_1^{(3)}(n)$. Thus as in problem 17 we introduced $g_2^{(3)}(n)$ and $g_3^{(3)}(n)$. $g_2^{(3)}(n)$ is the largest integer for which there is a set $\mathcal{S}_1 \in \mathcal{S}$, $|\mathcal{S}_1| = g_2^{(3)}(n)$ so that there is a monochromatic triple system on $\mathcal{S}_1$ so that to every $x \in \mathcal{S}_1$, $y \in \mathcal{S}_1$ there is a $z \in \mathcal{S}_1$ for which (x,y,z) belongs to our monochromatic system. $g_3^{(3)}(n)$ is defined analogously only the condition $z \in \mathcal{S}_1$ is replaced by $z \in \mathcal{S}$. It is easy to see that $g_3^{(3)}(n) = \frac{2n}{3} + O(1)$. We did not try seriously to estimate $g_2^{(3)}(n)$.

For $r > 3$ even the determination of $g_3^{(r)}(n)$ leads to non-trivial problems. In fact Lovász and I proved

(1) $$g_3^{(4)}(n) \quad c\log n.$$

First I repeat the definition of $g_3^{(4)}(n)$. It is the largest integer so that if we colour the quadruplets of $\mathcal{S}$, $|\mathcal{S}| = n$ by two colours, there always is a subset $\mathcal{S}_1 \subset \mathcal{S}$, $|\mathcal{S}_1| = g_3^{(4)}(n)$ so that there is a monochromatic quadruple subsystem for which for every triple (x,y,z) of $\mathcal{S}_1$ there is a $w \in \mathcal{S}$ so that (x,y,z,w) is a quadruple of our monochromatic quadruple subsystem. It easily follows from the probability method that if one colours the edges of a $K(n)$ by two colours then every $K(\ell)$, $\ell \geq c\log n$ contains a monochromatic triangle. This colouring of the edges induces a colouring of the quadruples as follows: If a quadruple of

$\mathcal{S}$ contains one of the monochromatic triangles then it gets the same colour as the triangle, otherwise its colour is arbitrary. Observe that a quadruple cannot contain two monochromatic triangles of different colours, thus our procedure really gives a colouring of the quadruples. This colouring proves (1). It seems likely that $g_3^{(4)}(n) > c_1 \log n$ holds. Clearly many more problems remain, some of them may lead to interesting new phenomena.

22. Finally I state a few miscellaneous problems and conjectures.

Denote by $f(n;k,r)$ the smallest integer so that if F is any family of subsets of size k of a set of size n then if $|F| \geq f(n;k,r)$ there are two members of F having exactly r elements in common. V.T. Sós and I conjectured four years ago that if $k > 3$, $n > n_0(k)$ then

(1) $$f(n;k,1) = \binom{n-2}{k-2} + 1.$$

Katona (unpublished) proved this for $k = 4$. The proof does not seem to generalise for $k > 4$ and as far as I know our conjecture is still open for $k > 4$. V.T. Sós and I easily settled $k = 3$ here we have

(2) $$f(4n;3,1) = f(4n+1;3,1) = f(4n+2;3,1) = 4n + 1,$$
$$f(4n+3;3,1) = 4n + 2.$$

(2) can be proved by a simple induction.

(1) can be considered as a sharpening of our well known result with Ko and Rado. Perhaps for $n > n_0(k,r)$

(2) $$f(n;k,r) \leq \max\left(\binom{n-r-1}{k-r-1} + 1,\ \left[\binom{n}{k}\binom{k}{r}^{-1}\right] + 1\right).$$

The reason for the first term is clear, we take all subsets of size k containing the same $r + 1$ elements. The second term is explained as follows: $\left[\binom{n}{k}\binom{k}{r}^{-1}\right]$ is the upper bound for the number of k-sets so that every r-set is contained in at most one of them. I just thought of (2) while writing these lines and thus would not be surprised if it would be completely false or at best not completely accurate.

Denote now by $f(n;r)$ the smallest integer so that if F is any family of subsets of $\mathcal{S}$, $|\mathcal{S}| = n$ and $|F| \geq f(n;r)$ then there always are two members of F, A_1 and A_2 satisfying $|A_1 \cap A_2| = r$. (In the definition of $f(n;k,r)$ we further assumed that all members of F have k elements.) It is easy to see that $f(n;0) = 2^{n-1} + 1$ and it would not be perhaps difficult to determine $f(n;r)$ for fixed r as $n \to \infty$. For some time I conjectured the following: Let $\varepsilon n < r < (\frac{1}{2}-\varepsilon)n$. Then there is a $c_\varepsilon > 0$ so that

(3) $$f(n;r) < (2-c_\varepsilon)^n.$$

(3) would have immediate application to a geometric problem considered by Larman and Rogers, but unfortunately I was not able to prove (3). It would be very interesting to determine $f(n;r)$ explicitly for every n and r but perhaps this is hopeless, and asymptotic formulas or good inequalities may be almost equally useful. If $|A_i \cap A_j| = r$ is replaced by $|A_i \cap A_j| \geq r$ the problem has been completely solved by Katona.

G.Y. Katona, Intersection theorems for systems of finite sets, Acta Math. Acad. Sci. Hung. 15 (1964), 329-337.

P. Erdös, C. Ko and R. Rado, Intersection theorems for systems of finite sets, Quarterly J. Math. 12 (1961),

D.E. Larman and C.A. Rogers, The realisation of distances within sets in Euclidean space, Mathematika, 19 (1972), 1-24.

23. Denote by $g(n;k)$ the largest integer with the following property: Let $1 \leq a_1 < \ldots < a_n$ be any sequence of integers, consider all the integers $b_1 < b_2 < \ldots < b_r$ which can be written as the sum or product of k distinct a's. Then $\min r = g(n;k)$ where the minimum is taken over all sets of distinct integers $\{a_1, \ldots, a_n\}$. I conjecture that for every k and $n > n_0(k)$

(1) $$g(n;k) > n^{k-\varepsilon}.$$

I have not even been able to prove that $g(n;2) > n^{1+\varepsilon}$.

It does not even seem to be easy to prove that

(2) $$\lim_{n=\infty} g(n;2)/n \to \infty.$$

With Szemerédi we observed that (2) will follow from the results of Freiman. It is easy to see that

$$g(n;2) = o\left(\frac{n^2}{(\log n)^r}\right)$$

for every r and the true order of magnitude of $g(n;2)$ is perhaps $n^2 \exp\left(-c\frac{\log n}{\log\log n}\right)$. It is very likely that the value $g(n;2)$ does not change very much (perhaps not at all) if we permit the a's to be real numbers or more generally elements of a vector space.

Let $1 \le a_1 < \ldots < a_n$ be a set of n distinct integers. Let $u_1 < \ldots < u_s$ be the set of all integers which can be expressed as the sum or product of distinct a's. Put

$$g(n) = \min s$$

where the minimum is to be taken over all sequences of distinct integers. It seems certain that $g(n) > n^k$ for every k if $n > n_0(k)$ but it is not hard to see that

$$g(n) = o(\exp n^{\varepsilon})$$

for every $\varepsilon > 0$. The true order of magnitude of $g(n)$ is perhaps $\exp(\exp(\log n)^{\alpha})$ for some $\alpha < 1$.

E. Straus proved a few years ago that if we only take subset sums then we get the fewest distinct numbers if the a's form an arithmetic progression.

Finally we could consider all possible sums of products formed from the a's with each a occuring at most once and define $\mathcal{g}(n)$ as the smallest number of distinct integers which we can represent in this form. For n = 3 we have to consider the terms $a_1, a_2, a_3, a_1 + a_2, a_1 + a_3, a_2 + a_3, a_1 + a_2 + a_3,$ $a_1 \cdot a_2, a_1 \cdot a_3, a_2 \cdot a_3, a_1 a_2 a_3, a_1 + a_2 a_3, a_2 + a_1 a_3, a_3 + a_1 a_2$. I have no guess at the moment about the behaviour of $\mathcal{g}(n)$.

G.A. Freiman, Foundations of a structural theory of set addition, Amer. Math. Soc. translation of Math. Monographs vol. 37, 1973.

24. Let $1 \leq a_1 < \ldots < a_k$ be a set of k integers for which the sums $a_i + a_j$, $i \neq j$ are all distinct. Is it true that there is an n_k and a perfect difference set $b_1, \ldots, b_t$ mod n_k, $t^2 - t + 1 = n_k$, so that the a's are a subset of the b's?

An analogous result has been proved a few years ago by C. Treash. She proved that to every k there is an n_k so that every incomplete Steiner system on k elements can be imbedded in a Steiner system on n_k elements. A family of triples is an incomplete Steiner system if every two of them have at most one element in common.

C.C. Lindler very recently proved that $n_k \leq 6k + 3$. The best value of n_k is not yet known - it certainly must be greater than 2k.

C. Treash, The completion of finite incomplete Steiner triple systems with application to loop theory, J.C.T. Ser. A 10 (1971), 259-265.

C.C. Lindler, A partial Steiner triple system of order n can be embedded in a Steiner triple system of order 6n + 3, ibid 18 (1975), 349-351.

25. Let $\mathcal{G}(n)$ be a graph of n vertices. Is it true that if every induced (or spanned) subgraph of $\mathcal{G}(n)$ having $[\frac{n}{2}]$ vertices has more than $\frac{n^2}{50}$ edges then $\mathcal{G}$ contains a triangle? It is easy to see that $\frac{n^2}{50}$ if true is best possible. More generally denote by $f(\alpha,n)$ the smallest integer so that if every spanned subgraph of $\mathcal{G}(n)$ of $[\alpha n]$ vertices has at least $f(\alpha,n)$ edges then $\mathcal{G}(n)$ has a triangle. Determine or estimate $f(\alpha,n)$. By Turán's theorem $f(1,n) = [\frac{n^2}{4}] + 1$. If the determination of $f(\alpha,n)$ is too complicated it would be of interest to determine

$$\lim_{n = \infty} f(\alpha,n)/n = g(\alpha).$$

Thus the first step towards our conjecture would be to prove $g(\frac{1}{2}) = \frac{1}{50}$.

Clearly many generalisations are possible if the triangle is replaced by other graphs or hypergraphs. There is no doubt that one gets interesting and fruitful problems if the triangle is replaced by larger complete graphs. I am not sure if new phenomena occur if the triangle is replaced say by a bipartite graph.

26. Let $|\mathcal{S}| = 2n$, $A_i \subset \mathcal{S}$, $1 \le i \le t_n$. Assume that the number of pairs $1 \le i_1 < i_2 \le t_n$ with $A_{i_1} \cap A_{i_2} = \emptyset$ is at least 2^{2n}. Is it true that $t_n \ge (1+o(1))2^{n+1}$? Or in a sharper form determine the smallest possible value of t_n for which the number of the intersecting pairs is $\ge 2^{2n}$. It is easy to see that

$$2^{n+1} - \min t_n \to \infty.$$

Clearly this problem can be extended and generalised in many ways.

27. The following problem is due to Rothschild and myself. Let $\mathcal{G}(n)$ be a graph of n labelled vertices. Denote by $C_2(\mathcal{G}(n))$ the number of ways one can colour the edges of $\mathcal{G}(n)$ with two colours so that there should be no monochromatic triangle. Clearly $C_2(\mathcal{G}(n)) \le 2^{e(\mathcal{G}(n))}$. We conjecture that for $n > n_0$

(1) $$\max C_2(\mathcal{G}(n)) = 2^{[n^2/4]}.$$

Equality only for the Turán graph $K_2([\frac{n}{2}],[\frac{n+1}{2}])$.

Probably (1) holds already for quite small n, but it is easy to see that it does not hold for all n.

Clearly many generalisations are possible, the number of colours can be increased, the triangle can be replaced by a general graph or hypergraph, again there is no shortage of problems or conjectures but unfortunately we have no results as yet.

28. A group G is said to have property A(k) if it has at most k elements which pairwise do not commute. Determine or estimate the smallest f(k) (if it exists) so that every group with property A(k) is the union of f(k) or fewer Abelian groups. This problem is a finite modification of a problem of B. Neumann.

Isaac proved

$$(1+c)^k < f(k) < k!^{2+\varepsilon}.$$

The exact determination of f(k) will perhaps not be easy.

29. Let $f_1(n)$ be the smallest integer for which every $\mathcal{G}(n;f_1(n))$ contains two edge disjoint circuits C_{ℓ_1} and C_{ℓ_2} $\ell_1 \geq \ell_2$ for which the vertex set of C_{ℓ_2} is a subset of that of C_{ℓ_1}. $f_2(n)$ is the smallest integer for which every $\mathcal{G}(n;f_2(n))$ contains two edge disjoint circuits C_ℓ having the same vertex set. $f_3(n)$ is the smallest integer for which every $\mathcal{G}(n;f_3(n))$ contains two edge disjoint circuits C_{ℓ_1} and C_{ℓ_2} so that if $(x_1,x_2),\ldots,(x_{\ell_1-1},x_{\ell_1}),(x_{\ell_1},x_1)$ are the edge of C_{ℓ_1} then the edges of C_{ℓ_2} are

$(x_1,x_{i_1}),\ldots,(x_{i_{\ell_2-1}},x_{i_{\ell_2}}),(x_{i_{\ell_2}},x_1)$ with $1 < i_1 <\ldots< i_{\ell_2} < \ell_1$ (i.e. geometrically the edges of C_{ℓ_2} do not cross each other).

An old result of Pósa states that every $\mathcal{G}(n;2n-3)$ has a circuit with a diagonal, 2n - 3 is best possible. He has various refinements from which I think one can deduce $f_1(n) < cn$. I do not know about $f_2(n)$ and $f_3(n)$.

Denote by $g_i(n)$ the smallest integer for which every $\mathcal{G}(n;g_i(n))$ contains a C_ℓ with at least i diagonals emanating from one of its vertices. Pósa's result gives $g_1(n) = 2n - 3$ and the proof of Czipszer easily gives $g_i(n) \leq (i+1)n + c_i$, $g_2(n) = 3n - 8$. I conjectured that

(1) $$g_i(n) = (i+1)n - (i+1)^2 + 1.$$

(1) would follow if $g_i(2i) = i^2 + 1$ would hold, but M. Lecvin disproved this and thus (1) is in doubt. It is easy to see that if (1) holds then it is best possible.

Pósa's result appeared as a problem in Matematikai Lapok about twelve years ago. The proof of Czipszer appeared there too.