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On a Ramsey–Turán Type Problem

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Denote by $I(G)$ the maximal number of independent points in a graph G and let $\alpha(G) = I(\overline{G})$, where $\overline{G}$ is the complement of G. Thus $\alpha(G)$ is the maximal p for which G contains a K_p, a complete graph with p vertices. Denote by $f(n, k, l)$ the maximal m for which there is a graph with n points and m edges such that $\alpha(G) < k$ and $I(G) < l$. The function $f(n, k, l)$ was introduced and investigated by Erdös and Sós [1]. They proved that

$$f(n, 3, l) \leqslant nl/2$$

and so

$$f(n, 3, l) = o(n^2) \qquad \text{if} \quad l = o(n).$$

Erdös and Sós also proved that if $l = o(n)$ then

$$f(n, 5, l) = (1 + o(1))(n^2/4)$$

and

$$f(n, 4, l) \leqslant (1 + o(1))(n^2/6).$$

The last inequality was improved by Szemerédi [2] who showed that if $l = o(n)$ then

$$f(n, 4, l) \leqslant (1 + o(1))(n^2/8). \tag{1}$$

The question remained whether or not $l = o(n)$ implies $f(n, 4, l) = o(n^2)$. We shall prove that this implication does not hold and, even more, equality holds in (1).

THEOREM. *If $l = o(n)$ then*

$$f(n, 4, l) = (1 + o(1))(n^2/8).$$

Proof. Let $\gamma > 0$ and $\delta > 0$. We shall show that if n is sufficiently large then there exists a graph with $2n$ ponits and at least $(1 - \gamma)(n^2/2)$

edges for which $\alpha(G) < 4$ and $I(G) < \delta n$. Consider the following integrals ($0 < \epsilon < 1$, k is a natural number):

$$A = \int_0^1 (1 - m^2)^{k/2}\, dm,$$

$$B = \int_0^{m_1} (1 - m^2)^{k/2}\, dm, \qquad \text{where} \qquad m_1 = \epsilon/k^{1/2},$$

$$C = \int_{m_2}^1 (1 - m^2)^{k/2}\, dm, \qquad \text{where} \qquad (1 - m_2{}^2)^{1/2} = 1 - (\epsilon/4(k)^{1/2}), \quad 0 < m_2 .$$

It is easily seen that one can choose $\epsilon > 0$ so small and then k so large that $B/A < \gamma$ and $C/A < \delta$. (Note that $m_2 \sim (1/2^{1/2})\, \epsilon^{1/2} k^{-1/4}$.) Choose such a pair ϵ, k and fix it.

If n is a sufficiently large natural number the $k + 1$ dimensional unit sphere $S^{k+1} = \{x \in R^{k+2}: |x| = 1\}$ can be divided into n sets having equal measure and diameter at most $\epsilon/10(\bar{k})^{1/2}$. Choose a point from each set and let S be the set of these points.

Let $V = V_1 \cup V_2$, $V_1 \cap V_2 = \varnothing$, $|V_1| = |V_2| = n$ and let ϕ_i: $V_i \to S$ be a bijection $i = 1, 2$. We shall define a graph G with point set V. Join a point $x \in V_1$ to a point $y \in V_2$ if $|\phi_1(x) - \phi_2(y)| < 2^{1/2} - \epsilon/k^{1/2}$. If $x, y \in V_i$, join x to y if $|\phi_i(x) - \phi_i(y)| > 2 - \epsilon/k^{1/2}$.

Let us check first that G does not contain a complete quadrilateral. For if G did have a K_4 then S would contain four points, say a_1, a_2, b_1, and b_2 such that $|a_1 - a_2| > 2 - \epsilon/k^{1/2}$, $|b_1 - b_2| > 2 - \epsilon/k^{1/2}$ and $|a_i - b_j| < 2^{1/2} - \epsilon/k^{1/2}$, $i, j = 1, 2$. Then

$$\max\{(a_1, a_2), (b_1, b_2)\} < -1 + (2\epsilon/k^{1/2}) - (\epsilon^2/2k)$$

and

$$2(a_i, b_j) > 2(2)^{1/2}(\epsilon/k^{1/2}) - (\epsilon^2/k),$$

where, as usual, (x, y) denotes the inner product. Thus

$$\begin{aligned} 0 &\leqslant |a_1 + a_2 - b_1 - b_2|^2 \\ &< 4 + 4(-1 + (2\epsilon/k^{1/2}) - (\epsilon^2/2k)) - 8(2)^{1/2}(\epsilon/k^{1/2}) + 4(\epsilon^2/k) \\ &= -8(2^{1/2} - 1)(\epsilon/k^{1/2}) + 2(\epsilon^2/k). \end{aligned}$$

This contradiction shows that G does not contain a K_4.

Denote by t_i the maximal number of independent points in V_i, by $2A_1$ the measure of S^{k+1} and by C_1 the measure of the cap of S^{k+1} with diameter $2 - (\epsilon/2(k)^{1/2})$. Clearly t_i is at most the number of points of S on a spherical cap with diameter $2 - (2\epsilon/3(k)^{1/2})$. Furthermore, the measure of a d dimensional sphere is proportional with the dth power of the radius. Thus

$$t_i \leqslant (C_1/A_1)(n/2) < (C/2A)n < (\delta/2)n.$$

Therefore G contains at most δn independent points.

It is also easy to check that the degree of a point of G is at least $((A - B)/A)(n/2) > (1 - \gamma)(n/2)$, and so G has at least $(1 - \gamma)(n^2/2)$ edges. This completes the proof of the theorem.

There remains the following problem. Does there exist a $G(n, [n^2/8])$ without a K_4 and at most $o(n)$ independent points? (As usual, $G(n, m)$ denotes a graph with n points and m edges.) At present we do not see a promising line of attack. The most we could hope for is the following. For every $\eta > 0$ there exists an $\epsilon > 0$ such that whenever n is sufficiently large, some $G = G(n, [(n^2/8)(1 + \epsilon)])$ satisfies $I(G) < \eta n$ and $\alpha(G) < 4$. Our method does not seem suitable for such a construction.

Naturally it is possible that such a graph does not exist and the result of Szemerédi can be extended as follows. There exists a constant $c > 0$ with the following property. For every $\epsilon > 0$ there exists an $n_0 = n_0(\epsilon, c)$ such that if $n \geqslant n_0$, $G = G(n, [(n^2/8)(1 + \epsilon)])$ and $\alpha(G) < 4$ then $I(G) > cn$.

References

1. P. Erdös and V. T. Sós, Some remarks on Ramsey's and Turán's theorem, *Colloq. Math. Balatonfüred* (1971), 395–404.
2. E. Szemerédi, Graphs without complete quadrilaterals (in Hungarian), *Mat. Lapok* **23** (1973), 113–116.

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