MATHEMATICAL NOTES

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ON A PROBLEM OF HIRSCHHORN

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Introduction. Hirschhorn gave the following problem [1]: Let $q_1 > 1$ be given and

$$q_{n+1} = q_n + \prod_{i \leqq n} \left(1 - \frac{1}{q_i}\right)^{-1}. \tag{1}$$

is it true that $q_n = (1 + o(1))\, n \log n$?

The background of this problem is that, if p_n denotes the nth prime, then the well-known sieve method gives that the number of integers between a and b which are not divisible by any of $p_1, \cdots, p_n$, is approximately

$$(b - a) \prod_{i \leqq n} \left(1 - \frac{1}{p_i}\right).$$

The interval $(p_n, p_{n+1}]$, contains exactly one prime, i.e., exactly one integer not divisible by any p_i $(i \leqq n)$. This suggests that

$$p_{n-1} - p_n = \prod_{i \leqq n} \left(1 - \frac{1}{p_i}\right)^{-1}.$$

We know in this special case by the prime number theorem that $q_n = (1 + o(1))\, n \log n$. This shows why we are interested in this particular sequence. (Of course, the argument is only a heuristic one. the sieve method cannot be applied to short intervals and from our point of view $(p_n, p_{n-1}]$ is too

(2) $$(q_{n+1}-q_n)/\log n \to 1$$

and, consequently,

(3) $$q_n = (1+o(1))n\log n,$$

so that the conjecture of Hirschhorn is proved.

REMARK. We can prove (2) and (3) in two different ways, and the way we shall prove them is the shorter, more formal one. But just because of this we first give a short heuristic argument why (1) implies (2).

Let

(4) $$r_n = \prod_{i\le n}\left(1-\frac{1}{q_i}\right)^{-1}, \qquad r_0 = 1.$$

Then

(5) $$q_{n+1} = q_n + r_n$$

and

(6) $$r_{n+1} = r_n\left(1-\frac{1}{q_{n+1}}\right)^{-1} = r_n + \frac{r_n}{q_{n+1}} + O\left(\frac{r_n}{q^2_{n+1}}\right).$$

Now, r_n and q_n form a "self-regulating" system in the following sense: if q_n were essentially larger than $n\log n$ for $n\in(a,b)$ and the interval (a,b) were long enough, then, by (4) r_n should become much smaller than $\log n$ and, by (5), after a while q_n would become smaller than $n\log n$. Similarly, if q_n were essentially smaller than $n\log n$ for a long period, then r_n would become larger than $\log n$ and, consequently, q_n also would become larger than $n\log n$. Of course, this argument does not exclude the possibility that r_n and q_n are oscillating about $\log n$ and $n\log n$ respectively, but, because of (6) the "inertia" of r_n is too great, more exactly, r_n changes only very slowly and does not "feel" minor changes in q_n. Thus the system $\{r_n, q_n\}$ is unable to oscillate.

The exact proof. We introduce two new sequences: $s_n = nr_n$ and $d_n = s_n - q_n$. Since

$$\frac{1}{q_n}-\frac{1}{q_{n+1}} = \frac{r_n}{q_nq_{n+1}}$$

and q_n is monotone increasing, (6) can be replaced by the more convenient

(7) $$r_{n+1} = r_n + \frac{r_n}{q_n} + O\left(\frac{r_n}{q^2_n}\right) = r_n + \frac{s_n-q_n}{nq_n} + \frac{1}{n} + O\left(\frac{r_n}{q^2_n}\right).$$

Further by (7)

(8) $$\begin{aligned} d_{n+1}-d_n &= (s_{n+1}-s_n)-(q_{n+1}-q_n) = n(r_{n+1}-r_n)+r_{n+1}-r_n \\ &= (n+1)(r_{n+1}-r_n) = \left(1+\frac{1}{n}\right)\frac{s_n}{q_n} + O\left(\frac{s_n}{q^2_n}\right). \end{aligned}$$

(A) First we need that q_n/n tends to infinity. A trivial induction gives that $q_n > n$ and it is also trivial that r_n is monotone increasing. From the left side of (7) we get

(9) $$r_{n+1}-r_n = (1+o(1))\frac{r_n}{q_n} = (1+o(1))\frac{s_n}{nq_n}.$$

Further,

$$q_n = q_1 + \sum_{i=1}^{n-1} r_i < q_1 + (n-1) r_n < s_n + q_1.$$

Thus

$$r_{n+1} - r_n \geqq (1 + o(1)) \frac{1}{n}.$$

This implies that $r_n \geqq \log n - o(\log n)$ and, consequently,

(10) $$q_n \geqq n \log n - o(n \log n).$$

(B) By (10) $s_n/q_n < 2s_n/n \log n = o(r_n)$, i.e., $s_n/q_n = o(q_{n+1} - q_n)$, and it follows from (8) that

(11) $$s_{n+1} - s_n = (1 + o(1))(q_{n+1} - q_n).$$

Since s_n and q_n tend to infinity, (11) yields

(12) $$s_n/q_n \to 1.$$

Now, applying (12) to (9) we obtain

(13) $$r_{n+1} - r_n = (1 + o(1))/n.$$

Thus

(14) $$r_n = (1 + o(1)) \log n \quad \text{and} \quad q_n = (1 + o(1))\, n \log n$$

which proves (2) and (3).
(C) (14) can be improved in the following way: By (12) and (8)

(15) $$d_{n+1} - d_n = 1 + o(1),$$

and hence

(16) $$d_n = n + o(n).$$

But (16) and (7) give

$$r_{n+1} - r_n = \frac{1}{n} + (1 + o(1)) \frac{1}{n \log n} + O\left(\frac{1}{n^2}\right).$$

Thus

(17) $$r_n = \log n + (1 + o(1)) \log \log n$$

and

(18) $$q_n = n \log n + (1 + o(1)) n \log \log n.$$

Here we used $q_n = \sum_{i=1}^{n-1} r_i + q_1$ and

(19) $$\sum_{i=1}^{n} \log i = n \log n - n + O(\log n),$$

(20) $$\sum_{i=2}^{n} \log \log i = n \log \log n + O(n/\log n).$$

(D) Iterating the method of (C) we can improve (17) and (18) in the following way: From (8)

$$d_{n+1} - d_n = (1 + 1/n)\left(1 + O\left(\frac{\log \log n}{\log n}\right)\right) = 1 + O\left(\frac{\log \log n}{\log n}\right).$$