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DECOMPOSITION OF SPHERES IN HILBERT SPACES

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Abstract: A simple construction of a graph with $\aleph_2$ vertices and with the chromatic number $\aleph_1$ whose every subgraph spanned by $\aleph_1$ vertices has chromatic number $\le \aleph_0$ is given.

Key word: Chromatic number of a graph.

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Assume the generalized continuum hypothesis. Consider the unit sphere of the Hilbert space of $\aleph_{\alpha+2}$ dimensions. We join two of its points by an edge if their distance is greater than $\frac{3}{2}$. Since $\frac{3}{2} < \sqrt{3}$ the chromatic number of this graph is by the following theorem $\aleph_{\alpha+1}$ (a graph is called m-chromatic if one can color its vertices by m colors so that two vertices which get the same color are not joined, but one cannot do this with fewer than m colors). On the other hand every subgraph spanned by $\aleph_{\alpha+1}$ vertices has again by the following theorem chromatic number $\le \aleph_\alpha$. A different construction of such graphs is given in [1].

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Theorem. Let $\aleph_0 \leq n < m$ be cardinal numbers. Then (i) - (iii) are equivalent and imply (iv), moreover, under generalized continuum hypothesis they are equivalent to (iv).

(i) For every $c > \sqrt{2}$ the unit sphere in a Hilbert space of m dimensions can be written as a union of n sets with diameter $< c$.

(ii) There is a number $c \in (\sqrt{2}, \sqrt{3})$ such that the unit sphere in $\ell_2(m)$ can be written as a union of n sets with diameter $< c$.

(iii) There is a family $\mathcal{C}$ of subsets of m such that card $(\mathcal{C}) \leq n$ and $\mathcal{C}$ separates points of m (i.e. for $\alpha, \beta \in m, \alpha \neq \beta$ there is a set $C \in \mathcal{C}$ with card $(C \cap \{\alpha, \beta\}) = 1$).

(iv) $m \leq 2^n$

Proof. The implications (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious. (ii) $\Rightarrow$ (iii): Let $\{A_\sigma; \sigma \in n\}$ be sets in $\ell_2(m)$ with diameter $< \sqrt{3}$ covering the unit sphere in $\ell_2(m)$. For $\alpha, \beta \in m, \alpha \neq \beta$ put

$$x_{\alpha,\beta}(\gamma) = {}^{1}/\sqrt{2} \text{ for } \gamma = \alpha$$

$$= {}^{-1}/\sqrt{2} \text{ for } \gamma = \beta$$

$$0 \text{ otherwise.}$$

Put $C_\sigma = \{\alpha \in m;$ there exists $\beta \in m, \beta \neq \alpha$ such that $x_{\alpha,\beta} \in A_\sigma\}$.

If $\alpha, \beta \in m, \alpha \neq \beta$ then there is a σ such that $x_{\alpha,\beta} \in A_\sigma$. Consequently, $\alpha \in C_\sigma$ and $\beta \notin C_\sigma$ since $\| x_{\alpha,\beta} - x_{\beta,\gamma} \| \geq \sqrt{3}$ for any γ. Therefore the family $\{C_\sigma; \sigma \in n\}$ separates points in m.

(iii) $\Longrightarrow$ (i): Let $0 < \varepsilon < \frac{1}{2}$. Let $\mathcal{A}$ be a family of subsets of m separating points of m. We may and will suppose that $\mathcal{A}$ is closed under complements and finite intersections. Let $\mathcal{B}$ be the system of all pairs of finite sequences $\{(A_1,\dots,A_p), (r_1,\dots,r_p)\}$ where $A_1,\dots,A_p \in \mathcal{A}$ are nonempty and disjoint and $r_1,\dots,r_p$ are rational numbers that $1 > \sum_{i=1}^{p} r_i^2 > (1-\varepsilon)^2$. For $\sigma \in \mathcal{B}$, $\sigma = \{(A_1,\dots,A_p), (r_1,\dots,r_p)\}$ put $C_\sigma = \{x \in \ell_2(m);\ \|x\| = 1$ and there are $\alpha_i \in A_i$ such that $\sum_{i=1}^{p} (x(\alpha_i) - r_i)^2 < \varepsilon^2\}$. First prove that the family $\{C_\sigma;\ \sigma \in \mathcal{B}\}$ covers the unit sphere in $\ell_2(m)$. If $x \in \ell_2(m)$, $\|x\| = 1$ find $\alpha_1,\dots,\alpha_p$ such that $\|y - x\| < \varepsilon$ where $y(\alpha_i) = x(\alpha_i)$ and $y(\alpha) = 0$ for all other α. Since $\mathcal{A}$ is closed under complements and finite intersections, we can find disjoint sets $A_i \in \mathcal{A}$, $i = 1,\dots,p$ such that $\alpha_i \in A_i$. Choosing r_i sufficiently close to $x(\alpha_i)$, we obtain $x \in C_\sigma$, where $\sigma = \{(A_1,\dots,A_p), (r_1,\dots,r_p)\}$.

Let us estimate the diameter of C_σ. If $x,y \in C_\sigma$, choose $\alpha_i \in A_i$, $\beta_i \in A_i$, $(i = 1,\dots,p)$ such that

$$\sum_{i=1}^{p} (x(\alpha_i) - r_i)^2 < \varepsilon^2 \quad\text{and}\quad \sum_{i=1}^{p} (y(\beta_i) - r_i)^2 < \varepsilon^2 .$$

Put $x_1(\alpha_i) = x(\alpha_i)$, $x_2(\alpha_i) = r_i$ for $i = 1,\dots,p$,

$x_1(\alpha) = x_2(\alpha) = 0$ for all other α,

$y_1(\beta_i) = y(\beta_i)$, $y_2(\beta_i) = r_i$ for $i = 1,\dots,p$,

$y_1(\beta) = y_2(\beta) = 0$ for all other β.

Then $1 = \|x - x_1\|^2 + \|x_1\|^2 \geq \|x - x_1\|^2 + (\|x_2\| - \|x_1 - x_2\|)^2 \geq \|x - x_1\|^2 + (1 - 2\varepsilon)^2$

thus $\|x - x_1\|^2 \leq 4\varepsilon - 4\varepsilon^2 \leq 4\varepsilon$;

similarly we prove that $\| y - y_1 \| \leq 2\sqrt{\varepsilon}$, therefore $\| x - y \| \leq \| x - x_1 \| + \| x_1 - x_2 \| + \| x_2 - y_2 \| + \| y_2 - y_1 \| + \| y_1 - y \| \leq \sqrt{2} + 4\sqrt{\varepsilon} + 2\varepsilon$.

(iv) $\Longrightarrow$ (iii): We can suppose that $m = 2^n$ and n is a set of ordinals such that card $T_\alpha < n$ for any $\alpha \in n$. For $\alpha \in n$ and $B \subset T_\alpha$ put $A_{\alpha, B} = \{ C \subset n;\ C \cap T_\alpha = B \}$. The family $\{ A_{\alpha,B} ;\ \alpha \in n,\ B \subset T_\alpha \}$ separates points in 2^n and, since $2^{\text{card } T_\alpha} \leq n$, its cardinality is $\leq n$.

Remark 1: Not using the continuum hypothesis we can prove (in the same way as in (iv) $\Longrightarrow$ (iii)) that (iii) holds for such cardinals n, m that

(a) $m \leq 2^n$

(b) If $n' < n$ then $2^{n'} \leq n$.

Remark 2: If $\aleph_0 \leq n < m$ are cardinal numbers satisfying the condition (iii) of the theorem and if $n^{\aleph_0} = n$ then the unit sphere in $\ell_2(m)$ can be written as a union of n sets with diameter $\leq \sqrt{2}$. (One can take the covers $\mathcal{C}_p$ with diameter $< \sqrt{2} + \frac{1}{p}$ and put $\mathcal{C} = \{ \bigcap_{p=1}^{\infty} A_{n_p} ;\ A_{n_p} \in \mathcal{C}_p \}$.) Therefore the graphs obtained by joining two points of the $\aleph_{\alpha+2}$ -dimensional Hilbert space if their distance is $> \sqrt{2}$ has the chromatic number $\aleph_{\alpha+1}$.

R e f e r e n c e

[1] P. ERDÖS and A. HAJNAL: On chromatic number of graphs and set-systems, Acta Math. Acad. Sci. Hung. 17 (1966), 61-99.

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