COLLOQUIA MATHEMATICA SOCIETATIS JÁNOS BOLYAI
10. INFINITE AND FINITE SETS, KESZTHELY (HUNGARY), 1973.

SPLITTING ALMOST-DISJOINT COLLECTIONS OF SETS INTO SUBCOLLECTIONS ADMITTING ALMOST-TRANSVERSALS

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1. INTRODUCTION

Let A be a collection of denumerable sets, and suppose that

(1.1) $\quad |A \cap B| \leqslant 1 \quad (A, B \in \mathscr{A};\ A \neq B)$,

(1.2) $\quad |\mathscr{A}| = \aleph_0$.

Does $\mathscr{A}$ necessarily admit a 1-transversal, that is, does there exist a set S satisfying the following condition?

(1.3) $\quad |A \cap S| = 1 \qquad (A \in \mathscr{A})$.

The answer is negative, but becomes affirmative if (1.3) is weakened to $1 \leqslant |A \cap S| \leqslant 2$ $(A \in \mathscr{A})$. Moreover we can always split $\mathscr{A}$ into two subcollections each admitting a 1-transversal, and this remains true if (1.1) is replaced by $|A \cap B| \leqslant 2$ (but not $|A \cap B| \leqslant 3$) or (1.2) by $|A| = \aleph_1$ (but not $|A| = \aleph_2$), but not both. We shall prove a rather wide generalization of these facts.

Definitions. A collection $\mathscr{A}$ of sets is *m-almost-disjoint* (m-a-d) if

$$A', A'' \in \mathscr{A} \ \& \ A' \neq A'' \Rightarrow |A' \cap A''| < m + 1 .$$

A set S is an *n-transversal* of a collection $\mathscr{A}$ of sets if

$$A \in \mathscr{A} \Rightarrow 1 \leqslant |A \cap S| < n + 1 .$$

We may, and shall, suppose that $S \subseteq \dot{\bigcup} \mathscr{A}$.

Given a non-empty index set I, cardinal numbers $n_i \geqslant 1$ $(i \in I)$, a non-negative integer m, and ordinal numbers μ, ν, we shall denote by $P((n_i), \aleph_\nu \to m, \aleph_\mu)$ the following proposition:

> *every* m-a-d *collection* $\mathscr{A}$ *of* $\aleph_\mu$ *sets of cardinality* $\aleph_\nu$ *can be split into subcollections* $\mathscr{A}_i$, $i \in I$, *such that* $\mathscr{A}_i$ *admits an* n_i*-transversal.*

(We shall show in Theorem 2 (of §2) that the parameters are correctly distributed on either side of the arrow in the symbol $P((n_i), \aleph_\nu \to m, \aleph_\mu)$.)

Problem. When is $P((n_i), \aleph_\nu \to m, \aleph_\mu)$ true?

The case when $|I| = 1$ was dealt with by Erdős and Hajnal [1], who proved a group of results that can be formulated as follows:

Theorem A. *For the truth of the proposition* $P(n, \aleph_\nu \to m, \aleph_\mu)$ *it is necessary and sufficient that at least one of the following conditions be satisfied:*

(i) $\mu < \nu$,

(ii) $\mu = \nu + r$ (r *finite*) *and* $n + 1 \geqslant mr + m + 2$,

(iii) $n \geqslant \aleph_0$.

The generalized continuum hypothesis is used in the proof of necessity when $\mu > \nu$.

We shall prove the appropriate generalization of Theorem A, in one direction:

Theorem 1. *For the truth of the proposition* $P((n_i), \aleph_\nu \to m, \aleph_\mu)$ *it is sufficient that at least one of the following conditions be satisfied:*

(i) $\mu < \nu$,

(ii) $\mu = \nu + r$ *(r finite) and* $\sum (n_i + 1) \geqslant mr + m + 2$,

(iii) $\sum (n_i + 1) \geqslant \aleph_0$.

This follows immediately from Theorem A in case (i), and also in the subcase of case (iii) when one of the n_i's is infinite. The other subcase of case (iii), when I is infinite, will be dealt with in §3. The more elaborate proof for case (ii) is given in §4.

We conjecture that our criterion is necessary as well as sufficient (assuming the generalized continuum hypothesis); some incomplete results in this direction will be presented in §5 (the case $r = 0$), §6 (the case $m = 1$, equal n_i's, with the consequence that condition (iii) is necessary when $\mu \geqslant \nu + \omega_0$), and §7 (falseness of $P((1, 1), \aleph_\nu \to 2, \aleph_{\nu+1})$). The case $m = 0$ (disjoint sets) is trivial. Unfortunately, even the case of equal n_i's is not clear for general m.

The axiom of choice will be assumed throughout. Always m will be understood to denote a non-negative integer.

Note. We have obtained no significant results about collections that include sets of different cardinalities, when the situation becomes more complicated. For example, the 1-a-d collection consisting of the $\aleph_0$ sets $E_\alpha = \{(\alpha, \beta)\colon 0 \leqslant \beta < \omega_1\}$ $(0 \leqslant \alpha < \omega_0)$ of cardinality $\aleph_1$, together with the $\aleph_1$ sets $F_\beta = \{(\alpha, \beta)\colon 0 \leqslant \alpha < \omega_0\}$ $(0 \leqslant \beta < \omega_1)$ of cardinality $\aleph_0$, does not admit an $\aleph_0$-transversal. By an inductive construction, it is not hard to show (assuming the generalized continuum hypothesis) that for every $n < \omega_0$ there is a 1-a-d collection consisting of sets of cardinalities $\aleph_0, \ldots, \aleph_n$ that cannot be split into $n - 1$ subcollections each admitting an $\aleph_0$-transversal. We do not know whether every 1-a-d collection of sets of $\aleph_1$ different cardinalities can be split into $\aleph_0$ subcollections each admitting a 1-transversal; by Theorem A (iii) this is possible when there are only $\aleph_0$ different cardinalities.

2. MONOTONICITY

Theorem 2. *For a given index-set* I, *if* $n_i' \geqslant n_i$ $(i \in I)$, $\nu' \geqslant \nu$, $m \geqslant m'$, *and* $\mu \geqslant \mu'$, *then*

$$P((n_i), \aleph_\nu \to m, \aleph_\mu) \Rightarrow P((n_i'), \aleph_{\nu'} \to m', \aleph_{\mu'}) .$$

Proof. (a) Monotonicity with respect to the system (n_i) is obvious, because if $n' > n$ then an n-transversal is automatically also an n'-transversal.

(b) Monotonicity with respect to m is obvious, because if $m > m'$ then every m'-a-d collection is automatically m-a-d.

(c) Monotonicity with respect to μ is obvious because if $\mu > \mu'$ then every m-a-d collection of $\aleph_{\mu'}$ sets of cardinality $\aleph_\nu$ can be extended to such a collection of $\aleph_\mu$ sets by adjoining $\aleph_\mu$ additional sets which are disjoint from one another and from all the original sets.

(d) Monotonicity with respect to ν is less obvious; it can easily be seen to follow, however, from the following lemma.

Lemma 1. *Given any* m-a-d *collection* $\mathscr{A}$ *of* $\aleph_\mu$ *sets of cardinality* $\aleph_{\nu'}$, $\nu' > \nu$, *we can find a set* $T \subseteq \bigcup \mathscr{A}$ *such that* $|T \cap A| = \aleph_\nu$ *for all* $A \in \mathscr{A}$.

Proof. Define sets T_α, $0 \leqslant \alpha < \omega_\nu$, satisfying the conditions $T_\alpha \subseteq \bigcup \mathscr{A}$, $T_\alpha \cap \big(\bigcup_{\beta < \alpha} T_\beta \big) = \phi$, $1 \leqslant |T_\alpha \cap A| \leqslant \aleph_0$ for all $A \in \mathscr{A}$, by transfinite induction, as follows. Given any α, let $\mathscr{A}_\alpha = \{A \setminus T_\alpha^*\colon A \in \mathscr{A}\}$, where $T_\alpha^* = \bigcup_{\beta < \alpha} T_\beta$. The collection $\mathscr{A}_\alpha$ satisfies the same conditions as $\mathscr{A}$. By Theorem A (iii), we can find a set $T_\alpha \subseteq \bigcup \mathscr{A}_\alpha$ such that $1 \leqslant |T_\alpha \cap A| \leqslant \aleph_0$ for all $A \in \mathscr{A}_\alpha$. Since T_α satisfies the three conditions, the induction can proceed. Finally, we set $T = \bigcup \{T_\alpha\colon 0 \leqslant \alpha < \omega_\nu\}$.

3. INFINITE INDEX-SET

In view of Theorem 2, the proof that $P((n_i), \aleph_\nu \to m, \aleph_\mu)$ is true when the index-set I is infinite reduces to the proof of the following.

Proposition A. *Every* m-a-d *collection* $\mathscr{A}$ *of denumerable sets can be split into denumerably many subcollections* $\mathscr{A}_i$, $i = 1, 2, \ldots$, *such that* $\mathscr{A}_i$ *admits a* 1-*transversal.*

The proof will use the notion of a closed subcollection, and two lemmas.

Definition. Let $\mathscr{A}$ be an m-a-d collection of sets of cardinality $\aleph_\nu$. A subcollection $\mathscr{B}$ of $\mathscr{A}$ is *closed* if

$$A \in \mathscr{A} \ \& \ |A \cap \bigcup \mathscr{B}| > m \Rightarrow A \in \mathscr{B} .$$

Obviously there exists a smallest closed subcollection containing any given subcollection $\mathscr{C}$; it will be denoted by $\bar{\mathscr{C}}$.

Lemma 2. *Under the conditions of the above definition, if* $|\mathscr{C}| \geqslant \aleph_\nu$ *then* $|\bar{\mathscr{C}}| = |\mathscr{C}|$.

Proof. For any $\mathscr{B} \subseteq \mathscr{A}$, let $\varphi(\mathscr{B}) = \{A \in \mathscr{A}: |A \cap \bigcup \mathscr{B}| > m\}$. Since any $(m + 1)$-element set is contained in at most a single set $A \in \mathscr{A}$, if $|\mathscr{B}| \geqslant \aleph_\nu$ we have $|\varphi(\mathscr{B})| \leqslant |\bigcup \mathscr{B}|^{m+1} = |\mathscr{B}|$. The result now follows from the obvious fact that $\bar{\mathscr{C}} = \mathscr{C} \cup \varphi(\mathscr{C}) \cup \varphi(\varphi(\mathscr{C})) \ldots$.

Lemma 3. *Every* m-a-d *collection* $\mathscr{A}$ *of denumerable sets admits a well ordering* $\prec$ *such that*

$$(3.1) \qquad A \in \mathscr{A} \Rightarrow |A \cap \bigcup\{B: B \prec A\}| < \aleph_0 .$$

Proof. We use transfinite induction on $|\mathscr{A}|$, starting with $|\mathscr{A}| = \aleph_0$, in which case any ω_0-ordering of $\mathscr{A}$ has the property. Now let $|\mathscr{A}| = \aleph_\theta > \aleph_0$, and suppose the result true for smaller cardinalities. Write

$$\mathscr{A} = \{A_\alpha: 0 \leqslant \alpha < \omega_\theta\}; \quad \mathscr{A}_\alpha = \{A_\beta: 0 \leqslant \beta < \alpha\} .$$

By Lemma 2 we have $|\bar{\mathscr{A}}_\alpha| = |\mathscr{A}_\alpha| = \bar{\alpha} < \aleph_\theta$ for $\omega_0 \leqslant \alpha < \omega_\theta$, and so by the induction hypothesis there is a well ordering $\prec_\alpha$ of $\mathscr{A}_\alpha$ such that

$$(3.2) \qquad A \in \bar{\mathscr{A}}_\alpha \Rightarrow |A \cap \bigcup\{B \in \bar{\mathscr{A}}_\alpha: B \prec_\alpha A\}| < \aleph_0 .$$

For any $A \in \mathscr{A}$ denote by $\alpha(A)$ the least ordinal $\alpha \geqslant \omega_0$ such that $A \in \bar{\mathscr{A}}_\alpha$, and define a relation $\prec$ on $\mathscr{A}$ as follows: $A \prec B$ if and only if *either* $\alpha(A) < \alpha(B)$ *or* $\alpha(A) = \alpha(B) = \alpha$, say, and $A \prec_\alpha B$. This is easily seen to be a well ordering of $\mathscr{A}$; we show that (1) is satisfied.

Let $A \in \mathscr{A}$, and suppose that $\alpha(A) = \alpha$. For each element $a \in A$, denote by $\beta(a)$ the least ordinal $\beta \geq \omega_0$ such that $a \in \bigcup \bar{\mathscr{A}}_\beta$, and let the $m+1$ elements of A with respective least values for β be $a_1, \ldots$ $\ldots, a_{m+1}$. Thus

$$(3.3) \qquad \begin{aligned} &\beta(a_1) \leq \ldots \leq \beta(a_{m+1}) \leq \beta(a) \leq \alpha \qquad \text{for} \\ &a \in A \setminus \{a_1, \ldots, a_{m+1}\}. \end{aligned}$$

Let $\beta(a_{m+1}) = \beta$; then since $\bigcup \bar{\mathscr{A}}_\beta$ is clearly non-decreasing as a function of β, we have $\{a_1, \ldots, a_{m+1}\} \subseteq \bigcup \bar{\mathscr{A}}_\beta$, whence by the definition of closure $A \in \bar{\mathscr{A}}_\beta$; consequently $\alpha \leq \beta$, and by (3.3) $\alpha = \beta$. In view of (3.3) we see that $\beta(a) = \alpha$ for $a \in A \setminus \{a_1, \ldots, a_m\}$; thus

$$(3.4) \qquad |A \cap \bigcup\{\bar{\mathscr{A}}_\gamma\colon \gamma < \alpha\}| \leq |\{a_1, \ldots, a_m\}| = m .$$

On the other hand, by (3.2)

$$(3.5) \qquad \begin{aligned} &|A \cap \bigcup\{B \in \bar{\mathscr{A}}_\alpha \setminus \bigcup_{\beta < \alpha} \bar{\mathscr{A}}_\beta\colon B \prec A\}| \leq \\ &\leq |A \cap \bigcup\{B \in \bar{\mathscr{A}}_\alpha\colon B \prec_\alpha A\}| < \aleph_0 , \end{aligned}$$

and by (3.4) and (3.5), since if $B \prec A$ then certainly $B \in \bigcup_{\beta \leq \alpha} \bar{\mathscr{A}}_\beta$, we conclude as required that $|A \cap \bigcup\{B\colon B \prec A\}| < \aleph_0$.

Proof of Proposition A. Let $\prec$ be a well ordering of $\mathscr{A}$ such that relation (3.1) of Lemma 3 holds. By (3.1) we can select for each set $A \in$ $\in \mathscr{A}$ an element $a(A) \in A \setminus \bigcup\{B\colon B \prec A\}$. Clearly, these elements are distinct, and

$$(3.6) \qquad A \cap \{a(B)\colon A \prec B\} = \phi .$$

Using (3.1) again, by transfinite induction we can select for each set $A \in$ $\in \mathscr{A}$ a positive integer $i(A)$ in such a way that

$$(3.7) \qquad B \prec A \ \& \ a(B) \in A \Rightarrow i(A) \neq i(B) .$$

Define $\mathscr{A}_i = \{A \in \mathscr{A}: i(A) = i\}$, $S_i = \{a(A): A \in \mathscr{A}_i\}$, $i = 1, 2, \ldots$. Now if $i(A) = i$ then $a(A) \in A \cap S_i$, so $|A \cap S_i| \geq 1$; on the other hand if $a(B) \in A$ for some $B \neq A$ then $B \prec A$ by (3.6) and $i(B) \neq i$ by (3.7), so that $B \notin \mathscr{A}_i$ and $a(B) \notin S_i$. Hence $|A \cap S_i| = 1$, and $\mathscr{A}_i$ admits the 1-transversal S_i.

Remark. The following slightly stronger result is true.

Proposition A′. *Every* m-a-d *collection* $\mathscr{A}$ *of denumerable sets can be split into denumerably many subcollections* $\mathscr{A}_i$, $i = 1, 2, \ldots$, *such that* $\mathscr{A}_i$ *admits a* 1-*transversal* S_i *and* $\bigcup_{i=1}^{\infty} S_i = \bigcup \mathscr{A}$.

We shall not give the details of the proof, which is a minor elaboration of that of Proposition A. One assigns *each* element of $\bigcup \mathscr{A}$ to one of the sets S_i, by transfinite induction, dealing with the sets $A \in \mathscr{A}$ in order according to the well ordering $\prec$ given by Lemma 3.

4. PROOF OF THEOREM 1 IN CASE (ii)

We restate what is to be proved.

Proposition B. *Let* $\mathscr{A}$ *be an* m-a-d *collection of* $\aleph_{\nu+r}$ *sets of cardinality* $\aleph_\nu$ *(r finite), and suppose that* $\sum(n_i + 1) \geq mr + m + 2$. *Then* $\mathscr{A}$ *can be split into subcollections* $\mathscr{A}_i$, $i \in I$, *such that* $\mathscr{A}_i$ *admits an* n_i-*transversal.*

The proof will use the notions of strong and weak precedence, and two additional lemmas.

Definitions. Given a well ordering $\prec$ of $\mathscr{A}$, we shall say that a set $B \in \mathscr{A}$ *strongly precedes* a set $A \in \mathscr{A}$, and write $B \prec\prec A$, if $|\{C: B \preceq C \prec A\}| \geq \aleph_\nu$. We shall say that B *weakly precedes* A, and write $B \precsim A$, if B precedes A but not strongly. We observe that $|\{B: B \precsim A\}| < \aleph_\nu$, for any $A \in \mathscr{A}$, because if B_0 is the first (with respect to $\prec$) set satisfying $B_0 \precsim A$ then

$$|\{B: B \precsim A\}| = |\{C: B_0 \preceq C \prec A\}| < \aleph_\nu .$$

Lemma 4. *There is a well ordering* $\prec$ *of* $\mathscr{A}$ *such that*

$$A \in \mathscr{A} \Rightarrow |A \cap \bigcup \{B\colon B \prec\prec A\}| \leqslant mr .$$

Proof. We use induction on r, starting with $r = 0$, in which case any ω_ν-ordering of $\mathscr{A}$ has the property. Now let $|\mathscr{A}| = \aleph_{\nu+r}$, and suppose the result true for $r-1$, where $r \geqslant 1$. We write

$$\mathscr{A} = \{A_\alpha\colon 0 \leqslant \alpha < \omega_{\nu+r}\} ; \quad \mathscr{A}_\alpha = \{A_\beta\colon 0 \leqslant \beta < \alpha\} .$$

By Lemma 2, for $\omega_{\nu+r-1} \leqslant \alpha < \omega_{\nu+r}$ we have $|\bar{\mathscr{A}}_\alpha| = |\mathscr{A}_\alpha| = |\alpha| = \aleph_{\nu+r-1}$, and so by the induction hypothesis there is a well ordering $\prec_\alpha$ of $\bar{\mathscr{A}}_\alpha$ such that

$$A \in \bar{\mathscr{A}}_\alpha \Rightarrow |A \cap \bigcup \{B \in \bar{\mathscr{A}}_\alpha\colon B \prec_\alpha\prec_\alpha A\}| \leqslant m(r-1) .$$

For any $A \in \mathscr{A}$ denote by $\alpha(A)$ the least ordinal $\alpha \geqslant \omega_{\nu+r-1}$ such that $A \in \bar{\mathscr{A}}_\alpha$, and define a relation $\prec$ on $\mathscr{A}$ as follows: $A \prec B$ if and only if *either* $\alpha(A) < \alpha(B)$ *or* $\alpha(A) = \alpha(B) = \alpha$, say, and $A \prec_\alpha B$. This is easily seen to be a well ordering of $\mathscr{A}$; we show that it has the required property.

Let $A \in \mathscr{A}$, and suppose that $\alpha(A) = \alpha$. For each element $a \in A$, denote by $\beta(a)$ the least ordinal $\beta \geqslant \omega_{\nu+r-1}$ such that $a \in \bigcup \bar{\mathscr{A}}_\beta$, and let the $m+1$ elements of A with respective least values for β be $a_1, \dots, a_{m+1}$. Thus

$$(4.1) \qquad \beta(a_1) \leqslant \dots \leqslant \beta(a_{m+1}) \leqslant \beta(a) \leqslant \alpha \quad \text{for} \quad a \in A \setminus \{a_1, \dots, a_{m+1}\} .$$

Let $\beta(a_{m+1}) = \beta$; then since $\bigcup \bar{\mathscr{A}}_\beta$ is clearly non-decreasing as a function of β, we have $\{a_1, \dots, a_{m+1}\} \subseteq \bigcup \bar{\mathscr{A}}_\beta$, whence by the definition of closure $A \in \bar{\mathscr{A}}_\beta$; consequently $\alpha \leqslant \beta$, and by (8) $\alpha = \beta$. In view of (4.1) we see that $\beta(a) = \alpha$ for $a \in A \setminus \{a_1, \dots, a_m\}$; thus $|A \cap \bigcup \{\bar{\mathscr{A}}_\gamma\colon \gamma < \alpha\}| \leqslant |\{a_1, \dots, a_m\}| = m$, that is, $|A \cap \bigcup \{B\colon \alpha(B) < \alpha\}| \leqslant m$. On the other hand, $|A \cap \bigcup \{B \in \bar{\mathscr{A}}_\alpha\colon B \prec_\alpha\prec_\alpha A\}| \leqslant m(r-1)$, and since if $B \prec\prec A$ it is clear that either $\alpha(B) < \alpha$ or $B \prec_\alpha\prec_\alpha A$, we conclude from the last two inequalities that $|A \cap \bigcup \{B\colon B \prec\prec A\}| \leqslant mr$, as required.

Remark. Lemma 4 becomes false if $\prec\prec$ is replaced by $\prec$.

Lemma 5. *If* $\prec$ *is the well ordering of Lemma* 4, *then we can select in each set* $A \in \mathscr{A}$ *an element* $a(A) \in A \setminus \bigcup\{B\colon B \prec A\}$ *in such a way that*

$$A \in \mathscr{A} \Rightarrow |A \cap \{a(B)\colon B \prec A\}| \leq mr + m + 1 .$$

Proof. We use transfinite induction with respect to the well ordering $\prec$. Suppose that $a(B)$ has been selected for all $B \prec A$. Let

$$\mathscr{B}(A) = \{B \in \mathscr{A}\colon |B \cap \{a(C)\colon C \stackrel{\sim}{\prec} A\}| \geq m + 1\},$$

and observe that $|\mathscr{B}(A)| < \aleph_\nu$, since there are fewer than $\aleph_\nu$ sets $C \stackrel{\sim}{\prec} A$, therefore fewer than $\aleph_\nu$ elements $a(C)$, $C \stackrel{\sim}{\prec} A$, and therefore fewer than $\aleph_\nu$ $(m + 1)$-tuples of them; and any $(m + 1)$-tuple is in at most a single set $B \in \mathscr{A}$. Select as $a(A)$ any element of the set

$$A \setminus \bigcup\{C\colon C \stackrel{\sim}{\prec} A\} \setminus \bigcup[\mathscr{B}(A) \setminus \{A\}] \setminus \bigcup\{B\colon B \prec\prec A\} .$$

Such an element exists, because each of the unions has fewer than $\aleph_\nu$ elements in common with A, in view of the m-a-d property of $\mathscr{A}$, the property of $\prec$ given in Lemma 4, and the inequalities $|\{C\colon C \stackrel{\sim}{\prec} A\}| < \aleph_\nu$, $|\mathscr{B}(A)| < \aleph_\nu$.

It will now (again in view of the property of $\prec$) be sufficient to show that

$$|A \cap \{a(C)\colon C \stackrel{\sim}{\prec} A\}| \leq m + 1 .$$

Suppose if possible that this is false; then there exist sets $C_1, \ldots, C_{m+2} \in \mathscr{A}$ such that $C_1 \stackrel{\sim}{\prec} C_2 \stackrel{\sim}{\prec} \ldots \stackrel{\sim}{\prec} C_{m+2} \stackrel{\sim}{\prec} A$ and $a(C_\mu) \in A$ for $\mu = 1, 2, \ldots, m + 2$. Then in particular $A \in \mathscr{B}(C_{m+2})$, and so $a(C_{m+2}) \in A \in \mathscr{B}(C_{m+2})$; whence

$$a(C_{m+2}) \in \bigcup[\mathscr{B}(C_{m+2}) \setminus \{C_{m+2}\}] ,$$

which contradicts the way $a(C_{m+2})$ was chosen.

Proof of Proposition B. Let $\prec$ be the well ordering of $\mathscr{A}$ of Lemma 4, and let $a(A)$, $A \in \mathscr{A}$, be the elements selected in accordance with

Lemma 5. By transfinite induction with respect to $\prec$, we now define for each set $A \in \mathscr{A}$ an index $i(A) \in I$ and an index $\epsilon(A)$ equal to 0 or 1, such that if we set

$$(4.2) \qquad S_j(A) = \{a(B)\colon B \preceq A \ \&\ i(B) = j \ \&\ \epsilon(B) = 1\}$$

then

$$(4.3) \qquad B \preceq A \Rightarrow 1 \leqslant |B \cap S_{i(B)}(A)| < n_{i(B)} + 1 .$$

Suppose this done for all $A \prec A_0$. By the property of the elements $a(A)$ given in Lemma 5, there are at most $mr + m + 1$ sets $A \prec A_0$ for which $a(A) \in A_0$. Hence there exists at least one index $i \in I$ such that $i(A) = i$ for no more than n_i of these sets $A \prec A_0$. (Otherwise their total number would be at least $\sum (n_i + 1) \geqslant mr + m + 2$.) We let $i(A_0) = i$ for some such index i; if $i(A) = i$ for *no* set $A \prec A_0$ with $a(A) \in A_0$, then we set $\epsilon(A_0) = 1$; otherwise, we set $\epsilon(A_0) = 0$.

Let us show that (4.3) holds for $A = A_0$. If $B \prec A_0$, we observe that since $a(C) \notin B$ for $B \prec C \preceq A_0$ (see Lemma 5), $B \cap S_{i(B)}(A_0) = B \cap S_{i(B)}(B)$, and so the truth of the conclusion of (4.3) for $A = A_0$ follows from its truth for $A = B$. If $B = A_0$, then it is clear from the way that $i(A_0)$ and $\epsilon(A_0)$ have been chosen that we have the required inequalities $1 \leqslant |A_0 \cap S_{i(A_0)}(A_0)| < n_{i(A_0)} + 1$.

Now define

$$\mathscr{A}_i = \{A \in \mathscr{A} : i(A) = i\}\,, \quad S_i = \bigcup \{S_i(A)\colon A \in \mathscr{A}\} \quad \text{for} \quad i \in I\,.$$

It follows at onee from these definitions and (4.2), (4.3) that

$$1 \leqslant |A \cap S_i| < n_i + 1$$

for all sets $A \in \mathscr{A}_i$, and thus $\mathscr{A}_i$ admits the n_i-transversal S_i and the proof is complete.

5. COUNTER-EXAMPLES: $r = 0$

Theorem 3. *Theorem* 1 *is best possible in the case* $r = 0$; *more precisely, the following proposition is valid for* $m = 0, 1, 2, \ldots$.

(A_m) *There exists an* m-a-d *collection* $\mathscr{A}^m$ *of* $\aleph_\nu$ *sets of cardinality* $\aleph_\nu$, *such that whenever* $\mathscr{A}^m$ *is split into subcollections* $\mathscr{A}_i^m$, $i \in I$, *where* $\mathscr{A}_i^m$ *admits an* n_i*-transversal, we have* $\sum(n_i + 1) \geqslant m + 2$.

Proof. We use induction on m, starting with the trivial case $m = 0$: a collection of $\aleph_\nu$ disjoint sets of cardinality $\aleph_\nu$ will serve as $\mathscr{A}^0$. Given $\mathscr{A}^m$, let $\mathscr{A}^{m+1}$ consist of $m + 2$ copies $\mathscr{A}^m(1), \ldots, \mathscr{A}^m(m + 2)$ of $\mathscr{A}^m$, with disjoint unions, together with a set $B(x_1, \ldots, x_{m+2})$ for each $(m + 2)$-tuple $(x_1, \ldots, x_{m+2})$ with $x_\mu \in \bigcup \mathscr{A}^m(\mu)$ $(\mu = 1, \ldots \ldots, m + 2)$, formed by extending $x_1, \ldots, x_{m+2}$ to a set of cardinality $\aleph_\mu$. (Here and later, when sets are spoken of as being extended to sets of larger cardinality, it is understood that the added parts are disjoint from one another and from all sets already defined.)

Suppose that $\mathscr{A}^m$ has the asserted property, and (if possible) that $\mathscr{A}^{m+1}$ does not. Then $\mathscr{A}^{m+1}$ can be split into subcollections $\mathscr{A}_i^{m+1}$, $i \in I$, where $\mathscr{A}_i^{m+1}$ admits an n_i-transversal S_i and $\sum(n_i + 1) \leqslant m + 2$. Since $\mathscr{A}^{m+1}$ includes a copy of $\mathscr{A}^m$, we have $\sum(n_i + 1) = m + 2$; moreover each S_i meets every set $\bigcup \mathscr{A}^m(\mu)$. Write $\{1, \ldots, m + 2\} = \bigcup\{J_i : i \in I\}$, where $|J_i| = n_i + 1$, and select elements $x_1, \ldots, x_{m+2}$ such that $x_\mu \in \bigcup \mathscr{A}^m(\mu) \cap S_i$ if $\mu \in J_i$. Now $B = B(x_1, \ldots, x_{m+2})$ satisfies $|B \cap S_i| \geqslant n_i + 1$ for all $i \in I$, which contradicts the fact the $B \in \mathscr{A}_i^{m+1}$ for some i.

6. COUNTER-EXAMPLES: $m = 1$, EQUAL n_i

Given n, we shall describe $\mathscr{S} = (S_i)_{i \in I}$ as a *transversal system for* $\mathscr{A}$ if $|I| < \aleph_0$ and $1 \leqslant |A \cap S_i| \leqslant n$ for some i whenever $A \in \mathscr{A}$. We may and shall suppose that $\bigcup \mathscr{S} \subseteq \bigcup \mathscr{A}$. Let $f(r)$ denote the integer part of $(r + 2)/(n + 1)$.

Theorem 4. *The generalized continuum hypothesis implies that Theorem* 1 *is best possible in the case* $m = 1$, $\mu = \nu + r$ *(*r *a non-negative integer),* $n_i = n$ *(a positive integer) for all* $i \in I$*; more precisely, that the following proposition is valid for* $r = 0, 1, 2, \ldots$.

(P_r) *There exists a* 1-a-d *collection* $\mathscr{A}$ *of* $\aleph_{\nu+r}$ *sets of cardinality* $\aleph_\nu$, *such that any transversal system for* $\mathscr{A}$ *includes more than* $f(r)$ *sets of cardinality* $\aleph_{\nu+r}$.

We shall say that $(S_i)_{i\in I}$ *properly covers* a collection $\mathscr{C}$ of t-tuples (t finite) if there are indices $i_1, \dots, i_t$, no $n+2$ of which are equal, such that each set $C \in \mathscr{C}$ can be written as $\{x_1, \dots, x_t\}$ with $x_\tau \in S_{i_\tau}$ for $\tau = 1, \dots, t$.

In our inductive proof of Proposition (P_r) we shall need the following auxiliary proposition.

(Q_r^s) *There exists a* 1-a-d *collection* $\mathscr{B}$ *consisting of*

(i) *a collection* $\mathscr{D}$ *of* $\aleph_{\nu+r}$ *sets of cardinality* $\aleph_\nu$, *together with*

(ii) $\aleph_{\nu+r}$ *disjoint* *s-element collections of* $(r+2)$*-tuples* $\mathscr{C}_\beta$, $0 \leqslant \beta < \omega_{\nu+r}$, *such that any transversal system for* $\mathscr{D}$ *properly covers some* $\mathscr{C}_\beta$.

We shall prove successively $(\forall s)(Q_0^s)$, $(Q_r^1) \Rightarrow (P_r)$, and $(\forall s)(Q_r^s)$ & & $(P_r) \Rightarrow (\forall s)(Q_{r+1}^s)$, and thereby Theorem 4 will be established.

Lemma 6. *Proposition* (Q_0^s) *is true for all* $s = 1, 2, \dots$.

Proof. Choose $\aleph_\nu$ disjoint sets D^α, $0 \leqslant \alpha < \omega_\nu$, of cardinality $\aleph_\nu$, and for each s-tuple $(x^1, \dots, x^s)$, where $x^1 \in D^{\alpha_1}, \dots, x^s \in D^{\alpha_s}$ and $\alpha_1 < \dots < \alpha_s$, choose a set $D(x^1, \dots, x^s)$ of cardinality $\aleph_\nu$, all these sets being disjoint from one another and from $\bigcup D^\alpha$. Let the collection $\mathscr{D}$ consist of all these sets $D(x^1, \dots, x^s)$ together with all the sets D^α. For each element $x \in D(x^1, \dots, x^s)$ let $\mathscr{C}(x, x^1, \dots, x^s) = \{\{x^1, x\}, \dots, \{x^s, x\}\}$. Let $\mathscr{B}$ consist of $\mathscr{D}$ together with these $\aleph_\nu$ disjoint s-element collections of pairs. Clearly $\mathscr{B}$ is 1-a-d.

Now let $\mathscr{S} = (S_i)_{i\in I}$ be a transversal system for $\mathscr{D}$. At least one of the sets S_i must meet s different sets D^α, that is, we can find an index i, indices $\alpha_1 < \dots < \alpha_s$, and elements $x^\sigma \in A^{\alpha_\sigma} \cap S_i$ $(\sigma = 1, \dots, s)$; moreover we can then find an index j and an element $x \in D(x^1, \dots, x^s) \cap S_j$. Now $\mathscr{C}(x, x^1, \dots, x^s)$ is properly covered by $\mathscr{S}$, since there are two indices i, j (no $n+2$ equal, since $n \geqslant 1$) such that each pair in $\mathscr{C}(x, x^1, \ldots, x^s)$ can be written as $\{x^\sigma, x\}$ with $x^\sigma \in S_i$ and $x \in S_j$.

Lemma 7. *Proposition* (Q_r^1) *implies Proposition* (P_r).

Proof. According to (Q_r^1), there exists a 1-a-d collection $\mathscr{B}$ consisting of (i) a collection $\mathscr{D}$ of $\aleph_{\nu+r}$ sets of cardinality $\aleph_\nu$, together with (ii) a collection $\mathscr{C}$ of $\aleph_{\nu+r}$ distinct $(r+2)$-tuples, such that any transversal system for $\mathscr{D}$ properly covers $\{C\}$ for some $C \in \mathscr{C}$. Extend each set $C \in \mathscr{C}$ to a set C^+ of cardinality $\aleph_\nu$.

Let the collection $\mathscr{A}^*$ consist of $\mathscr{D}$ together with all the sets C^+, $C \in \mathscr{C}$. These are 1-a-d sets of cardinality $\aleph_\nu$. It will be sufficient to show that any transversal system for $\mathscr{A}^*$ includes more than $f(r)$ non-empty sets, because for $\mathscr{A}$ we may then take the union of $\aleph_{\nu+r}$ "copies" of $\mathscr{A}^*$ (with mutually disjoint unions).

Suppose if possible that there exists a transversal system $\mathscr{S}$ for $\mathscr{A}^*$ that includes no more than $f(r)$ non-empty sets S_i $(S_i \neq \phi,\ i \in I;$ $|I| \leqslant f(r)$. In particular $\mathscr{S}$ is a transversal system for $\mathscr{D}$, and therefore properly covers $\{C\}$ for some $C \in \mathscr{C}$. Writing $C = \{x_1, \ldots, x_{r+2}\}$, we conclude that there are indices $i_1, \ldots, i_{r+2} \in I$, no $n+2$ of which are equal, such that $x_\rho \in S_{i_\rho}$ for $\rho = 1, \ldots, r+2$. Now if some index $i \in I$ occurred less than $n+1$ times as an i_ρ it would follow that

$$r + 2 \leqslant n + (|I| - 1)(n+1) \leqslant (n+1)f(r) - 1 \ ;$$

this being false by the definition of $f(r)$, we conclude that each index $i \in I$ occurs exactly $n+1$ times as an i_ρ, and consequently $|C^+ \cap S_i| \geqslant$ $\geqslant n+1$ for all $i \in I$. This contradicts the fact that $\mathscr{S}$ is a transversal system for $\mathscr{A}^*$, and the proof of Lemma 7 is complete.

Lemma 8. *The proposition* $(\forall s)(Q_r^s)$ & (P_r) *implies* $(\forall s)(Q_{r+1}^s)$.

Proof. Let s be a positive integer, let $t = f(r) + 1$, let $\mathscr{A}$ be a 1-a-d collection of $\aleph_{\nu+r}$ sets of cardinality $\aleph_\nu$ satisfying the conditions of (P_r), and list all the distinct t-tuples of disjoint subsets of cardinality

$\aleph_{\nu+r}$ in $\bigcup \mathscr{A}$ as

$$(A^1(\alpha), \ldots, A^t(\alpha)), \qquad 0 \leqslant \alpha < \omega_{\nu+r+1}$$

(there are $\aleph_{\nu+r+1}$ of them according to the generalized continuum hypothesis). Any transversal system $(S_i)_{i\in I}$ for $\mathscr{A}$ includes more than $f(r)$ sets of cardinality $\aleph_{\nu+r}$, and it follows at once that for some t distinct indices $j_1, \ldots, j_t \in I$ and some α $(0 \leqslant \alpha < \omega_{\nu+r+1})$ we have

$$(6.1) \qquad A^1(\alpha) \subseteq S_{j_1} \;\&\ldots\&\; A^t(\alpha) \subseteq S_{j_s} .$$

With each α associate a collection $\mathscr{B}(\alpha)$ satisfying the conditions of Proposition (Q_r^{st}), the sets $\bigcup \mathscr{B}(\alpha)$ being disjoint from one another and from $\bigcup \mathscr{A}$. In the notation of (Q_r^{st}), $\mathscr{B}(\alpha)$ is a 1-a-d collection consisting of (i) a collection $\mathscr{D}(\alpha)$ of $\aleph_{\nu+r}$ sets of cardinality $\aleph_\nu$, together with (ii) $\aleph_{\nu+r}$ disjoint st-element collections of $(r+2)$-tuples $\mathscr{C}_\beta(\alpha)$, $0 \leqslant \beta < \omega_{\nu+r}$, such that any transversal system for $\mathscr{D}(\alpha)$ properly covers some $\mathscr{C}_\beta(\alpha)$. We can write

$$\mathscr{C}_\beta(\alpha) = \{C(\alpha, \beta, \sigma, \tau)\colon \sigma = 1, \ldots, s;\ \tau = 1, \ldots, t\} ,$$

each $C(\alpha, \beta, \sigma, \tau)$ being an $(r+2)$-tuple. For each $\tau = 1, \ldots, t$ index the $\aleph_{\nu+r}$ elements of $A^\tau(\alpha)$ as

$$a^\tau(\alpha, \beta, \sigma),\ 0 \leqslant \beta < \omega_{\nu+r},\ 1 \leqslant \sigma \leqslant s .$$

Consider the collection $\mathscr{B}$ consisting of (i) all the sets of $\mathscr{A}$ and of all the $\mathscr{D}(\alpha)$'s, $0 \leqslant \alpha < \omega_{\nu+1}$, together with the $\aleph_{\nu+r+1}$ disjoint s-element collections of $(r+3)$-tuples $\mathscr{C}(\alpha, \beta, \tau)$ $(0 \leqslant \alpha < \omega_{\nu+r+1}$, $0 \leqslant \beta < \omega_{\nu+r}$, $1 \leqslant \tau \leqslant t)$, where

$$(6.2) \qquad \mathscr{C}(\alpha, \beta, \tau) = \{C(\alpha, \beta, \sigma, \tau) \cup \{a^\tau(\alpha, \beta, \sigma)\}\colon 1 \leqslant \sigma \leqslant s\} .$$

We omit the straightforward verification that $\mathscr{B}$ is a 1-a-d collection. To prove that $\mathscr{B}$ satisfies the condition of Proposition (Q_{r+1}^s), we let $\mathscr{S} = (S_i)_{i\in I}$ be any transversal system for $\mathscr{B}$, and observe that in particular it is one for $\mathscr{A}$. Hence there are t distinct indices $j_1, \ldots, j_i \in I$ and an index α $(0 \leqslant \alpha < \omega_{\nu+r+1})$ such that (6.1) holds. Now $\mathscr{S}$ is also a transversal system for $\mathscr{D}(\alpha)$, and therefore by hypothesis properly covers some $\mathscr{C}_\beta(\alpha)$, $0 \leqslant \beta < \omega_{\nu+r}$. Thus there are indices $i_1, \ldots, i_{r+2}$, no $n+2$ of which are equal, such that each set $C \in \mathscr{C}_\beta(\alpha)$ can be written as $\{x_1, \ldots, x_{r+2}\}$ with $x_\rho \in S_{i_\rho}$ for $\rho = 1, \ldots, r+2$. But no more than $(r+2)/(n+1)$ indices can occur as many as $n+1$ times among $i_1, \ldots, i_{r+2}$, and $t > (r+2)/(n+1)$, and therefore for some τ, $1 \leqslant \tau \leqslant t$, the index j_τ does not occur $n+1$ times. Consider the s sets of which $\mathscr{C}(\alpha, \beta, \tau)$ is composed (see (6.2)). Each of them can be written as $\{x_1, \ldots, x_{r+2}, a^\tau(\alpha, \beta, \sigma)\}$, where

$$x_1 \in S_{i_1}, \ldots, x_{r+2} \in S_{i_{r+2}}, \; a^\tau(\alpha, \beta, \sigma) \in S_{j_\tau},$$

and no index occurs more than $n+1$ times among $i_1, \ldots, i_{r+2}, j_\tau$. Thus $\mathscr{C}(\alpha, \beta, \tau)$ is properly covered by S, and the proof is complete.

Theorem 5. *The generalized continuum hypothesis implies that Theorem* 1 *is best possible in the case when* $\mu \geqslant \nu + \omega_0$; *that is, that given any* ν *there exists a* 1-a-d *collection* $\mathscr{E}$ *of* $\aleph_{\nu+\omega_0}$ *sets of cardinality* $\aleph_\nu$ *such that whenever* $\mathscr{E}$ *is split into subcollections* $\mathscr{E}_i$, $i \in I$, *where* $\mathscr{E}_i$ *admits an* n_i*-transversal, we have* $\Sigma(n_i + 1) \geqslant \aleph_0$.

Proof. By Theorem 4, for each n and r we can construct a 1-a-d collection $\mathscr{A}(n, r)$ of $\aleph_{\nu+r}$ sets of cardinality $\aleph_\nu$, such that whenever $\mathscr{A}(n, r)$ is split into subcollections $A_i(n, r)$, $i \in I$, where $\mathscr{A}_i(n, r)$ admits an n-transversal, we have $(n+1)\ |I| > r+2$. We may suppose that the sets $\bigcup \mathscr{A}(n, r)$ are mutually disjoint, and it is then clearly sufficient to set $\mathscr{E} = \bigcup_{n,r} \{\mathscr{A}(n, r)\}$.

7. ANOTHER COUNTER-EXAMPLE

Theorem 6. *Assuming the generalized continuum hypothesis, the proposition* $P((1, 1), \aleph_\nu \to 2, \aleph_{\nu+1})$ *is false; that is, there exists a* 2-a-d *collection* $\mathscr{A}$ *of* $\aleph_{\nu+1}$ *sets of cardinality* $\aleph_\nu$ *that cannot be split into two subcollections each admitting a* 1*-transversal.*

Proof. We already know (Theorem A (ii)) that there exists a 1-a-d collection of $\aleph_\nu$ sets of cardinality $\aleph_\nu$, not admitting a 1-transversal;

let us refer to this as a 1-system. Take $\aleph_{\nu+1}$ 1-systems $\mathscr{A}_\gamma$ $(0 \leqslant \gamma < \omega_\nu)$ and $\mathscr{B}_\alpha$ $(0 \leqslant \alpha < \omega_{\nu+1})$, with disjoint unions. Let $X = \bigcup_\gamma \bigcup \mathscr{A}_\gamma$ and let $X_\alpha = \bigcup \mathscr{B}_\alpha$. The generalized continuum hypothesis implies that we can list all pairs of disjoint subsets of X, each of cardinality $\aleph_\nu$, as (Y_α, Z_α), $0 \leqslant \alpha < \omega_{\nu+1}$. List the elements of Y_α and Z_α^* as

$$y_{\alpha\beta} \quad (0 \leqslant \beta < \omega_\nu)\,, \quad z_{\alpha\beta} \quad (0 \leqslant \beta < \omega_\nu)\,.$$

List the pairs of elements of X_α as $(u_{\alpha\beta}, v_{\alpha\beta})$, $0 \leqslant \beta < \omega_\nu$, and extend each quadruple $\{y_{\alpha\beta}, z_{\alpha\beta}, u_{\alpha\beta}, v_{\alpha\beta}\}$ to a set $E_{\alpha\beta}$ of cardinality $\aleph_\nu$.

Let $\mathscr{A}$ denote the collection consisting of all the sets of all the 1-systems $\mathscr{A}_\gamma, \mathscr{B}_\alpha$, together with all the sets $E_{\alpha\beta}$. It is easy to verify that $\mathscr{A}$ is a 2-a-d collection (of $\aleph_{\nu+1}$ sets of cardinality $\aleph_\nu$). Suppose if possible that $\mathscr{A}$ can be split into two subcollections $\mathscr{A}', \mathscr{A}''$ admitting respective 1-transversals S, T Since $\mathscr{A}_\gamma$ does not admit a 1-transversal, both $S \cap \bigcup \mathscr{A}_\gamma$ and $T \cap \bigcup \mathscr{A}_\gamma$ are non-empty for every γ, and it follows at once that for some α, $0 \leqslant \alpha < \omega_{\nu+1}$, we have $S \supseteq Y_\alpha$ and $T \supseteq Z_\alpha$. Since $\mathscr{B}_\alpha$ does not admit a 1-transversal, for some β, $0 \leqslant \beta < \omega_\nu$, we have $u_{\alpha\beta} \in S$ and $v_{\alpha\beta} \in T$. Thus both $|S \cap E_{\alpha\beta}| \geqslant 2$ and $|T \cap E_{\alpha\beta}| \geqslant 2$, which is impossible because either $E_{\alpha\beta} \in \mathscr{A}'$ or $E_{\alpha\beta} \in \mathscr{A}''$.

REFERENCE

[1] P. Erdős – A. Hajnal, On a property of families of sets, *Acta Math. Hung.*, 12 (1961), 87-123.