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RAMSEY THEOREMS FOR MULTIPLE COPIES OF GRAPHS

BY

S. A. BURR, P. ERDÖS AND J. H. SPENCER(1)

ABSTRACT. If G and H are graphs, define the *Ramsey number* $r(G, H)$ to be the least number p such that if the edges of the complete graph K_p are colored red and blue (say), either the red graph contains G as a subgraph or the blue graph contains H. Let mG denote the union of m disjoint copies of G. The following result is proved: Let G and H have k and l points respectively and have point independence numbers of i and j respectively. Then $N - 1 \leqslant r(mG, nH) \leqslant N + C$, where $N = km + ln - \min(mi, mj)$ and where C is an effectively computable function of G and H. The method used permits exact evaluation of $r(mG, nH)$ for various choices of G and H, especially when $m = n$ or $G = H$. In particular, $r(mK_3, nK_3) = 3m + 2n$ when $m \geqslant n$, $m \geqslant 2$.

1. Introduction. Let G and H be graphs without isolated points. Following Chvátal and Harary [1], define the Ramsey number $r(G, H)$ to be the least integer n such that if the edges of K_n (the complete graph on n points) are two-colored, say red and blue, either the red graph contains G as a subgraph or the blue graph contains H. Note that $r(K_k, K_l)$ is the "ordinary" Ramsey number $r(k, l)$ for which an extensive literature exists. The evaluation of $r(G, H)$ has received attention from several authors in recent years. An extensive survey is given in [2].

In this paper we will generally follow the notation of Harary [3]. In particular, let nG denote the union of n vertex-disjoint copies of G. In §2 we obtain surprisingly sharp and general upper and lower bounds for $r(nG, nH)$ for G and H fixed and n sufficiently large. In §3 we extend these results to $r(mG, nH)$, in §4 to k-graphs. In §5 we consider a related problem of J. W. Moon concerning the decomposition of a complete graph into complete monochromatic subgraphs of prescribed size. Finally, in §6 we give exact values for various cases.

2. The Ramsey numbers $r(nG, nH)$. Again following [3], let $p(G)$ denote the number of points of G and let $\beta_0(G)$ denote the number of points in a maximal independent set in G. As a special notation, let $[X]^2$ denote the complete graph on X and XY denote the complete bipartite graph on X and Y. Also, let $r(G) = r(G, G)$; these we call the diagonal Ramsey numbers.

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THEOREM 1. *Let* $p(G) = k$, $p(H) = l$, *and* $i = \min(\beta_0(G), \beta_0(H))$. *Then*

$$(k + l - i)n - 1 \leqslant r(nG, nH) \leqslant (k + l - i)n + C, \tag{1}$$

where C is a constant depending only on G and H.

We first prove Theorem 1 for $G = H = K_3$. The more general proof to follow will then have clearer intuitive appeal. In fact, we show the following stronger result, which has been shown independently by Seymour at Oxford (personal communication).

THEOREM 2. *For* $n \geqslant 2$, $r(nK_3) = 5n$.

Before turning to the proof of Theorem 2, we prove the following simple result which will be used several times in the sequel.

LEMMA 1. *Let F, G, and H be graphs, with* $p(G) = k$, $p(H) = l$. *Then, if* $m, n \geqslant 1$,

$$r(G, F \cup H) \leqslant \max(r(G, F) + l, r(G, H)),$$

$$r(mG, nH) \leqslant r(G, H) + (m - 1)k + (n - 1)l.$$

PROOF. Let a complete graph on $\max(r(G, F) + l, r(G, H))$ points be two-colored. If there is no red G, then there is certainly a blue H. Remove the l points of H from the graph. Among the remaining points there must be a blue F. Hence the original graph contains either a red G or a blue $F \cup H$, and the first inequality follows. The second inequality follows from repeated application of the first. Q.E.D.

PROOF OF THEOREM 2. We show $r(nK_3) \geqslant 5n$ by exhibiting the coloring of Figure 1:

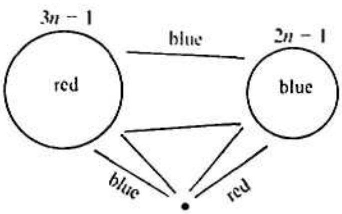

FIGURE 1

Formally, let $|A| = 3n - 1$, $|B| = 2n - 1$, $|C| = 1$, with A, B, C disjoint. Color $[A \cup B \cup C]^2$ by coloring $[A]^2$ red, $[B]^2$ blue, AB blue, AC blue, and BC red. The reader can easily show that this is a two-coloring of K_{5n-1} without a monochromatic nK_3.

We show $r(nK_3) \leqslant 5n$ by induction. The finite demonstration for $n = 2$ is given in §6. Now let $n \geq 3$, and fix a two-coloring of K_{5n}. We need to show the existence of a monochromatic nK_3. Since $5n \geq 6$, there exists a monochromatic, say red, K_3.

Assume there is no blue K_3 on the remaining $5n - 3$ points. Then, since $r(K_3) = 6$, by Lemma 1 we have $r(nK_3, K_3) \leq 3n + 3 \leq 5n - 3$, because $n \geq 3$. Since there is no blue K_3, there must be n disjoint red K_3's as desired.

Now assume, on the other hand, that there is a blue K_3 which is vertex-disjoint from the red K_3. That is $|A| = |B| = 3$, $A \cap B = \varnothing$, $[A]^2$ red, $[B]^2$ blue. (See Figure 2(a).)

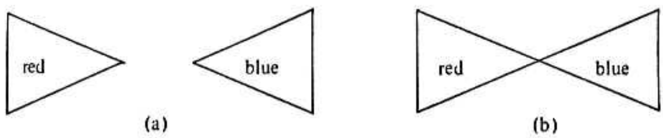

FIGURE 2

Of the nine lines of AB at least five must be one color, say red. Then $\exists a \in A$ and $b_1, b_2 \in B$, such that $\{a, b_1\}$, $\{a, b_2\}$ are red. This yields (see Figure 2(b)) a "bowtie": two K_3, one red, one blue, with one common vertex. Deleting the bowtie we find, by the induction hypothesis, a monochromatic $(n-1)K_3$. Adding the appropriately colored K_3 from the bowtie yields a monochromatic nK_3 in the full graph. Q.E.D. (Theorem 2).

The next result, which is essentially the lower bound in Theorem 1, provides a very useful lower bound for Ramsey numbers in general and is therefore given separately.

LEMMA 2. *If $p(G) = k$, $p(H) = l$, then*

$$r(G, H) \geq k + l - \min(\beta_0(G), \beta_0(H)) - 1.$$

PROOF. We form a graph on $k + l - \beta_0(G) - 2$ points containing neither a red G nor a blue H. Let $|A| = k - \beta_0(G) - 1$, $|B| = l - 1$, $A \cap B = \varnothing$, and color $[B]^2$ blue, all else red (see Figure 3).

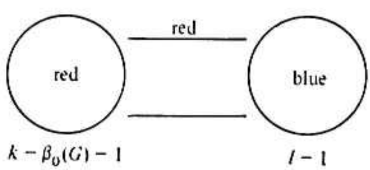

FIGURE 3

It is clear that there is no blue H; but a red G would have to use $\beta_0(G) + 1$ points of B, which is impossible since they would all be independent in the red graph. Thus $r(G, H) \geq k + l - \beta_0(G) - 1$; but by symmetry $r(G, H) \geq k + l - \beta_0(H) - 1$, and combining the two inequalities yields the desired result. Q.E.D.

PROOF OF THEOREM 1. Without loss of generality, we may assume $\beta_0(G) \geqslant \beta_0(H) = i$. For the lower bound of (1) employ Lemma 2, with G replaced by nG and H replaced by nH. Then k and l become nk and nl. Thus, since $\beta_0(nG) = n\beta_0(G)$ and $\beta_0(nH) = n\beta_0(H)$, we have

$$r(nG, nH) \geqslant nk + nl - \min(\beta_0(nG), \beta_0(nH)) - 1$$
$$= n(k + l - i) - 1.$$

We now show the upper bound in (1). Let n_0 be a constant, dependent only on G and H, to be described later. We find $C \geqslant 0$ so that (1) holds for $n \leqslant n_0$, and now wish to apply induction. Fix a two-coloring χ of the edges of the complete graph on $(k + l - i)(n + 1) + C$ points; we need to show the existence of a red $(n + 1)G$ or a blue $(n + 1)H$. Here $n \geqslant n_0$, and we know $r(nG, nH) \leqslant (k + l - i)n + C$.

By a "bowtie" we mean in this case a two-colored graph on $\leqslant k + l - i$ points containing a red G and a blue H. An example of such a graph is one on the set of points $S = R \cup N \cup B$ where $|R| = k - i$, $|B| = l - i$, $|N| = i$, all disjoint, $[R]^2$ red, $[B]^2$ blue, RN red, BN blue, and the other edges unspecified. (See Figure 4.)

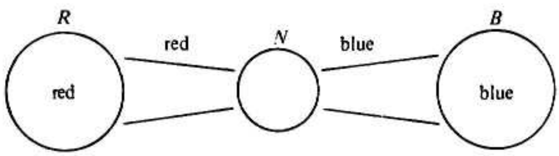

FIGURE 4. Bowtie

If the graph contains a bowtie then, deleting the bowtie, we find by induction a monochromatic nG or nH, to which we add the appropriately colored G or H from the bowtie giving a red $(n + 1)G$ or a blue $(n + 1)H$.

Let $M \geqslant \max(k, l) - i$ be such that if $|A| = |D| = M$, $A \cap D = \emptyset$, and AD is two-colored, there exist $A_1 \subseteq A$, $D_1 \subseteq D$ with $|A_1| = |D_1| = \max(k, l) - i$ and A_1D_1 monochromatic. (The existence of such an M is not hard to verify; one may take $M = 2^{\max(k,l)+1}$ for definiteness.) Assume, given a coloring χ, that there exist A, D with $|A| = |D| = M$, $A \cap D = \emptyset$, $[A]^2$ red, $[D]^2$ blue. We find A_1, D_1, with A_1D_1 monochromatic. (See Figure 5.)

If A_1D_1 is red, we form a bowtie by taking R to be $k - i$ points out of A_1, N to be i points out of D_1, and B to be $l - i$ other points of D. If A_1D_1 is blue, we proceed similarly, taking N out of A_1.

Finally, assume there do not exist monochromatic vertex-disjoint K_M of

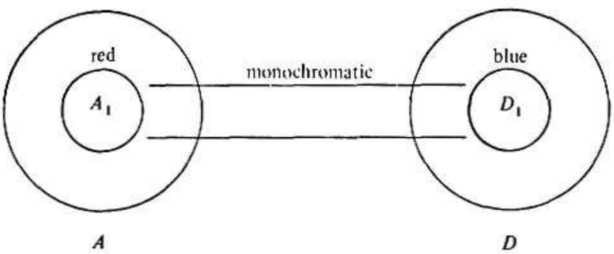

FIGURE 5. Large red and blue K_M yielding bowtie

different colors. For convenience, let $y = (n + 1)(k + l - i) + C$ = number of points. If $y \geqslant r(M, M)$ we find a monochromatic K_M, say red. By Lemma 1,

$$r((n + 1)G, K_M) \leqslant r(G, K_M) + nk \leqslant r(k, M) + nk.$$

Thus, provided $y \geqslant r(M, M)$ and $y - M \geqslant r(k, M) + nk$, we either have a red $(n + 1)G$, in which case we are done, or else we have a blue K_M disjoint from the red one, a contradiction. Now we specify n_0 to be an integer such that

$$(n_0 + 1)(k + l - i) \geqslant r(M, M),$$

$$(n_0 + 1)(k + l - i) - M - r(k, M)/p \geqslant n_0 k.$$

Note that the definition of n_0 depends only on G and H–in fact, only on k and l. (Since C does not appear in the definition of n_0, we have avoided circularity.) To summarize, given $n \geqslant n_0$ and a coloring χ on the complete graph on $(k + l - i)(n + 1) + C$ points, one of the following holds:

(a) We find two monochromatic K_M of different colors, in which case there exists a bowtie so, by induction, a monochromatic $(n + 1)G$; or

(b) there is no K_M in one color, in which case the use of Lemma 1 gives an $(n + 1)G$ in the other color. Q.E.D. (Theorem 1).

Note that in our proof of Theorem 1 we actually showed

(1) $(k + l - i)n - 1 \leqslant r(nG, nH)$,

(2) $r((n + 1)G, (n + 1)H) \leqslant r(nG, nH) + (k + l - i)$ for $n \geqslant n_0$.

The function $g(n) = r(nG, nH) - (k + l - i)n$ is integral, nonincreasing for $n \geqslant n_0$, and bounded from below. Consequently, it is eventually constant. Hence, the following theorem.

THEOREM 3. *Under the assumptions of Theorem* 1, *there exist* n_1 *and* C_1 *such that*

$$r(nG, nH) = (k + l - i)n + C_1 \quad \textit{for } n \geqslant n_1.$$

We note that we have not been able to find any upper bound on n_1. That is, we have not been able to show that n_1 is a recursive function of G.

3. **The Ramsey numbers** $r(mG, nH)$.

THEOREM 4. *Let* $p(G) = k$, $p(H) = l$, $\beta_0(G) = i$, *and* $\beta_0(H) = j$. *Then*

$$km + ln - \min(mi, nj) - 1 \leqslant r(mG, nH)$$
$$\leqslant km + ln - \min(mi, nj) + C,$$

where C *is a constant depending only on* G *and* H.

PROOF. The lower bound follows directly from Lemma 2. We will prove the upper bound by induction on $m + n$. Starting the induction is trivial (for a suitable value of C). Now assume the result to have been proved for all cases in which $m + n$ is less than some value N, and consider a case in which $m + n = N$. If either m or n is no greater than $\max(i, j)$, then by Lemma 1 the desired inequality holds, with some new value for C.

By a bowtie we will now mean a two-colored graph on $kj + li - ij$ joints containing simultaneously a red jG and a blue iH. Suppose now we have a two-coloring on $km + ln - \min(mi, nj) + C$ points which contains a bowtie. On removing the bowtie one has a graph on

$$km + ln - \min(mi, nj) + C - kj - li + ij$$
$$= k(m - j) + l(n - i) - \min((m - j)i, (m - i)j) + C$$

points. By the induction hypothesis, this graph contains either a red $(m - j)G$ or a blue $(n - i)H$, and hence the original graph contains either a red mG or a blue nH. The argument is almost the same as that in the proof of Theorem 1 and will only be sketched. Take $M = 2^{\max(kj, li)+1}$. Then, if C has been chosen large enough, the graph contains a monochromatic (say red) K_M. By Lemma 1, and again assuming C is large enough, the rest of the graph contains either a red mG or a blue K_M. In the former case we are done immediately; in the latter case we have a monochromatic complete bipartite graph joining the two K_M sufficiently large to guarantee the existence of a bowtie. Q.E.D.

It is possible to prove a considerably more general result than Theorem 4, which may be stated as follows: Let G and H be disjoint unions of graphs chosen from a finite set G of graphs, and let $p(G) = k$, $p(H) = l$, $\beta_0(G) = i$, and $\beta_0(H) = j$. Then $k + l - \min(i, j) - 1 \leqslant r(G, H) \leqslant k + l - \min(i, j) + C$, where C depends only on G. Although the ideas involved in the proof are essentially the same as those in this paper, the details are tedious and will be omitted.

4. *k*-**graphs.** In this section we partially extend the results of Theorem 1 to k-graphs. A k-graph is defined as a set V of vertices and a set E of "edges" where each edge $e \in E$ is a subset of V of cardinality k. It is clear that general-

ized Ramsey theory can be extended to k-graphs, and indeed has been discussed in [4]. Our proofs will be more sketchy than in the previous sections; also, for clarity's sake, only diagonal numbers will be considered.

THEOREM 5. *Let G be a k-graph with no isolated points. Then*

$$Dn - 1 \leqslant r(nG) \leqslant Dn + C,$$

where $D = D(G)$ will be defined in the proof and C is a constant depending only on G.

PROOF. Let G have p points. Let $[X]^k = \{Y \subseteq X: |Y| = k\}$ denote the complete k-graph on X. Let A, B be disjoint sets. A coloring c of $[A \cup B]^k$ is called canonical if the color of $e \in [A \cup B]^k$ is dependent only on $|e \cap A|$. There are only 2^{k+1} canonical colorings, corresponding to $(k + 1)$-tuples of red and blue. Let c be such a coloring where $|A|$, $|B|$ are sufficiently large, with $[A]^k$ red, $[B]^k$ blue. Let r_c be the least integer such that r_c points from A and $p - r_c$ points from B contain a red G ($r_c \leqslant p$, since $[A]^k$ is red). Let b_c be the largest integer such that b_c points from A and $p - b_c$ points from B contain a blue G (note that $b_c \geqslant 0$). Let

$$D_c = \begin{cases} p + r_c - b_c & \text{if } r_c \geqslant b_c, \\ p & \text{if } r_c \leqslant b_c. \end{cases}$$

If $D_c > p$ then r_c points from A together with $p - b_c$ points from B contain both a red and a blue G. That is, there is a bowtie on D_c points. If $D_c = p$, so $r_c \leqslant b_c$, then $r_c b_c$ points from A together with $(p - r_c)b_c$ points from B form both a red b_cG (each G split r_c, $p - r_c$) and a blue b_cG (r_c G's split b_c, $p - b_c$; the other $(b_c - r_c)$ G's entirely in B). That is, there are pb_c points that contain a red and a blue b_cG. We shall call this a multibowtie.

We define $D = \min D_c$ over the 2^{k-1} possible c with $[A]^k$ red and $[B]^k$ blue. The lower bound of Theorem 5 is trivial if $D = p$. If not, let c be such that $D = D_c$. Split $nD - 2$ points into A, B, $|A| = nr_c - 1$, $|B| = nb_c - 1$, and color $[A \cup B]^k$ canonically by c.

The upper bound is by induction, beginning with n sufficiently large. Fix a coloring c. Then the k-graph so colored must contain disjoint sets X, Y, colored red and blue, where $|X|$, $|Y|$ are arbitrarily large (though independent of n); for otherwise, by analogy with Lemma 1 the points may be split into G's of the same color with a bounded number of points left over–so $r(nG)$ would be $\leqslant np + C$. We pick $|X|$, $|Y|$ so large that there must be $A \subseteq X$, $B \subseteq Y$ so that $A \cup B$ is colored canonically. (This step requires strong use of Ramsey's theorem and yields absurdly high bounds.) Say $A \cup B$ is colored canonically by c; of course $D_c \geqslant D$. We find a bowtie of D points (or perhaps, if $D_c = p$, a multibowtie). Deleting it and applying induction, we arrive at a proof of Theorem 5. Q.E.D.

We note that the off-diagonal number could also be easily found.

COROLLARY. *Let $K_p^{(k)}$ denote the complete k-graph on p points. Then*

$$(2p - (k-1))n - 1 \leqslant r(nK_p^{(k)}) \leqslant (2p - (k-1))n + C.$$

We suppress the proof which involves only a calculation of D.

5. **Decomposition of K_n into monochromatic K_k.** The following question was first raised for the case $k = 3$ by J. W. Moon [5]: What is the minimal integer $f(n, k)$, $k < n$, such that given a two-coloring of K_n it is possible to find vertex-disjoint monochromatic K_k with $\leqslant f(n, k)$ points left over? Note that the K_k may be different colors. We are interested in k fixed, n large. Clearly $f(n, k) \leqslant r(k, k) - 1$, as given any coloring of K_n we may delete monochromatic K_k until there are $< r(k, k)$ points left.

THEOREM 6. *If k is given, then for sufficiently large n,*

$$f(n, k) = r(k, k-1) - 1 + \text{rem}(n - r(k, k-1) + 1, k),$$

where rem(a, b) *is the remainder when a is divided by b.*

PROOF. We may color (see Figure 6) K_n by

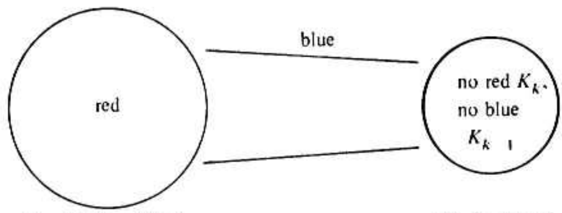

FIGURE 6

letting $|B| = r(k, k-1) - 1$, $|A| = n - |B|$, $A \cap B = \varnothing$, coloring $[B]^2$ with no red K_k and no blue K_{k-1}, coloring AB blue and $[B]^2$ red. Then no point of B can be part of a monochromatic K_k so the K_k must be all from A, leaving $|B| +$ rem$(|A|, k)$ points.

Now set $u = (k-1)(r(k, k) - r(k, k-1)) + (k-1)(k-2) + 1$, and fix any 2-coloring of K_n, where $n \geqslant r(u, u)$. We find a monochromatic, say red, K_u on a set C of points. We decompose the remaining points into monochromatic K_k until a set D, $|D| < r(k, k)$, is remaining with no monochromatic K_k. Assume D contains a blue K_{k-1} on a set E. For $x \in E$ let $C_x = \{y \in C\colon \{x, y\} \text{ is red}\}$. If any $|C_x| \geqslant k - 1$ we take x and $k - 1$ points of C_x, inducing a red K_k; delete this and continue. If all $|C_x| < k - 1$, then

$$\left|C \sim \bigcup_{x \in E} C_x\right| \geq |C| - \sum_{x \in E} |C_x| = |C| - (k-1)(k-2) > 0,$$

so there exists $y \in C$ such that $\{y, x\}$ is blue for $x \in E$. Thus $E \cup \{y\}$ gives a blue D_k; delete this and continue.

We may continue in this manner ($|C|$ will decrease but u has been chosen sufficiently large so that the above counting arguments continue to hold) until D has been reduced to D_1, $|D_1| < r(k, k-1)$, and C to some C_1. Now $D_1 \cup C_1$ are our leftovers—but we further delete red K_k from C_1 until a set C_2, $|C_2| < k$ remains. As $|D_1| + |C_1| \equiv n \pmod{k}$,

$$|C_1| \equiv n - |D_1| \pmod{k},$$

so

$$|C_2| = \operatorname{rem}(n - |D_1|, k);$$

therefore

$$\begin{aligned} f(n, k) &\leq |D_1| + |C_2| = |D_1| + \operatorname{rem}(n - |D_1|, k) \\ &\leq r(k, k-1) - 1 + \operatorname{rem}(n - r(k, k-1) + 1, k). \qquad \text{Q.E.D.} \end{aligned}$$

It would be of interest to try to extend this result to k-graphs.

6. **Some exact values.** In this section we will consider primarily some special cases of $r(nG, nH)$. To find exact values for such numbers by the methods of this paper, four steps are necessary. First, one must find a lower bound for $r(nG, nH)$ of the form $(k + l - i)n + C$. Second, one must evaluate $r(n_0G, n_0H)$ for some value of n_0 for which the lower bound is achieved. Third, one must show that $r((n+1)G, (n+1)H) \leq r(nG, nH) + k + l - i$ for $n \geq n_0$. Finally, one must evaluate $r(nG, nH)$ for $n < n_0$. Of these four steps, no general methods of carrying out the second and fourth are known, and each problem must be met on an ad hoc basis. About the first and third, however, it is possible to say something general of substance, albeit not as much as one would like.

Lemma 2 gives a very useful lower bound on $r(nG, nH)$, one which probably determines its ultimate value in a great many cases. But as Theorem 2 shows, it is sometimes possible to do better. The lower bound of Theorem 2 can be generalized, although somewhat clumsily. We first note that Ramsey numbers still make sense if one or both arguments are replaced by some class of graphs; such a generalization is indicated in [1].

LEMMA 3. *Let G and H be graphs with $p(G) = k$. Let $\mathcal{H}$ be the class of* *maximal graphs formed by nH by removing at most* $mk - 1$ *independent points in all possible ways. Then* $r(mG, nH) \geqslant mk + r(G, \mathcal{H}) - 1$.

PROOF. Obvious.

Although this result appears awkward to apply, this is not always the case. For instance, in Theorem 2, in which $m = n$ and $G = H = K_3$, $\mathcal{H}$ consists of the single graph nK_2, and $r(G, \mathcal{H})$ is easily seen to be $2n + 1$, yielding $r(nK_3) \geqslant 5n$ as desired.

To prove that $r((m + 1)G, (n + 1)H) \leqslant r(mG, nH) + k + l - i$ for appropriate m and n, it is sufficient to prove that a two-colored complete graph on that many points has either a red $(m + 1)G$, a blue $(n + 1)H$, or a bowtie. It turns out that in many cases it can be shown that if one has a red G and a disjoint blue H one must have a bowtie between them. In addition it is easy to see that any graph on that many points must have a red $(m + 1)G$, a blue $(n + 1)H$, or a red G and a blue H disjoint from each other. This leads us immediately to the following result.

LEMMA 4. *Let* $p(G) = k$, $p(H) = l$, $i = \min(\beta_0(G), \beta_0(H))$, *and suppose that any two-colored graph containing a mutually disjoint red G and blue H contains a bowtie. Then, if* $m \geqslant 1, n \geqslant 1$,

$$r((m + 1)G, (n + 1)H) \leqslant r(mG, nH) + k + l - i.$$

Unfortunately the conditions of Lemma 4 are not always met, for instance for $G = H = K_4$; nevertheless they often are, as will be seen. We will now begin the study of specific cases by completing the proof of Theorem 2, and generalizing it.

THEOREM 7. *Let* $m \geqslant n \geqslant 1, m \geqslant 2$. *Then* $r(mK_3, nK_3) = 3m + 2n$.

PROOF. We first apply Lemma 3. $\mathcal{H}$ consists of the single graph nK_2, and it is easy to see that $r(K_3, nK_2) \geqslant 2n + 1$, so $r(mK_3, nK_3) \geqslant 3m + 2n$.

The hypotheses of Lemma 4 have been shown, by the proof of Theorem 2, to be satisfied. Thus $r((m + 1)K_3, (n + 1)K_3) \leqslant r(mK_3, nK_3) + 5$. Hence the desired result will follow from the initial conditions $r(2K_3) \leqslant 10$, $r(mK_3, K_3) \leqslant 3m + 2$. We first show $r(2K_3) \leqslant 10$, which also completes the proof of Theorem 2.

Fix a two-coloring of K_{10} and assume there is no monochromatic $2K_3$. We easily see that if there were no monochromatic $2K_3$, there would exist a bowtie, which we now fix. The five points not in the bowtie must not contain a monochromatic K_3, so our K_{10} must be as in Figure 7.

By symmetry we may assume three of the edges $0i$, $5 \leqslant i \leqslant 9$, are blue. If $0i_1$,

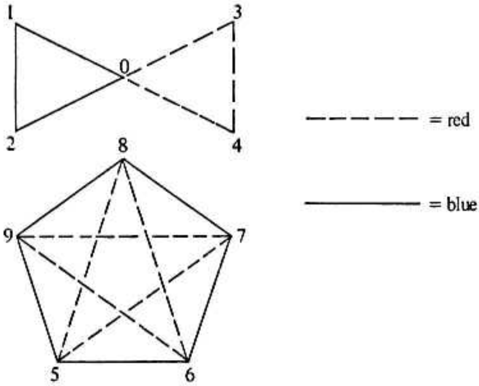

FIGURE 7

$0i_2$, $0i_3$ are blue, two of i_1, i_2, i_3 are adjacent in the blue graph so, by symmetry, we assume 05, 06 blue. Now 56034 is a bowtie so 12789 must be two-colored without a monochromatic K_3, so the blue lines must form a pentagon. As 12, 78, 89 are already blue either 19, 27 or 17, 29 are blue. The cases are symmetric, so we assume the latter, with the remaining edges of 12789 red. Now, 19, 96 red implies 61 blue (otherwise 169, 034 red) so 61034 is a bowtie, hence 25789 is a blue pentagon; but 92, 95, 98 are already blue, a contradiction.

We next show that $r(2K_3, K_3) \leqslant 8$. In [5] it was shown that if K_8 is two-colored, it contains two disjoint monochromatic triangles. If both are red, we are done; if one is blue, we are also done. Therefore $r(2K_3, K_3) \leqslant 8$; but then successive applications of the first part of Lemma 1 yield $r(mK_3, K_3) \leqslant 3m + 2$ for $m \geqslant 3$ as desired. Q.E.D.

THEOREM 8.

$$r(K_{1,3}) = 6,$$

$$r(mK_{1,3}, nK_{1,3}) = 4m + n - 1, \qquad m \geqslant n,\ m \geqslant 2.$$

PROOF. The first part is easy; see [6]. In [7] it is established that $r(2K_{1,3}) = 9$, and Lemma 2 shows that $r(mK_{1,3}, nK_{1,3}) \geqslant 4m + n - 1$. Also, an easy calculation, which we omit, shows that $r(2K_{1,3}, K_{1,3}) = 8$. Successive applications of the first part of Lemma 1 establish that $r(mK_{1,3}, K_{1,3}) \leqslant 4m$. Another easy calculation shows that the conditions of Lemma 4 are satisfied, so that $r((m + 1)K_{1,3}, (n + 1)K_{1,3}) \leqslant r(mK_{1,3}, nK_{1,3}) + 5$, and the theorem follows by induction. Q.E.D.

THEOREM 9. *If* $p(G) = k$, *then* $r(nG, nK_2) = (k + 1)n - 1$.

PROOF. By Lemma 2, $r(nG, nK_2) \geqslant (k + 1)n - 1$. To prove the other half of the result it suffices to let $G = K_k$. Certainly $r(K_k, K_2) = k$; moreover, the conditions of Lemma 4 are clearly satisfied, so the result follows immediately. Q.E.D.

Let P_3 denote a path on three points (not edges). Chvátal and Harary [8] have shown that if $p(G) = k$, then $r(G, P_3) = k$ if $\bar{G}$ has a 1-factor, and $r(G, P_3) = 2k - 2\beta_1(\bar{G}) - 1$ otherwise, where $\beta_1(\bar{G})$ is the number of lines in a maximal independent set in $\bar{G}$. In our final theorem, we extend this result.

THEOREM 10. *If $p(G) = k$, then, provided $n \geq 2$,*

$$r(nG, nP_3) = \begin{cases} (k+2)n - 1 & \text{if } G = K_k, \\ (k+1)n - 1 & \text{if } G \neq K_k. \end{cases}$$

PROOF. The lower bound follows easily from Lemma 2. For the upper bound, first assume $G = K_k$. Fix a two-coloring of K_{2k+3}. By the result of Chvátal and Harary quoted above, $r(2K_k, P_3) = 2k$, so we can assume we have a blue P_3. Remove these three points, leaving a two-coloring of K_{2k}. But by the same result there must now exist a red $2K_k$ or another blue P_3. Thus $r(2K_k, 2P_3) = 2k$. It is easy to see that the conditions of Lemma 4 are satisfied; thus we are immediately led to the result $r(nK_k, nP_3) = (k+2)n - 1$.

We now must show $r(nG, nP_3) \leq (k+1)n - 1$ if $G \neq K_k$; we will omit many details. We may assume that $G = K_k - x$; that is, the graph formed by removing one edge from K_k. Again, Lemma 4 is applicable, leaving only the problem of establishing that $r(2(K_k - x), 2P_3) = 2k + 1$. Fix a two-coloring on K_{2k+1} and assume that there is neither a red $2(K_k - x)$ nor a blue $2P_3$. Consider the largest component C of the blue subgraph. All other blue components have one or two points. Let v be a point of maximal (blue) degree in C. The degree of v must be at least 3. Now consider the graph formed by removing v from C. This graph must contain a P_3. At most one blue line emanating from v can go to a point not in that P_3. This leads to two cases; each one can easily be seen to lead to a blue graph whose complement contains a red $2(K_k - x)$, a contradicition. Q.E.D.

An exact result of particular interest to obtain would be the value of $r(nK_4)$, at least for large n. It is easy to see, for instance by Lemma 3, that $r(mK_k, nK_l) \geq km + ln - \min(m, n) + r(k-1, l-1) - 2$ in general. It is conjectured that for large m and n equality is achieved; so perhaps $r(nK_4) = 7n + 4$ for large n.

The results of this section are of interest in themselves, but they are also significant in that they indicate a deficiency in the rest of the paper. The bounds on C and n_0 that come out of the proof of Theorems 1 and 4 are very large, being essentially double exponentials, and the n_1 of Theorem 3 has no known bound at all. These bounds essentially come from the proof of the existence of a bowtie. In this section we have seen that often it requires relatively few points to force a bowtie. This is also true in [9], where similar ideas occur. It seems quite possible that more reasonable general bounds can be found for the constants in Theorems 1 and 4, or even for Theorem 3.

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BELL TELEPHONE LABORATORIES, 175 PARK AVENUE, MADISON, NEW JERSEY 07940

HUNGARIAN ACADEMY OF SCIENCES, BUDAPEST, HUNGARY (Current address of P. Erdös)

DEPARTMENT OF MATHEMATICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MASSACHUSETTS 02139

Current address (S. A. Burr): A.T.&T. Long Line, 110 Belmont Drive, Somerset, New Jersey 08873

Current address (J. H. Spencer): Department of Mathematics, SUNY at Stony Brook, Stony Brook, New York 11790