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ON SET-SYSTEMS HAVING LARGE CHROMATIC NUMBER AND NOT CONTAINING PRESCRIBED SUBSYSTEMS

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§0. INTRODUCTION

In this paper we are going to present several new results concerning chromatic numbers of set-systems. As a starting point for our investigation we can take the paper [5] where the chromatic number of a set-system is defined and the following theorem is proved (see [5] p. 72, Theorem 5.5).

Theorem A (Erdős – Hajnal). *Let $\kappa \geqslant \omega$ be an infinite cardinal. Suppose $\mathcal{G}$ is a graph of chromatic number $> \kappa$. Then $\mathcal{G}$ contains a complete bipartite graph $K(t, \kappa^+)$ for every $t < \omega$.*

In [5] a false generalization of this theorem was claimed for n-tuple systems with $3 \leqslant n \leqslant \omega$. The simplest special instance of the false theorem said that if a triple system has chromatic number $> \aleph_0$ then it contains two triples with a common edge (see [5] p. 92, Theorem 12.1).

It was discovered in [12] that this holds only if the set of vertices has cardinality $\leqslant \aleph_1$ and otherwise there are triple systems of arbitrarily large chromatic number consisting of edge disjoint triples.

This led us to the following problem in [12]. To have a short notation let us say that $\mathcal{S}$ is an (n, i, λ)-system if $\mathcal{S}$ consists of n-tuples such that every $i+1$ set is contained in at most λ members of $\mathcal{S}$. $(n, i, 1)$-systems will be briefly called (n, i)-systems.

It was proved in [12] that for $1 \leqslant i \leqslant n < \omega \leqslant \kappa$ there are (n, i)-systems $\mathcal{S}$ with chromatic number $> \kappa$ and that the cardinality of such an $\mathcal{S}$ must be large (depending on n, i and κ) but in [12] we could not tell how large it must be.

One of the main aims of this paper is to settle this problem assuming G.C.H. We are going to prove

Theorem B.

a) *Assume* $mi + 2 \leq n < \aleph_0$. *Then every* $(n, i, \aleph_\alpha)$*-systems* $\mathcal{S}$ *of cardinality* $\aleph_{\alpha + m}$ *has chromatic number at most* $\aleph_\alpha$.

b) *Assume* $2 \leq n < mi + 2 < \aleph_0$ *and that* G.C.H. *holds. Then there is an* (n, i)*-system* $\mathcal{S}$ *of cardinality* $\aleph_{\alpha + m}$ *and chromatic number* $> \aleph_\alpha$ *(see Corollaries* 2.1 *and* 12.4).

As to b) we prove a theorem in ZFC which yields it if G.C.H. is assumed. In fact we will show that b) is not a Theorem of ZFC. See Theorem 5.6 and 5.7. E.g. we will prove that $MA_\kappa \Rightarrow$ Every (3, 1)-system of cardinality κ has chromatic number $\leq \aleph_0$. On the other hand in §§12-15 we will prove in ZFC a number of (incomparable) results working in the direction of Theorem B b) and we will summarize the situation in §16.

Theorem B answers the question if there is an n-tuple system of size λ and of chromatic number $> \kappa$ not containing the special n-tuple systems which consist of two n-tuples having $\geq i + 1$ elements in common. The general problem now arises

(I) Given κ, λ and $n \geq 2$, characterize those finite n-tuple systems which are contained in all n-tuple systems of chromatic number $> \kappa$ and cardinality λ.

Of course we could omit "finite" from (I) but we are not prepared to do so. In fact we will not in general discuss (I) but we will focus our attention on the following problems (II), (III), (IV), where (III) is an instance of (I), (II) and (IV) are suggested by (I).

(II) Characterize those finite n-tuple systems $\mathcal{S}$ for which $\kappa \to \to (\kappa, \mathcal{S})^n$ holds.

(III) Characterize those finite n-tuples systems which are contained in all n-tuple systems of cardinality and chromatic number κ.

(IV) Characterize those finite n-tuple systems which are contained in all n-tuple systems of chromatic number $> \kappa$.

The reason that one dares to ask such general questions at all is that in case $n = 2$ all of them can be answered.

Theorem C (Erdős – Hajnal). *For every* $\kappa \geq \omega$ *and for every* $0 < i < \omega$ *there exists a graph* $\mathcal{G}$ *of* κ *vertices and of chromatic number* κ *which does not contain* C_{2j+1} *for* $0 < j \leq i$ (see [5] p. 76. Theorem 7.4).

Theorem A and C together give that in case $n = 2$ the answer for both (III) and (IV) is the class of finite bipartite graphs. On the other hand the well-known Erdős – Dushnik – Miller theorem $\kappa \to (\kappa, \aleph_0)^2$ for $\kappa \geq \aleph_0$ gives that in case $n = 2$ (II) is the class of all finite graphs.

Of course this does not exhaust all problems which can be asked in case $n = 2$. One of the most interesting questions which remains open is due to W. Taylor and really calls for the characterization of those classes of finite subgraphs which are the finite subgraphs of a given graph of chromatic number $> \aleph_0$. However presently we do not go in this direction. The little we know about this is published in [13].

Unfortunately the problems (II), (III), (IV) become very difficult for $n \geq 3$ and strictly speaking we only have partial results even for $\kappa = \omega_1$, $n = 3$.

Our knowledge about (II) and (III) will be summarized at the end of §§10-11 respectively. (IV) will be discussed in §14 and we will restrict our attention mainly to triple systems.

The first part of our paper, §§2-5, contains Theorems going in the direction of Theorem B a), the rest is mainly devoted to constructions going in the other direction.

Corresponding to (II), (III) and (IV) we can define functions $h_n(t, \alpha)$, $\hat{g}_n(t, \alpha)$, $g_n(t, \alpha)$ as follows:

$h_n(t, \alpha) = \min\{m$: There is an n-tuple system $\mathcal{S}$ on $\aleph_{\alpha+1}$ without free $\aleph_{\alpha+1}$-sets such that all subsystems induced by t points contain at most m n-tuples of $\mathscr{S}$}.

$\hat{g}_n(t, \alpha) = \min\{m$: There is an n-tuple system $\mathscr{S}$ on $\aleph_{\alpha+1}$ of chromatic number $\aleph_{\alpha+1}$ such that all subsystems induced by t points contain at most m n-tuples of $\mathscr{S}$}.

$g_n(t, \alpha) = \min\{m$: There is an n-tuple system $\mathscr{S}$ of chromatic number $> \aleph_\alpha$ such that all subsystems induced by t points contain at most m n-tuples of $\mathscr{S}$}.

We will be able to prove that

(II′) G.C.H. $\Rightarrow h_3(t, \alpha) = \left[\frac{(t-1)^2}{4}\right]$.

(III′) G.C.H. $\Rightarrow \hat{g}_3(t, \alpha) = \left[\frac{t^2}{8}\right]$.

(IV′) $\left(\frac{t}{3}\right)^{\frac{3}{2}} - t \leqslant g_3(t, \alpha) \leqslant \left(\frac{t}{3}\right)^{\frac{3}{2}}$,

$$g_n(t, \alpha) = \left(\frac{t}{n}\right)^{\frac{n}{n-1}} + o\left(t^{\frac{n}{n-1}}\right) \text{ for } n \geqslant 4.$$

See Corollaries 10.7, 11.14 and Theorems 14.4, 14.6. Note also that $h_2(t, \alpha) = \binom{t}{2}$, $\hat{g}_2(t, \alpha) = g_2(t, \alpha) = \left[\frac{t^2}{4}\right]$ because of the remarks made after Theorem C, and because of the well-known theorem of Turán that a graph of t vertices not containing a triangle has at most $\left[\frac{t^2}{4}\right]$ edges. G.C.H. is used only in the upper estimates in (II′) and (III′). These are the most general theorems we can prove concerning problems (II), (III), (IV). There is another Taylor type problem which arises in connection with (IV).

(V) Determine the smallest cardinal λ with the following property:

If a finite triple system $\mathscr{T}$ occurs in all triple-systems having chromatic number $> \aleph_0$ and cardinality $\leqslant \lambda$ then it occurs in all triple systems of chromatic number $> \aleph_0$.

It is fairly obvious that such a λ exists, and the example of two triangles with a common edge shows that $\lambda \geqslant \aleph_2$. This problem was already asked in [13] Problem 5. There we claimed that in this paper we will exhibit a finite triple system which does not occur in a triple system of cardinality $\left(2^{2^{\aleph_0}}\right)^+$ and of chromatic number $> \aleph_0$ and for which we can not improve this estimate. Working through the material of this paper we killed all these candidates. We may conjecture but we have no hape to prove that $\lambda \leq 2^{2^{\aleph_0}}$. We are going to state more problems of this type at the end of §14.

Our inquiry led us to some other questions which are more or less independent of the main lines of the paper described above. Results concerning these problems will be included as well, but they will be summarized in the respective chapters only. For the convenience of the reader we will state in detail most of the results we use from our earlier papers on this subject and sometimes we even give proofs for them.

§1. NOTATION

In what follows we work in ZFC. Our notation will be standard. In particular, ordinals are identified with the sets of their predecessors, and cardinals with their initial numbers. Greek lower case letters denote ordinals. $i, j, n, k, l, m, n, r, s$ denote non negative integers. We use both $\aleph_\alpha$ and ω_α to denote cardinals. κ^+ is the immediate cardinal successor of κ.

We use the well-known partition relations, the "ordinary partition relation", the "polarized partition relation" and the "square-bracket partition relation". Since they are not our main subject in this paper we do not give the definitions. We offer [8] as reference where the definitions can easily be found.

By a *set-system* we mean a set of sets $\mathcal{S}$ such that $|A| \geq 2$ for all $A \in \mathcal{S}$. The purpose of this convention to make the following definition possible.

Definition. Let $\mathcal{S}$ be a set-system. The chromatic number of $\mathcal{S}$ is the smallest cardinal κ for which there is a partition of length κ of $\bigcup \mathcal{S}$, $\bigcup \mathcal{S} = \bigcup_{\nu < \kappa} P_\nu$ such that $A \not\subset P_\nu$ for all $\nu < \kappa$ and $A \in \mathcal{S}$.

The chromatic number of a set-system $\mathscr{S}$ will be denoted by $\mathrm{Chr}(\mathscr{S})$.

Obviously $\mathrm{Chr}(\mathscr{S}) \leq |\bigcup \mathscr{S}|$ for all set-systems $\mathscr{S}$. (See [5] p. 66).

This is a generalization of the chromatic number of a graph. We say that a set-system $\mathscr{S}$ is a λ-tuple system if $|A| = \lambda$ holds for all $A \in \mathscr{S}$. A graph $\mathscr{G}$ is a 2-tuple system. That means we identify a graph with the set of its edges. From our point of view it will be usually irrelevant if the set of vertices is $\bigcup \mathscr{G}$ or any set containing $\bigcup \mathscr{G}$. If nothing else is said and $\mathscr{S}$ is a set-system then $\bigcup \mathscr{S}$ will be called the set of its vertices.

If $\mathscr{S}$ is a set-system and X a set we say that X is a *free set* for $\mathscr{S}$ if no element of $\mathscr{S}$ is a subset of X.

Assume $\mathscr{S}$ is a set-system, λ a cardinal. *The λ-tuple system induced by $\mathscr{S}$* is defined as

$$\{Y: |Y| = \lambda \wedge \exists A (A \in \mathscr{S} \wedge Y \subset A)\}.$$

If $\mathscr{S}$ is a set-system, and X a subset of its vertices. The *sub-set-system* of $\mathscr{S}$ *induced* (or spanned) *by* X is the set-system $\{Y: Y \in \mathscr{S} \wedge Y \subset X\} = \mathscr{S} \cap \mathscr{P}(X)$.

Where X is a set, λ a cardinal we put as usual $[X]^\lambda = \{Y \subset X: |Y| = \lambda\}$, $[X]^{<\lambda} = \{Y \subset X: |Y| < \lambda\}$.

If X_i $(i < n)$ are pairwise disjoint sets, λ_i $(i < n)$ cardinals then

$$[X_0, \ldots, X_{n-1}]^{\lambda_0, \ldots, \lambda_{n-1}} = \{Y \subset \bigcup_{i<n} X_i: |X_i \cap Y| = \lambda_i \text{ for } i < n\}.$$

A $K(\lambda, \kappa)$-bipartite graph is a graph of the form $[X_0, X_1]^{1,1}$ where $|X_0| = \lambda$, $|X_1| = \kappa$.

A *complete κ-graph* is a graph of the form $[X]^2$ where $|X| = \kappa$.

C_j, $3 \leq j < \omega$ denotes a graph which is a circuit of length j.

If we say e.g. that a graph $\mathscr{G}$ contains a circuit C_j we mean that

$\mathcal{G}$ contains a subgraph isomorphic to a C_j. We use similar conventions for other classes of graphs or set-systems whenever there is no danger of misunderstanding.

Finally we mention two conventions about ordered sets. Let $R, \prec$ be an ordered set. We put $R \mid \prec x = \{y \in R: y \prec x\}$. For $A, B \subset R$ we write $A \prec B$ iff $\forall a, b(a \in A \wedge b \in B \Rightarrow a \prec b)$.

Though we will try to make the rest of the paper self contained, a knowledge of the introductory parts of [5] might be helpful.

§2. A THEOREM RESTRICTING THE CHROMATIC NUMBERS OF RELATIVELY SMALL n-TUPLE SYSTEMS

In this chapter a generalization of Theorem A is proved for n-tuple systems which yields the only if part of Theorem B. The method of proof is the same which was used to establish results of similar character in [6] §5, [5] §4, and [12]. In its simplest form this method is due to E. W. Miller (see [21]).

First we state our theorem in its general form.

Theorem 2.1. *Let* $k, i_j \in \omega$, $1 \leqslant j < \omega$ *be such that* $k = \sum_{j=1}^{\infty} i_j$, *and* $i_j = 0$ *implies* $i_m = 0$ *for* $j < m$. *Suppose* $\mathcal{S}$ *is a* $k + 2$-*tuple system with chromatic number* $> \aleph_\alpha$. *Then there is a positive integer* m *such that*

(1) *for each* $t < \omega$ *there are pairwise disjoint* $i_m + 1$ *sets* A_s $(s < t)$ *and an* $\aleph_{\alpha+m}$ *set* B *such that* $\forall s < t \, \forall b \in B \, \exists X \in \mathcal{S} (A_s \cup \{b\} \subset X)$. *Hence in particular, there exists* $\mathcal{S}' \subset \mathcal{S}$ *with* $|\mathcal{S}'| = \aleph_{\alpha+m}$ *and* $|\bigcap \mathcal{S}'| \geqslant i_{m+1}$.

Note that for a given sequence of i_j's we get the best result if we arrange them decreasingly.

Before turning to the proof first we show the only if part of Theorem B follows from this.

Corollary 2.2. *Let* $n \geqslant mi + 2$. *If* $\mathscr{S}$ *is an* $(n, i, \aleph_\alpha)$*-system on* $\aleph_{\alpha+m}$ *vertices then* $\mathrm{Chr}(\mathscr{S}) \leqslant \aleph_\alpha$.

Proof. Since the chromatic number of the induced $mi + 2$-tuple system is not smaller than $\mathrm{Chr}(\mathscr{S})$ we may assume $n = mi + 2$.

Let $k = mi$, $i_j = i$ for $1 \leqslant j \leqslant m$ and $i_j = 0$ for $j > m$. Now by Theorem 2.1, either $\mathscr{S}$ is not an $(n, i, \aleph_\alpha)$-system or the graph induced by $\mathscr{S}$ contains a point of valency $\aleph_{\alpha+m+1}$, a contradiction.

For the proof of Theorem 2.1 we need a sequence of lemmas.

Lemma 2.3. *Suppose* $f\colon [\lambda]^{<\omega} \to [\lambda]^{\leqslant\kappa}$ *where* λ *is an uncountable cardinal and* $\kappa < \lambda$. *Then there is a decomposition*

$$\lambda = \bigcup_{\nu < \mathrm{cf}(\lambda)} S_\nu$$

where the sets S_ν *are disjoint,* $|S_\nu| < \lambda$ *and* $X \in \left[\bigcup_{\mu<\nu} S_\mu\right]^{<\omega}$ *implies* $f(X) \subset \bigcup_{\mu<\nu} S_\mu$ *for* $\nu < \mathrm{cf}(\lambda)$.

Proof. It is an easy exercise or else it follows from the Löwenheim – Skolem theorem.

Definition 2.4. Let $\mathscr{S}$ be a set-system with set of vertices V. The *strong coloring number* of $\mathscr{S}$ is the smallest cardinal λ such that there exists a well-ordering $\prec$ of V satisfying the following:

For all $x \in V$ and for all systems $\mathscr{S}'$ of pairwise disjoint subsets of $V|\prec x$ with $A \in \mathscr{S}' \Rightarrow A \cup \{x\} \in \mathscr{S}$, $|\mathscr{S}'| < \lambda$ holds.

The strong coloring number of $\mathscr{S}$ will be denoted by $\mathrm{Col}^*(\mathscr{S})$. This concept was first introduced in [5] 13.7 where we called it quasi-coloring number.

Lemma 2.5 (see [5] 3.8). *Assume* $\mathscr{S}$ *is a set-system of finite sets. Then* $\mathrm{Chr}(\mathscr{S}) \leqslant \mathrm{Col}^*(\mathscr{S})$.

Proof. Let $\prec$ be a well-ordering of the set of vertices V of $\mathscr{S}$ satisfying the requirement of the above definition with $\lambda = \mathrm{Col}^*(\mathscr{S})$.

One can easily define a mapping $f\colon V \to \lambda$ by transfinite induction on $\prec$ in such a way that $f^{-1}(\{\nu\})$ is a free set of $\mathcal{S}$ for all $\nu < \lambda$.

Lemma 2.6. *Suppose* $0 < r < n < \omega < \lambda$. *Set* $\mathcal{S}$ *be an* n*-tuple system with set of vertices* λ *and* $\operatorname{Chr}(\mathcal{S}) > \aleph_\alpha$. *Then one of the following conditions holds:*

(1) *For any* $t < \omega$ *and* $\tau < \lambda$ *there are pairwise disjoint sets* $A_s \in \in [\lambda]^r$, $(s < t)$ *such that* $|\{b \in \lambda\colon \forall s < t\, \exists X \in \mathcal{S}\, (A_s \cup \{b\} \subset X)\}| > \tau$.

(2) *The* $(n - r + 1)$*-tuple system induced by* $\mathcal{S}$ *has a subsystem of cardinality* $< \lambda$ *with chromatic number* $> \aleph_\alpha$.

Proof. Let $t < \omega$ and $\omega \leqslant \tau < \lambda$ be a pair for which (1) fails and assume indirectly that (2) is false as well. Define $f(X)$ for $X \in [\lambda]^{<\omega}$ as follows. Assume $X \in [\lambda]^{tr}$. Let $y \in f(X)$ iff there are sets $A_s \in [X]^r$, $(s < t)$, $\bigcup_{s<t} A_s = X$ such that for all $s < t$ there is a $Y \in \mathcal{S}$ with $A_s \cup \{y\} \subset Y$. Put $f(X) = \phi$ in the other cases. By the indirect assumption that (1) fails we know that $|f(X)| \leqslant \tau$ for $X \in [\lambda]^{<\omega}$. Hence applying Lemma 2.3 we get a decomposition

$$\lambda = \bigcup_{\nu < \operatorname{cf}(\lambda)} S_\nu$$

satisfying the requirements of Lemma 2.3.

Now $n - r + 1 \geqslant 2$, and we may consider the $(n - r + 1)$-tuple systems induced by $\mathcal{S}$ on the sets S_ν for $\nu < \operatorname{cf}(\lambda)$. Since $|S_\nu| < \lambda$ and (2) fails they all have chromatic number $\leqslant \aleph_\alpha$, hence there are sets $S_{\nu,\rho}$, $\rho < \omega_\alpha$ such that

$$S_\nu = \bigcup_{\rho < \omega_\alpha} S_{\nu,\rho} \quad \text{and}$$

no $(n - r + 1)$-set of $S_{\nu,\rho}$ is contained in an element of $\mathcal{S}$ for $\nu < < \operatorname{cf}(\lambda)$, $\rho < \omega_\alpha$.

Put $D_\nu = \bigcup_{\nu < \operatorname{cf}(\lambda)} S_{\nu,\rho}$ for $\rho < \omega_\alpha$. This is a decomposition of λ into the union of $\aleph_\alpha$ sets and thus we will be done if we show that $\mathcal{S}$ is $\leqslant \aleph_\alpha$ chromatic on each set D_ρ, $\rho < \omega_\alpha$.

Let now ρ be fixed and put briefly $D_\rho = D$, $S_{\nu,\rho} = Z_\nu$ for $\nu < < \mathrm{cf}(\lambda)$. For $X \in \mathscr{S}$, $X \subset D$ let $\nu = \max\{\mu: Z_\mu \cap X \neq \phi\}$. Then, by the above mentioned properties of $S_{\nu,\rho}$, $|X \cap \bigcup_{\mu<\nu} Z_\mu| \geqslant r$.

Let now

$$\hat{\mathscr{S}} = \left\{ Y \in [D]^{r+1}: \exists X \in \mathscr{S}\ \exists \nu < \mathrm{cf}(\lambda) \right.$$
$$\left. \left(Y \subset X \wedge |Y \cap Z_\nu| = 1 \wedge |Y \cap \bigcup_{\mu<\nu} Z_\mu| = r \right) \right\}.$$

Since for all $X \in \mathscr{S}$, $X \subset D$ there is $Y \in \hat{\mathscr{S}}$, $Y \subset X$ it is sufficient to see that $\mathrm{Chr}(\hat{\mathscr{S}}) \leqslant \aleph_\alpha$.

Let now $\prec$ be a well-ordering of D such that $Z_\mu < Z_\nu$ for $\mu < < \nu < \mathrm{cf}(\lambda)$. By the definition of f, for all $x \in D$ there are at most $t-1$ pairwise disjoint sets $T \subset D | \prec x$ such that $T \cup \{x\} \in \hat{\mathscr{S}}$. This means according to definition 2.4 that $\mathrm{Col}^*(\hat{\mathscr{S}}) \leqslant t$, hence by Lemma 2.5, $\mathrm{Chr}(\hat{\mathscr{S}}) \leqslant t \leqslant \aleph_\alpha$.

The following is an easy corollary of Lemma 2.6.

Corollary 2.7. *Assume $\mathscr{S}$ is an n-tuple system with chromatic number $> \aleph_\alpha$ and let κ be an infinite cardinal. Either the graph induced by $\mathscr{S}$ contains a $K(t, \kappa^+)$ for every $t < \omega$ or else $\mathscr{S}$ has a subsystem on $\leqslant \kappa$ vertices with chromtic number $> \aleph_\alpha$.*

Proof. Assume that for some $t < \omega$ the induced graph does not contain a $K(t, \kappa^+)$ subgraph. By minimizing we may assume the existence of a subsystem such that the set of vertices is λ, and all subsystems spanned by a set of smaller cardinality have chromatic number $\leqslant \aleph_\alpha$. If $\lambda > \kappa$, then using Lemma 2.6 with $\tau = \kappa$, $r = 1$ we get a contradiction.

Note that Theorem A stated in the introduction follows from Corollary 2.7 if we put $\kappa = \aleph_\alpha$, $n = 2$.

Now we can give the

Proof of Theorem 2.1. Let $l = \min\{j - 1: i_j = 0\}$. If (1) is false for $m = l + 1$, then by Corollary 2.7 we may assume that the set of vertices has cardinality $\leqslant \aleph_{\alpha+l}$. Now we apply induction on k. For $k=0$ we have $l=0$ and by now the statement is trivial. Assume $k>0$ and the statement is true for all $k'<k$. Then $l>0$. Put $k'=k-i_l$. Using Lemma 2.6 with $r=i_l+1$ we get that either (1) holds with $m=l$ or the $k'+2$-tuple system induced by $\mathscr{S}$ has a subsystem $\mathscr{S}'$ of chromatic number $>\aleph_\alpha$ having at most $\aleph_{\alpha+l-1}$ vertices. Let $i'_j=i_j$ for $j<l$, $i'_j=0$ for $j\geqslant l$, $k'=\sum_{j=1}^{\infty} i'_j$. By the induction hypothesis $\mathscr{S}'$ satisfies (1) of Theorem 2.1 with some m and $m<l$ for this m because the cardinality of $\mathscr{S}$ is small. Hence $\mathscr{S}$ satisfies (1) as well.

§3. COROLLARIES TO THEOREM 2.1. THE LOWER ESTIMATES FOR $h_3(t,\alpha)$, $\hat{g}_3(t,\alpha)$, $g_n(t,\alpha)$.

The following is an obvious corollary to Theorem 2.1.

Corollary 3.1. *Let* $\mathscr{S}$ *be a triple system with* $\mathrm{Chr}(\mathscr{S})>\aleph_\alpha$. *Then, either the induced graph contains a* $K(t,\aleph_{\alpha+2})$ *for every* $t<\omega$, *or else for every* $t<\omega$ *there are pairwise disjoint 2-sets* A_s $(s<t)$ *and an* $\aleph_{\alpha+1}$ *set* B *such that* $A_s\cup\{b\}\in\mathscr{S}$ *for all* $s<t$, $b\in B$.

Now we draw the first corollary for $\hat{g}_3(t,\alpha)$.

Definition 3.2. Let $\hat{g}_3(t,\alpha)=\min\{m$: There is a triple system $\mathscr{S}$ on $\aleph_{\alpha+1}$ points with $\mathrm{Chr}(\mathscr{S})=\aleph_{\alpha+1}$ such that all subsystems induced by t points have at most m triples$\}$.

Corollary 3.3. *If* $\mathscr{S}$ *is an* $\aleph_{\alpha+1}$*-chromatic triple system on* $\aleph_{\alpha+1}$ *points, then for all* $t<\omega$ *there are disjoint 2-sets* A_s $(s<t)$ *and an* $\aleph_{\alpha+1}$ *set* B *such that* $A_s\cup\{b\}\in\mathscr{S}$ *for all* $s<t$, $b\in B$. *Hence for all* $t>\omega$ *there are* t *points contining* $\left[\frac{t^2}{8}\right]$ *triples i.e.* $\hat{g}_3(t,\alpha)\geqslant\left[\frac{t^2}{8}\right]$.

Proof. The first statement follows from 3.1. The second follows from this with an easy discussion.

3.3 will be shown to be best possible using G.C.H. (see §11. Theorem 11.13).

Now first we deduce a stronger result for triple system on $\aleph_{\alpha+1}$ without free $\aleph_{\alpha+1}$-sets.

Definition 3.4. Let $h_3(t, \alpha) = \min\{m$: There is a triple system $\mathscr{S}$ on $\aleph_{\alpha+1}$ without free $\aleph_{\alpha+1}$-sets, and such that all subsystems induced by t points have at most m triples$\}$.

Lemma 3.5. *Let $\mathscr{S}$ be a triple system on $\omega_{\alpha+1}$ without a free $\aleph_{\alpha+1}$-set. Then there is a set $A \subset \omega_{\alpha+1}$, $|A| = \aleph_0$ such that*

(1) $\qquad |\{Y \in \mathscr{S}: X \subset Y\}| = \aleph_{\alpha+1}$ for all $X \in [A]^2$.

Proof. Let $\mathscr{G}$ be the graph on $\omega_{\alpha+1}$ whose edges are the $X \in$ $\in [\omega_{\alpha+1}]^2$ satisfying (1). If $\mathscr{G}$ contains a complete $\aleph_0$ graph we are done. If this is not the case, then by the Erdős – Dushnik – Miller theorem $\aleph_{\alpha+1} \to (\aleph_{\alpha+1}, \aleph_0)^2$ we obtain an $\aleph_{\alpha+1}$-set B such that (1) is false for all $X \in [B]^2$. Then, by 3.3, $\mathscr{S}$ is $\leqslant \aleph_\alpha$ chromatic on B hence there is an $\aleph_{\alpha+1}$-subset of B free for $\mathscr{S}$, a contradiction.

Corollary 3.6. *If $\mathscr{S}$ is a triple system on $\omega_{\alpha+1}$ with no free $\aleph_{\alpha+1}$-set then there are points α_i, β_i, $i < \omega$ such that $\alpha_i < \beta_i < \alpha_j$, $\{\alpha_i, \beta_i, \alpha_j\} \in \mathscr{S}$, $\{\alpha_i, \beta_i, \beta_j\} \in \mathscr{S}$ for all $i < j < \omega$.* Hence for all $t < \omega$ *there are t points containing $\left[\frac{(t-1)^2}{4}\right]$ triples i.e. $h_3(t, \alpha) \geqslant$ $\geqslant \left[\frac{(t-1)^2}{4}\right]$.*

Proof. To prove the first statement one chooses the points α_i, β_i by induction on $i < \omega$ using 3.5. It then follows that for every $t < \omega$ there are $2t$ points containing $t(t-1)$ triples. The rest is obvious.

For upper estimates see § 10 Theorem 10.5. The next corollary is included just to show the strength of Theorem 2.1 for those readers who prefer numerical examples.

Corollary 3.7. *Suppose $\mathscr{S}$ is a 4-tuple system with chromatic number $> \aleph_0$. Then the following two statements hold:*

(1) *There is $\mathscr{S}_1 \subset \mathscr{S}$ such that either*

$|\mathscr{S}_1| = \aleph_1, \ |\bigcap \mathscr{S}_1| \geqslant 3$ or else $|\mathscr{S}_1| = \aleph_2, \ |\bigcap \mathscr{S}_2| \geqslant 1$.

(2) *There is* $\mathscr{S}_2 \subset \mathscr{S}$ *such that either*

$|\mathscr{S}_2| = \aleph_1, \ |\bigcap \mathscr{S}_2| \geqslant 2$ *or else* $|\mathscr{S}_2| = \aleph_3, \ |\bigcap \mathscr{S}_2| \geqslant 1$.

Proof. Both statements follow from Theorem 2.1. To see the first choose $k_1 = 2, \ k_j = 0$ for $j > 1$, for the second one put $k_1 = k_2 = 1$ and $k_j = 0$ for $j > 2$.

At this point we have to confess to the reader that Theorem 2.1 is still not general enough to get the promised lower estimate of the $g_n(t, \alpha)$ functions. We now describe the proof of the necessary generalization for $n = 3$ only.

Theorem 3.8. *If a triple system* $\mathscr{S}$ *has chromatic number* $> \aleph_\alpha$ *then one of the following statements holds.*

(1) *For every* $t < \omega$ *there are* t *disjoint edges and* $\aleph_{\alpha+1}$ *points joined to all of them by a triple of* $\mathscr{S}$.

(2) *For every* $t < \omega$ *there is a set* $F, \ |F| = t^2$, *and there are* $\aleph_{\alpha+1}$ *vertex-disjoint* $K(t, t)$ *such that each edge in a* $K(t, t)$ *is joined by a triple of* $\mathscr{S}$ *to some point in* F.

Proof. Let λ be the smallest cardinal such that there is a triple system $\mathscr{S}$ with set of vertices λ, $\mathrm{Chr}(\mathscr{S}) > \aleph_\alpha$ for which (1) and (2) both fall say for t_1 and t_2. Then, by Corollary 3.3, $\lambda \geqslant \aleph_{\alpha+2}$.

We define two functions f_1, f_2 on $[\lambda]^{<\omega}$ as follows. For $Y \in [\lambda]^{2t_1}$ put $y \in f_1(Y)$ iff there are t_1 disjoint 2-sets A_s $(s < t)$ with $\bigcup_{s<t_1} A_s = Y$ such that $A_s \cup \{y\} \in \mathscr{S}$ for $s < t_1$. Put $f_1(Y) = \phi$ in all other cases. For each $Z \in [\lambda]^{\leqslant t_2^2}$ choose a maximal system $\mathscr{F}(Z)$ of $2t_2$-sets satisfying the following conditions:

(3) The elements of $\mathscr{F}(Z)$ are pairwise disjoint, and for $A \in \mathscr{F}(Z)$ there are $A_0, A_1 \in [A]^{t_2}$, $A_0 \cup A_1 = A$ such that for all $a_0 \in A_0$, $a_1 \in A_1$ there is a $z \in Z$ with $\{a_0, a_1, z\} \in \mathscr{S}$. Put $f_2(Z) = \bigcup \mathscr{F}(Z)$ and

let $f_2(Z) = \phi$ in case $t_2^2 < |Z| < \omega$. Let further $f(Z) = f_1(Z) \cup f_2(Z)$. By the assumption on t_1 and t_2 we know $|f(Z)| \leq \aleph_{\alpha+1}$ for $Z \in \in [\lambda]^{<\omega}$. By Lemma 2.3 we get a partition $\lambda = \bigcup_{\nu < \mathrm{cf}(\lambda)} S_\nu$ satisfying the requirements of Lemma 2.3. By the minimality of λ, $\mathcal{S}$ is $\leq \aleph_\alpha$-chromatic on the sets S_ν. Choose sets $S_{\nu,\rho}$, $\rho < \omega_\alpha$ free for $\mathcal{S}$ with $S_\nu = \bigcup_{\rho < \omega_\alpha} S_{\nu,\rho}$, and put $C_\rho = \bigcup_{\nu < \mathrm{cf}(\lambda)} S_{\nu,\rho}$ for $\rho < \omega_\alpha$. Again it suffices to see that $\mathcal{S}$ is $\leq \aleph_\alpha$-chromatic on each C_ρ.

Let $\rho < \omega_\alpha$ be fixed and assume $X \subset C_\rho$, $X \in \mathcal{S}$. Let $\nu(X) = \max\{\mu < \mathrm{cf}(\lambda): S_{\mu,\rho} \cap X \neq \phi\}$. By the choice of $S_{\nu,\rho}$, $1 \leq |X \cap S_{\nu(X),\rho}| \leq 2$. Put $\hat{\mathcal{S}}_i = \{X \in \mathcal{S}: X \subset C_\rho \wedge |X \cap S_{\nu(X),\rho}| = i\}$ for $i = 1, 2$. We now claim that the graph induced by $\hat{\mathcal{S}}_2$ on $S_{\nu,\rho}$ is $\leq \aleph_\alpha$-chromatic for all $\nu < \mathrm{cf}(\lambda)$. By Corollary 2.7 it is sufficient to see that it does not contain a $K(t_2, t_2)$.

Assume $A \subset S_{\nu,\rho}$, $|A| = 2t_2$, $|A_0| = |A_1| = t_2$, $A = A_0 \cup A_1$ and all the edges $\{a_0, a_1\}$, $a_0 \in A_0$, $a_1 \in A_1$ belong to the graph induced by $\hat{\mathcal{S}}_2$. For each such pair a_0, a_1 pick z with $\{a_0, a_1, z\} \in \mathcal{S}$ and let Z be the set of all z chosen this way. Then $|Z| \leq t_2^2$ and by the definition of $\hat{\mathcal{S}}_2$, $Z \subset \bigcup_{\mu < \nu} S_\mu$. By the choice of the set S_ν, then $f_2(Z) = \bigcup \mathcal{F}(Z) \subset \bigcup_{\mu < \nu} S_\mu$ and this contradicts the maximality of $\mathcal{F}(Z)$. It now follows that $S_{\nu,\rho} = \bigcup_{\sigma < \omega_\alpha} S_{\nu,\rho,\sigma}$ where the sets $S_{\nu,\rho,\sigma}$ are free sets for the graph induced by $\hat{\mathcal{S}}_2$. Put $D_{\rho,\sigma} = \bigcup_{\nu < \mathrm{cf}(\lambda)} S_{\nu,\rho,\sigma}$. Then the sets $D_{\rho,\sigma}$ are free sets for $\hat{\mathcal{S}}_2$ as well. It is now enough to see that each $D_{\rho,\sigma}$ is the union of at most $\aleph_\alpha$ subsets free with respect to $\hat{\mathcal{S}}_1$.

However just as in the proof of Lemma 2.6, this follows from the fact that by the choice of f_1, $\hat{\mathcal{S}}_1$ has strong coloring number at most t_1 on $D_{\rho,\sigma}$. This completes the proof.

Corollary 3.9. *If a triple system has chromatic number* $> \aleph_0$, *then for each* $t < \omega$ *there is a set of* $3t^2$ *points containing* $\geq t^3$ *triples of it and as a corollary of this* $g_3(t, \alpha) \geq \left(\frac{t}{3}\right)^{\frac{3}{2}} - t$ *for all* α.

Proof. If either of the conditions (1) or (2) holds we can choose t "edge-disjoint" $K(t, t)$'s and a set F of at most t^2 elements disjoint to the union of this $K(t, t)$'s so that every edge of a $K(t, t)$ is joined from a point of F by a triple of $\mathscr{S}$.

We now state a similar corollary for n-tuple system.

Corollary 3.10. *Let $\mathscr{S}$ be an n-tuple system of chromatic number $> \aleph_0$. Then for each $t < \omega$ there is a set of $n \cdot t^{n-1}$ points containing t^n n-tuples of $\mathscr{S}$,*

$$g_n(t, \alpha) \geqslant \left(\frac{t}{n}\right)^{\frac{n}{n-1}} + o\left(t^{\frac{n}{n-1}}\right) .$$

We have proved this for $n = 2$ and $n = 3$.

The general result is to be proved by a common generalization of Theorems 2.1 and 3.8. The proof does not require new ideas. We omit the details.

In §14 we will state an upper estimate for $g_n(t, \alpha)$ (see Theorem 14.6) for arbitrary α. However in what follows we will focus our attention on the numerous problems and difficulties arising for triple systems. Our results will determine $g_3(t, \alpha)$ for small values of t (see Theorem 14.4).

§4. A GENERAL THEOREM FOR SET-SYSTEMS

In §2 we have seen the use of the strong coloring number. There is a theorem for graphs both trivial and well-known saying that if the valency of each vertex of a graph is $\leqslant \aleph_\alpha$, then the graph has chromatic number at most $\aleph_\alpha$.

Let us make the following

Definition 4.1. Let $\mathscr{S}$ be a set-system. We say that a vertex p has strong valency $> \aleph_\alpha$ if there is an $\mathscr{S}' \subset \mathscr{S}$, $|\mathscr{S}'| > \aleph_\alpha$ such that $X \cap Y = \{p\}$ for all $X \neq Y \in \mathscr{S}'$.

To the best of our knowledge the following generalization of the graph theorem mentioned above is not stated in the literature.

Theorem 4.2. *Let* $\mathscr{S}$ *be a set-system of chromatic number* $> \aleph_\alpha$ *and consisting of sets of cardinality* $\leq \aleph_\alpha$. *Then there is a point of strong valency* $> \aleph_\alpha$.

Proof. Assume no point has strong valency $> \aleph_\alpha$. For each vertex p let $\mathscr{F}(p)$ be a maximal subfamily of $\mathscr{S}$ such that $p \in X$ for $X \in \mathscr{F}(p)$ and $X \neq Y \in \mathscr{F}(p)$ implies $X \cap Y = \{p\}$. Then $|\mathscr{F}(p)| \leq \aleph_\alpha$ and $|f(p)| \leq \aleph_\alpha$ for $f(p) = \bigcup \mathscr{F}(p)$. Let λ be the set of vertices. Let us say that $A \subset \lambda$ is closed if $f(x) \subset A$ for all $x \in A$. Obviously each $\xi \in \lambda$ is contained in a closed subset A_ξ of cardinality $\leq \aleph_\alpha$.

Let $B_\xi = A_\xi - \bigcup_{\eta < \xi} A_\eta$ for $\xi < \lambda$. Considering that $|B_\xi| \leq \aleph_\alpha$ there are sets C_ρ, $(\rho < \omega_\alpha)$ such that each C_ρ meets each B_ξ in at most one point and $\lambda = \bigcup_{\rho < \omega_\alpha} C_\rho$. We claim that each C_ρ is a free set for $\mathscr{S}$. Assume indirectly $X \in \mathscr{S}$, $X \subset C_\rho$. Put $\xi = \min\{\eta\colon X \cap B_\eta \neq \phi\}$. Let $\{\zeta\} = C_\rho \cap X \cap B_\xi$. Then $\zeta \in A_\xi$, $f(\zeta) \subset A_\xi$, $f(\zeta) \subset \bigcup_{\eta \leq \xi} B_\eta$, $X \cap \bigcup_{\eta \leq \xi} B_\eta = \{\zeta\}$. It follows that then $\mathscr{F}(\zeta) \cup \{X\}$ still has the property that the intersection of any two elements of it is $\{\zeta\}$ and this contradicts the maximality of $\mathscr{F}(\zeta)$. This proves the claim.

Remark. The assumption that the elements of $\mathscr{S}$ have cardinality $\leq \aleph_\alpha$ seems fairly natural. However, we can not prove it to be necessary.

Problem 1. Let $\mathscr{S} \subset [\omega_1]^{\aleph_1}$ be such that each vertex has strong valency $\leq \aleph_0$. Is it true that then $\mathrm{Chr}(\mathscr{S}) \leq \aleph_0$?

A. Máté pointed out to us that the answer is negative provided there is a non-trivial $\aleph_1$-complete ideal I on $\aleph_1$ that the Boolean factor algebra $P(\aleph_1)|I$ has a dense subset of cardinality $\aleph_1$. However, considering the simplicity of the question it would be nice to have more information.

In this chapter we only draw one corollary of Theorem 4.2. In what follows we will introduce notation for a number of special triple systems. We will usually give a diagram with the definition, and at the end of the paper we will give a list with reference to the place of its definition.

Let $\mathscr{T}_1$ be the triple system with six vertices and three triples having pairwise one element in common (see Diagram 1).

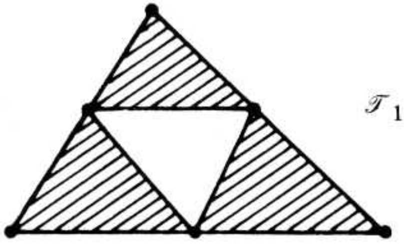


Diagram 1

Let $\mathscr{T}_2$ be the triple system with seven vertices and four triangles. Three of them have one point in common and the fourth meets each of these three in a point different from this point (see Diagram 2).

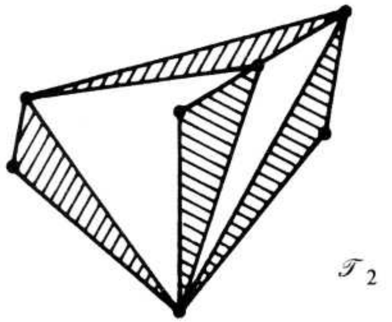


Diagram 2

Corollary 4.2.

(1) $\aleph_{\alpha+1} \rightarrow (\aleph_{\alpha+1}, \mathscr{T}_1)^3$,

(2) $\aleph_{\alpha+1} \rightarrow (\aleph_{\alpha+1}, \mathscr{T}_2)^3$.

Proof. Let $\mathscr{S}$ be a triple system on $\omega_{\alpha+1}$ with no free set of cardinality $\aleph_{\alpha+1}$. Then $\mathrm{Chr}(\mathscr{S}) = \aleph_{\alpha+1}$. By 4.1, there is a point with strong valency $> \aleph_\alpha$. That means there is a vertex p and $\aleph_{\alpha+1}$ disjoint

2-sets each joined to p by a triangle of $\mathscr{S}$. Since there is no free $\aleph_{\alpha+1}$-set, $\mathscr{S}$ must contain a triangle meeting three different 2-sets. Hence $\mathscr{S}$ contains a $\mathscr{T}_2$. This proves (2) and (1) is a corollary.

The point is that $\mathscr{T}_2$ is the simplest triple system $\mathscr{T}$ for which $\aleph_{\alpha+1} \to (\aleph_{\alpha+1}, \mathscr{T})^3$ holds and is not of the type obtained in Corollary 3.6.

§5. SOME CONSEQUENCES OF MARTIN'S AXIOM

Our first aim in this section is to prove Theorem 5.6. This is a result pointing in the main direction of this paper. Assuming MA and $2^{\aleph_0}$ is large, it gives a strengthening of a special case of Corollary 2.2, and shows that it can not be best possible in ZFC alone. However, we will generalize a theorem of Baumgartner and Hajnal as well. For Martin's axiom see [24], p. 232.

Definition 5.1. Let $\mathscr{S}$ be a set system with set of vertices κ. A partial function from κ into ω is said to be a good coloring if it is not constant on any $X \subset D(f)$, $X \in \mathscr{S}$.

Lemma 5.2. *Assume* MA_κ. *Let* $\mathscr{S}$ *be a system of finite sets on* κ, *with* $\mathrm{Chr}(\mathscr{S}) > \aleph_0$. *Then there is a sequence* f_α $(\alpha < \omega_1)$ *of good colorings such that* $f_\alpha \cup f_\beta$ *is not a good coloring for* $\alpha \neq \beta < \omega_1$ *and* $|f_\alpha| < \aleph_0$ *for* $\alpha < \omega_1$.

Proof. Let P be the set of good colorings f, $|f| < \aleph_0$ with the partial order $f \preceq g$ iff $f \supset g$. The sets $D_\xi = \{f \in P\colon \xi \in D(f)\}$ are obviously dense in P for $\xi < \kappa$. If the requirement of the lemma does not hold then P satisfies the countable chain condition. By MA_κ then there is a set G generic over the family of D_ξ's. Then $F = \bigcup G$ is a good coloring of κ, hence $\mathrm{Chr}(\mathscr{S}) \leqslant \aleph_0$, a contradiction.

As a corollary we get

Lemma 5.3. *Assume* MA_κ. *If* $\mathscr{G}$ *is a graph on* κ *vertices with* $\mathrm{Chr}(\mathscr{G}) > \aleph_0$, *then there are* $\aleph_1$ *"vertex-disjoint" finite subgraphs such that any two of them are joined by an edge.*

Proof. By 5.2 there is a sequence f_α $(\alpha < \omega_1)$ of good colorings such that $|f_\alpha| < \aleph_0$ and $f_\alpha \cup f_\beta$ is not a good coloring for $\alpha \neq \beta$. Let $A_\alpha = D(f_\alpha)$ for $\alpha < \omega_1$. By the Erdős – Rado theorem (see [15]) we may assume that the A_α from a "Δ-system" i.e. there is a set D such that $A_\alpha \cap A_\beta = D$ for all $\alpha \neq \beta$ and that $f_\alpha | D = f_\beta | D$ for all $\alpha, \beta < \omega_1$. If $\alpha \neq \beta$ then there must be an edge joining $A_\alpha - D$ and $A_\beta - D$ otherwise $f_\alpha \cup f_\beta$ is a good coloring.

Theorem 5.4. *Assume* $\mathrm{MA}_{\aleph_\alpha}$. *If* $\mathcal{G}$ *is a graph on an ordinal* $\beta < \omega_{\alpha+1}$ *with* $\mathrm{Chr}(\mathcal{G}) > \aleph_0$ *then* $\mathcal{G}$ *contains a* $K(\gamma, \omega_1)$ *for every* $\gamma < \omega_1$.

Note that as a corollary of this $\mathrm{MA}_{\aleph_1}$ implies $\omega_1 \to \left(\omega_1, \binom{\omega_1}{\gamma}\right)^2$ for all $\gamma < \omega_1$. This result was stated in [2] without proof. A proof of this is given in Laver's paper [20] in this volume even in case the first two ω_1 are replaced by a κ with $\mathrm{cf}(\kappa) > \omega$. The present result is much stronger, but the proofs are almost the same. They are based on one of the main lemmas of [2]. First we state the following immediate

Corollary 5.5. *Assume* $\mathrm{MA}_{\aleph_\alpha}$ *and* $r < \omega < \mathrm{cf}(\omega_\alpha)$. *Then* $\omega_\alpha^r \to \left(\omega_\alpha^r, \binom{\omega_1}{\gamma}\right)^2$ *and "stationary subset of* $\omega_\alpha \to \left(\text{stationary subset of } \omega_\alpha, \binom{\omega_1}{\gamma}\right)^2$" *hold for* $\gamma < \omega_1$.

Proof of Theorem 5.4. By 5.3, there are disjoint sets A_ν $(\nu < \omega_1) \subset \beta$ such that for $\nu \neq \mu$ there is an edge of $\mathcal{G}$ joining A_ν and A_μ. We may assume that γ is an indecomposable ordinal (i.e. of the form ω^σ for some $1 \leq \sigma < \omega_1$), and that $|A_\nu| = m$ for $\nu < \omega_1$.

Let $a_{\nu,i}$ denote i-th member of A_ν for $\nu < \omega_1$. We may assume $a_{\nu,i} < a_{\mu,i}$ for $\nu < \mu < \omega_1$, $i < m$. We are going to consider the complete bipartite graph $K(\gamma, \omega_1 - \gamma)$ and a partition of length m^2 of this graph. For $\mu < \gamma$, $\nu < \omega_1 - \gamma$ put $\{\mu, \nu\} \in P(i,j)$ iff $\{a_{\mu,i}, a_{\nu,j}\} \in \mathcal{G}$. By a result of [2] (see p. 202, Corollary 2), $\alpha > 0$ and $\mathrm{MA}_{\aleph_\alpha}$ implies that there is a $K(\gamma, \omega_1 - \gamma)$ homogeneous for this partition. That means there are sets $B \subset \gamma$, $C \subset \omega_1 - \gamma$, $\operatorname{tp} B = \gamma$, $\operatorname{tp} C = \omega_1$ and $\langle i, j\rangle \in {}^2m$ such that $\{\mu, \nu\} \in P(i, j)$ for all $\mu \in B$, $\nu \in C$. But then the complete bipartite graph spanned by the sets $\{a_{\mu,i}: \mu \in B\}$, $\{a_{\nu,j}: \nu \in C\}$ is a subgraph of $\mathscr{G}$, and $\operatorname{tp}\{a_{\mu,i}: \mu \in B\} = \gamma$, $\operatorname{tp}\{a_{\nu,j}: \nu \in C\} = \omega_1$.

We now prove

Theorem 5.6. *Assume* MA_κ. *Suppose* $1 \leqslant n < \omega$, $\mathscr{S} \subset [\kappa]^{2n+1}$, $\operatorname{Chr}(\mathscr{S}) > \aleph_0$. *Then there is* $X \in [\kappa]^{n+1}$ *such that* $|\{Y \in \mathscr{S}: X \subset Y\}| \geqslant \aleph_0$.

Proof. Let $\mathscr{S}$ be a set-system satisfying the conditions of the theorem. By 5.2, there are good colorings $|f_\alpha| < \aleph_0$ $(\alpha < \omega_1)$ such that $f_\alpha \cup f_\beta$ is not a good coloring for $\alpha \neq \beta < \omega_1$. Just as in the proof of 5.3, we may assume that $D(f_\alpha) = A_\alpha$, $A_\alpha \cap A_\beta = D$ for $\alpha \neq \beta$, $f_\alpha | D = f_\beta | D$. Let $B_\alpha = A_\alpha - D$ for $\alpha < \omega_1$. If $\alpha \neq \beta$ and $f_\alpha \cup f_\beta$ is not a good coloring then $D \cup B_\alpha \cup B_\beta$ must contain a $Y \in \mathscr{S}$ which is not contained either in A_α or in A_β. Note that in this case Y must meet both B_α and B_β, and one of the intersections $Y \cap A_\alpha$, $Y \cap A_\beta$ has at least $n+1$ elements.

We now assume that the conclusion of the theorem is false and define a set mapping $f\colon \omega_1 \to [\omega_1]^{<\omega}$ as follows.

For $\alpha < \omega_1$ let

$$f(\alpha) = \{\beta < \omega_1 : \beta \neq \alpha \wedge \exists X \in [A_\alpha]^{n+1}\, \exists Y \in \mathscr{S}$$

$$(X \subset Y \wedge Y \cap B_\beta \neq \phi)\}.$$

By the indirect assumption we have $|f(\alpha)| < \aleph_0$ for $\alpha < \omega_1$. It now follows that there is an independent pair $\alpha, \beta < \omega_1$, $\alpha \neq \beta$, for this mapping and this is a contradiction.

To clarify the situation let us point out some conclusions. Consider the case $n = 1$. Then the above theorem implies that if MA_κ holds then every system of edge disjoint triples of chromatic number $> \aleph_0$ has more than κ vertices, and thus MA implies that the number of vertices must be

$\geqslant 2^{\aleph_0}$ no matter how large $2^{\aleph_0}$ is. Corollary 2.2 which was our result proved in ZFC says only that the number of vertices is $\geqslant \aleph_2$.

If we look at the next simplest case where $n = 2$ we see that if $\mathscr{S}$ is a 5-tuple system such that any two 5-tuples have at most 2 points in common the above theorem says again that assuming MA the set of vertices must have cardinality $\geqslant 2^{\aleph_0}$. However, this result is not directly comparable with the corresponding result of Corollary 2.2, which says that a 5-tuple system of chromatic number $> \aleph_0$ must have at least $\aleph_2$ vertices even if the intersection of any two 5-tuples has cardinality at most 3.

The situation is complicated further by two more facts. One of them is that we do not know the answer to

Problem 2. Does Theorem 5.6 remain true if we replace the requirement by $|\{Y \in \mathscr{S}\colon X \subset Y\}| \geqslant \aleph_1$? To put the question in a simpler form:

Is it consistent with $\mathrm{ZFC} + 2^{\aleph_0} > \aleph_2$ that every $(3, 1, \aleph_1)$ system has cardinality $\geqslant 2^{\aleph_0}$?

The second fact is that the presence of MA changes the estimates obtained for the number of vertices of large chromatic set-systems even beyond $2^{\aleph_0}$ as it is shown by the following

Theorem 5.7. *Assume* MA. *Let* $2^{\aleph_0} = \aleph_\alpha$. *If* $1 \leqslant i < \omega$, $m < \omega$ *and* $n \geqslant (m+3)i + 1$, $\mathscr{S} \subset [\omega_{\alpha+m}]^n$ *and* $\mathrm{Chr}(\mathscr{S}) > \aleph_0$, *then there is an* $X \in [\omega_{\alpha+m}]^{i+1}$ *such that* $|\{Y \in \mathscr{S}\colon X \subset Y\}| \geqslant \aleph_0$.

(Here we do not know again if $\aleph_0$ can be replaced by $\aleph_1$.)

Proof. We may assume $n = (m+3)i + 1$. Our proof goes by induction on m. We now assume that if $m > 0$ we know the result for $m - 1$. Assume the conclusion is false and define an $f\colon [\omega_{\alpha+m}]^{<\omega} \to [\omega_{\alpha+m}]^{<\omega}$ as follows. For $X \in [\omega_{\alpha+m}]^{i+1}$, $f(X) = \bigcup\{Y \in \mathscr{S}\colon X \subset Y\}|$ and put $f(X) = \phi$ in the other cases. Note that $\omega_{\alpha+m} > \omega$ even for $m = 0$. By Lemma 2.3, we get that

$$\omega_{\alpha+m} = \bigcup_{\nu < \mathrm{cf}(\omega_{\alpha+m})} S_\nu$$

where the S_ν satisfy the requirements of Lemma 2.3. For $X \in \mathscr{S}$ let $\nu(X) = \nu = \max\{\mu: X \cap S_\mu \neq \phi\}$. By the construction of S_ν, $|X \cap S_{\nu(X)}| \geqslant \geqslant (m-1+3)i+1$. Now using our standard argument, to obtain a contradiction, it is sufficient to see that the $((m+2)i+1)$-tuple systems induced by $\mathscr{S}$ are $\leqslant \aleph_\alpha$-chromatic on each S_ν, $(\nu < \mathrm{cf}(\omega_{\alpha+m}))$. Let us remark that, by the construction $|S_\nu| < \omega_{\alpha+m}$.

Considering the indirect assumption, the claim follows from Theorem 5.6 in case $m = 0$ and from the induction hypothesis in case $m > 0$ respectively.

Again we point our one instance. Theorem 5.7 implies that if MA holds and $\mathscr{S}$ is a $> \aleph_0$ chromatic 4-tuple system and two 4-tuples have only one point in common then $\mathscr{S}$ has at least $2^{\aleph_0}$ vertices. Corollary 2.2 proved in ZFC gives that $\mathscr{S}$ has $\geqslant \aleph_3$ vertices.

We now leave the reader alone to ponder about the mess we are in until we give the "upper estimates" in § § 12-15. These will clarify things if we assume G.C.H. and make the matter worse without this assumption. See the problems stated in § 16.

§ 6. THE CONCEPT OF SIMULTANEOUS CHROMATIC NUMBER. A PROBLEM. A RESULT IN L.

In the rest of the paper we are going to construct 3-tuple systems and n-tuple systems having some specific properties and chromatic number $> \kappa$. In quite a few cases the constructions will be inductive using n'-tuple systems $2 \leqslant n' < n$ with large chromatic numbers already constructed. These proofs lead us to the following generalization of the chromatic number. The idea is that the stronger property supports inductions better.

Definition. 6.1. Let $\mathscr{S}_\nu$ $(\nu < \lambda)$ be a sequence of set-systems having the same set of vertices V. The system is said to have simultaneous chromatic number κ if κ is the smallest cardinal such that there is a partition of length κ of the vertices

$$V = \bigcup_{\xi < \kappa} P_\xi$$

such that for each $\xi < \kappa$ there is $\nu < \kappa$ such that P_ξ is a free set for $\mathscr{S}_\nu$.

Obviously the simultaneous chromatic number of the system is less than or equal to all $\mathrm{Chr}(\mathscr{S}_\nu)$, hence it is a very strong assumption to have a system with large simultaneous chromatic number.

The concept defined above will be most frequently used in the following form.

Definition 6.2. Let $\mathscr{S}$ be a set-system with set of vertices V. $P(\mathscr{S}, \lambda, \kappa)$ is said to hold if there is a partition of $\mathscr{S}$ into the union of λ disjoint set-systems $\mathscr{S}_\nu$, $(\nu < \lambda)$ in such a way that this system has simultaneous chromatic number $\geqslant \kappa$.

For the convenience of the reader we give a direct definition: $P(\mathscr{S}, \lambda, \kappa)$ holds iff

$$\exists f\colon \mathscr{S} \to \lambda\ \forall \kappa' < \kappa\ \forall g\colon V \to \kappa'\ \exists \xi < \kappa'\ \forall \nu < \lambda\ \exists X \in \mathscr{S}$$
$$(X \subset g^{-1}(\{\xi\}) \wedge f(X) = \nu)\ .$$

We will often use a stronger property which can not be put in terms of the simultaneous chromatic number.

Definition 6.3. Let $\mathscr{S}$ be a set-system with set of vertices V, and let λ, κ, r be cardinals. $P^*(\mathscr{S}, \lambda\ \kappa, r)$ is said to hold iff

$$\exists f\colon \mathscr{S} \to \lambda\ \forall \kappa' < \kappa\ \forall g\colon V \to \kappa'\ \exists \xi < \kappa'\ \exists A \in [V]^r\ \forall \nu < \lambda\ \exists X \in \mathscr{S}$$
$$(A \subset X \wedge X \subset g^{-1}(\{\xi\}) \wedge f(X) = \nu)\ .$$

Obviously $P^*(\mathscr{S}, \lambda, \kappa, 0) \Leftrightarrow P(\mathscr{S}, \lambda, \kappa)$ and $P^*(\mathscr{S}, \lambda, \kappa, r) \Rightarrow P(\mathscr{S}, \lambda, \kappa) \Rightarrow$ $\Rightarrow \mathrm{Chr}(\mathscr{S}) \geqslant \kappa$ for $\lambda \geqslant 1$.

In case $\mathscr{G}$ is a graph and $r = 1$ we write $P^*(\mathscr{G}, \lambda, \kappa)$ for $P^*(\mathscr{G}, \lambda, \kappa, 1)$.

Again it is obvious that $P(\mathscr{S}, 1, \kappa) \Leftrightarrow \mathrm{Chr}(\mathscr{S}) \geqslant \kappa$, and seemingly

$P(\mathcal{S}, 2, \kappa), \ldots, P(\mathcal{S}, |\mathcal{S}|, \kappa)$ are much stronger assumptions. In fact to prove the existence of a graph $\mathcal{G}$ with $\aleph_1$-vertices satisfying $P(\mathcal{G}, \aleph_1, \aleph_1)$ we need C.H.

In spite of this we do not know the answer to the following

Problem 3. If $\mathcal{G}$ is a graph with $\mathrm{Chr}(\mathcal{G}) = \kappa \geqslant \aleph_0$, does then $P(\mathcal{G}, \kappa, \kappa)$ hold?

If $\mathcal{G}$ is an $\aleph_1$-chromatic graph on ω_1, does $P(\mathcal{G}, 2, \aleph_1)$ hold?

Surprisingly enough, our partial results point to a yes answer. We can prove that most "known graphs" of chromatic number $\aleph_1$ "split". Those results will be given in the next chapters and will be used later to construct strange 3-tuple systems. However, we have hopes to prove a positive answer only if the $\aleph_1$-chromatic graph has some essential "large parts" to split. We will show in this section that this is not necessarily the case.

• **Definition 6.4.** A *Shelah-graph* is a graph $\mathcal{G}$ on set of vertices V, where $|V| = \aleph_1$ and there is $X \in [V]^{\aleph_0}$ such that $|\{x \in X: \{x, a\} \in \mathcal{G}\}| \geqslant \aleph_0$ for all $a \in V - X$.

Answering Problem 32 of [8] Shelah proved the following results.

Theorem 6.5 (See: S. Shelah [22]. Theorems 2.1, 2.4).

(A) *Assume* C.H. *If* $\mathcal{G}_\alpha$ $(\alpha < \omega_1)$ *is a system of Shelah graphs then* $\omega_1 \nrightarrow [\bigvee_{\alpha<\omega_1} \mathcal{G}_\alpha]^2_{\aleph_1}$.

As a corollary of this we know that there is an $\aleph_1$-chromatic graph on ω_1 containing none of the $\mathcal{G}_\alpha$.

(B) *Assume* $\mathrm{V} = \mathrm{L}$ *then* $\omega_1 \to (\mathcal{G})^2_2$ *holds iff*

$\mathrm{Col}(\mathcal{G}) \leqslant \omega$.

Note that all Shelah graphs have coloring number $> \omega$.

The coloring number of a graph $\mathcal{G}$ (see [5] p. 66. Definition 2.9) is the same as the strong coloring number of $\mathcal{G}$ as defined in 2.4.

Now we ask

Problem 4. Can it be proved in ZFC or does C.H. imply that there is an $\aleph_1$-chromatic graph on ω_1 containing no Shelah graphs?

We are going to prove

Theorem 6.6. *If* $V = L$, *then* $\omega_1 \nrightarrow$ (*stationary subset of* ω_1, *Shelah graph*)^2.

This result is also stated and proved in [19] and as a corollary of this the answer to Problem 4 is affirmative provided $V = L$. This is the result which makes Problem 3 awkward to answer.

Proof of Theorem 6.6. *By* $V = L$, $\diamond_{\omega_1}$ holds. That means there exists a sequence $S_\alpha \subset \mathscr{P}(\alpha)$, $|S_\alpha| \leqslant \aleph_0$ for $\alpha < \omega_1$ such that $\{\alpha < \omega_1 : X \cap \alpha \in S_\alpha\}$ contains a closed unbounded subset for all uncountable $X \subset \omega_1$.

We may also assume that for all limit α, $Y \in S_\alpha$ implies that Y is cofinal in α. Now for every limit number $\alpha < \omega_1$ we can choose a set $R_\alpha \subset \alpha$, $\operatorname{tp} R_\alpha = \omega$, R_α cofinal in α and such that $Y \cap R_\alpha \neq \phi$ for $Y \in S_\alpha$.

We define a graph $\mathscr{G}$ by

$$\mathscr{G} = \{\{x, \alpha\} : x \in R_\alpha \wedge \alpha \text{ a limit number } < \omega_1\}.$$

Obviously $\mathscr{G}$ does not contain a Shelah graph. To conclude we show there is no stationary subset X free for $\mathscr{G}$. Assume X is stationary. Then there is a limit $\alpha < \omega_1$ such that $X \cap \alpha \in S_\alpha$ and $\alpha \in X$. It follows that there is $x \in X \cap \alpha \cap R_\alpha$, hence $\{x, \alpha\} \subset X$, $\{x, \alpha\} \in \mathscr{G}$.

§7. SIMPLE PROPERTIES OF $P(\mathscr{S}, \lambda, \kappa)$, $P^*(\mathscr{S}, \lambda, \kappa, r)$. PRELIMINARY LEMMAS

Theorem 7.1.

(1) *If* $\kappa \geqslant \omega$, $\kappa > \lambda$ *then* $P(\mathscr{S}, \lambda, \kappa) \Rightarrow P^*(\mathscr{S}, \lambda, \kappa, 1)$.

(2) *If* $m, n < \omega$, *then* $P(\mathscr{S}, m, mn+1) \Rightarrow P^*(\mathscr{S}, m, n+1, 1)$.

Proof. Let V be the set of vertices of the set-system $\mathscr{S}$. Let us now assume that either $\kappa \geqslant \omega$ and $\kappa > \lambda$ or $\kappa = mn+1$, $\lambda = n+1$ and $P^*(\mathscr{S}, \lambda, \kappa, 1)$ fails. Choose an $f\colon \mathscr{S} \to \lambda$ establishing $P(\mathscr{S}, \lambda, \kappa)$. Then there are $\kappa' < \kappa$ and a partition $V = \bigcup_{\xi < \kappa'} T_\xi$ such that for all $\xi < \kappa'$, $p \in T_\xi$, there is a $\nu(p) < \lambda$ such that $p \in X \subset T_\xi \wedge f(X) = \nu(p)$ holds for no $X \in \mathscr{S}$. Define $T_{\xi,\nu} = \{p \in T_\xi : \nu(p) = \nu\}$. Then we get a contradiction since $V = \bigcup_{\xi < \nu'} \bigcup_{\nu < \lambda} T_{\xi,\nu}$ is a partition of length $\kappa'\lambda < \kappa$ or $\leqslant nm$ such that for all ξ, ν in question no $X \in \mathscr{S}$ with $f(X) = \nu$ is a subset of $T_{\xi,\nu}$. This proves both statements.

Theorem 7.2. *Let* $\mathscr{S}$ *be a set-system with set of vertices* V. *If* $\lambda \geqslant \omega$ *then* $P(\mathscr{S}, \lambda, \lambda^+)$ *holds if and only if*

$$\exists f\colon \mathscr{S} \to \lambda\, \forall g\colon V \to \lambda\, \exists X \in \mathscr{S}\, \forall x \in X (g(x) = f(X)) .$$

Proof. The "only if" is trivial, we prove the "if". Suppose we have an f such that $\forall g\colon V \to \lambda\, \exists X \in \mathscr{S}\, \forall x \in X (g(x) = f(X))$. Let $\lambda = \bigcup_{\nu < \lambda} N_\nu$ where each $|N_\nu| = \lambda$ and the N_ν are pairwise disjoint. Let $\mathscr{S}_\nu = \{X \in \mathscr{S} : f(X) \in N_\nu\}$. Then the sequence $\mathscr{S}_\nu$, $(\nu < \lambda)$ has simultaneous chromatic number $> \lambda$. Assume the contrary, then there is a partition $V = \bigcup_{\xi < \lambda} T_\xi$ and a function $\nu\colon \lambda \to \lambda$ such that for each $\xi < \lambda$, T_ξ is a free set for $\mathscr{S}_{\nu(\xi)}$. Choose a one-to-one function $h\colon \lambda \to \lambda$ with $h(\xi) \in N_{\nu(\xi)}$. Define $g\colon V \to \lambda$ so that $g(x) = h(\xi)$ for $x \in T_\xi$. By hypothesis, there exists $X \in \mathscr{S}$ such that $g(x) = f(X)$ for $x \in X$. It follows that $X \subset T_\xi$ for some ξ, and $f(X) = h(\xi) \in N_{\nu(\xi)}$ so $X \in \mathscr{S}_{\nu(\xi)}$. But this contradicts the assumption that T_ξ is a free set for $\mathscr{S}_{\nu(\xi)}$.

Theorem 7.3. *Let* $\lambda \geqslant \omega$. *Assume* $\mathscr{S}$ *is a set-system which consists of finite sets. Then* $P(\mathscr{S}, \lambda, \lambda^+) \Rightarrow P(\mathscr{S}, \lambda, \lambda^{++}) \vee P(\mathscr{S}, \lambda^+, \lambda^+)$.

Proof. Let $\mathscr{S}$ be a set-system with set of vertices V, $\mathscr{S} \subset [V]^{<\omega}$. Assume $P(\mathscr{S}, \lambda, \lambda^+)$. Let $f\colon \mathscr{S} \to \lambda$ be a mapping which establishes this fact. Assume now that $P(\mathscr{S}, \lambda, \lambda^{++})$ fails. Then, as a corollary of this, there exists a disjoint partition $V = \bigcup_{\xi < \lambda^+} T_\xi$ of the set of vertices such that for all $\xi < \lambda^+$ there is a $\nu(\xi) < \lambda$ satisfying $f(e) \neq \nu(\xi)$ for all $e \in [T_\xi]^{<\omega} \cap \mathscr{S}$.

We now define $\hat{f}\colon \mathscr{S} \to \lambda^+$ and show that $\hat{f}$ establishes $P(\mathscr{S}, \lambda^+, \lambda^+)$. To define $\hat{f}$ first we choose g_ξ $(\xi < \lambda^+)$ mapping λ onto $\xi + 1$, and we fix a well-ordering $\prec$ of V satisfying $T_\xi \prec T_\eta$ for $\xi < \eta < \lambda^+$. Let $e \in \mathscr{S}$, $y = \max_{\prec} e$. Put $\hat{f}(e) = g_\eta(f(e))$ for the η satisfying $y \in T_\eta$. Let now $V = \bigcup_{\mu < \lambda} R_\mu$ be a disjoint partition of V. Assume this partition is bad for $\hat{f}$. We will obtain a contradiction by exhibiting a partition of length λ which is bad for f too.

Let now $\mu < \lambda$. Put $R_\mu = R$. There is a σ such that $\hat{f}(e) \neq \sigma$ for all $e \in [R]^{<\omega}$. Let $T'_\xi = T_\xi \cap R$ for $\xi < \lambda^+$. Then

$$R = \bigcup_{\xi < \lambda^+} T'_\xi .$$

Let $P = \bigcup_{\sigma < \xi < \lambda} T'_\xi$. Then $R = \bigcup_{\xi \leq \sigma} T'_\xi \cup P$ and since all T'_ξ omit the color $\nu(\xi)$ for $\xi \leq \sigma$, we only have to define a bad partition of P.

Let $P_\nu = \{x \in P\colon x \in T'_\xi \wedge g_\xi(\nu) = \sigma\}$. Then $P = \bigcup_{\nu < \lambda} P_\nu$. Let $\nu < \lambda$, $e \subset P_\nu$, $e \in S$. We claim that $f(e) \neq \nu$. Otherwise there is a unique ξ with $\max_{\prec} e = y \in T'_\xi$, $g_\xi(\nu) = \sigma$ and $\hat{f}(e) = \sigma$, a contradiction.

Definition 7.4. Let $\mathscr{S}$ be a set-system, κ a cardinal. With some abuse of notation we denote by $\mathscr{S} \cdot \kappa$ a set-system which consists of κ "vertex-disjoint copies" of $\mathscr{S}$. If we claim a statement for $\mathscr{S} \cdot \kappa$ we mean that the statement holds for all set-systems which can be written in this form.

Lemma 7.5. *Assume* $P(\mathscr{S}, \aleph_\alpha, \delta)$ *and* $\delta \leq \aleph_{\alpha+1}$. *Then* $P(\mathscr{S} \cdot \aleph_{\alpha+n}, \aleph_{\alpha+n}, \delta)$ *holds for* $n < \omega$.

Proof. We prove that $P(\mathscr{S}, \kappa, \delta)$ implies $P(\mathscr{S} \cdot \kappa^+, \kappa^+, \delta)$ for $\kappa \geq \omega$, $\delta \leq \kappa^+$.

The lemma follows from this by induction on n. Assume $P(\mathscr{S}, \kappa, \delta)$ holds. Let V be the set of vertices of $\mathscr{S}$. Put $W_\xi = V \times \{\xi\}$,

$W = V \times (\kappa^+ - \kappa)$,

$$\mathscr{S}_\xi = \{X \subset W_\xi : D(X) \in \mathscr{S}\}, \quad \hat{\mathscr{S}} = \bigcup_{\kappa \leqslant \xi < \kappa^+} \mathscr{S}_\xi .$$

Then $\hat{\mathscr{S}}$ is $\mathscr{S} \cdot \kappa^+$ with set of vertices W. Let $f: \mathscr{S} \to \kappa$ be a coloring of the elements of $\mathscr{S}$ which establishes $P(\mathscr{S}, \kappa, \delta)$.

Let g_ξ be a one-to-one mapping of κ onto ξ for $\kappa \leqslant \xi < \kappa^+$. For $X \in \mathscr{S}_\xi$ put

$$\hat{f}(X) = g_\xi(f(D(X))) .$$

Assume now $W = \bigcup_{\nu < \sigma} T_\nu$ is a disjoint partition of W, for some cardinal $\sigma < \delta \leqslant \kappa^+$ such that for all $\nu < \sigma$ there is $\eta(\nu) < \kappa^+$ satisfying $\hat{f}(X) \neq \eta(\nu)$ for $X \subset T_\nu$, $X \in \hat{\mathscr{S}}$. Then there is a $\xi < \kappa^+$ with $\sup\{\eta(\nu): \nu < \sigma\} < \xi$. We now obtain a contradiction by showing that the partition $\bigcup_{\nu < \sigma} T_\nu$ induces a partition of the ξ-th copy of $\mathscr{S}$ which is bad for f. Namely $W_\xi = \bigcup_{\nu < \sigma} W_\xi \cap T_\nu$, $V = \bigcup_{\nu < \sigma} D(W_\xi \cap T_\nu)$, $f(X) \neq$ $\neq g^{-1}(\{\eta(\nu)\})$ for $X \subset D(W_\xi \cap T_\nu)$, $X \in \mathscr{S}$, $\nu < \sigma$.

We now state two lemmas without proofs.

Lemma 7.6 (A. Máté). *Let κ, λ be cardinals $\kappa \geqslant \lambda \geqslant \omega$, λ regular. Let $A = \bigcup_{\mu < \kappa} A_\mu$, $|A_\mu| > \kappa$ and let f be a set-mapping on A such that $x \in A \wedge \mu < \kappa \Rightarrow |f(x) \cap A_\mu| < \lambda$. Then there is a free set $X \subset A$ such that $|X \cap A_\mu| = \kappa$ for all $\mu < \kappa$.*

The proof of Lemma 7.6 is given in [19] 7.3 Lemma.

Lemma 7.7 (P. Erdős – G. Fodor). *Let κ be an infinite cardinal and let α be an ordinal less than κ. Let f be a set-mapping on κ such that $x \in \kappa \Rightarrow \mathrm{tp}\, f(x) < \alpha$.*

Suppose $A_\mu \in [\kappa]^\kappa$ for $\mu < \lambda$ for some cardinal $\lambda < \kappa$. Then there is a free set $X \subset \kappa$ such that $|X \cap A_\mu| = \kappa$ for all $\mu < \kappa$.

In [4] the theorem is proved assuming G.C.H. In [18] a proof is outlined without assuming G.C.H. In both papers α is assumed to be a cardinal. The idea of looking for free sets if α is an ordinal $< \kappa$ occurs first in [16]. The proof outlined in [18] works for this case without essential changes.

§8. GRAPH CONSTRUCTIONS

The next theorem contains one of the main ideas of several transfinite constructions to be given later.

Theorem 8.1. *Let* λ *be an infinite cardinal and let* δ *be the least cardinal such that* $\lambda^\delta > \lambda$. *Then there is a "triangle-free" graph* $\mathscr{G}$ *on* λ *such that* $P^*(\mathscr{G}, \lambda, \delta)$.

Proof. Note first that δ is a regular cardinal with $\omega \leqslant \delta \leqslant \mathrm{cf}(\lambda)$. First we split λ into the union of δ disjoint sets each of cardinality λ:

$$\lambda = \bigcup_{\xi < \delta} A_\xi .$$

Put $B_\xi = \bigcup_{\eta < \xi} A_\eta$ for $\xi < \delta$. We define the graph and a mapping f: $\mathscr{G} \to \lambda$ inductively by defining for each $x \in A_\xi$, $\xi < \delta$ a subset $G(x) \subset \subset B_\xi$. The intention is that $G(x) = \{y \in B_\xi : y$ is joined to x in $\mathscr{G}\}$, and the mapping $f \mid \{\{y, x\}: y \in G(x)\}$ will be defined for each x.

Assume that $\xi < \delta$ and this has been done for all $\eta < \xi$, $x \in A_\eta$ i.e. $\mathscr{G} \mid B_\xi$ is already defined.

Let $K_\xi = \{X \subset B_\xi : X$ is a partial transversal of the A_η, $(\eta < \xi)$ and X is a free set of $\mathscr{G} \mid B_\xi\}$.

Now considering that $\lambda^{|\xi|} = \lambda = |A_\xi|$ we can arrange matters so that $G(x) \in K_\xi$ should hold for all $x \in A_\xi$ and moreover for all $X \in K_\xi$, $g \in {}^X\lambda$, there are pairwise disjoint sets $A_\xi(X, g) \subset A_\xi$ satisfying $|A_\xi(X, g)| = \lambda$ and $G(x) = X$ for $x \in A_\xi(X, g)$. Finally for $y \in G(x)$, $x \in A_\xi(X, g)$ we put

$$f(\{x, y\}) = g(y).$$

This defines the graph $\mathscr{G}$, and a mapping f: $\mathscr{G} \to \lambda$ of it. We claim that this mapping establishes $P^*(\mathcal{G}, \lambda, \delta)$. It is clear from the construction that $\mathcal{G}$ contains no triangles.

Assume $\lambda = \bigcup_{\nu<\delta'} T_\nu$ for some $\delta' < \delta$ is a disjoint partition of the vertices of $\mathcal{G}$.

Assume further indirectly that for each $y \in T_\nu$, $\nu < \delta'$ there is a $\rho(y) < \lambda$ for which

(1) $\qquad f(\{x, y\}) \neq \rho(y)$ holds for all $x \in T_\nu$ with $\{x, y\} \in \mathcal{G}$.

Let $N = \{\nu < \delta': |\{\xi < \delta: |T_\nu \cap A_\xi| = \lambda\}| = \delta\}$ and put $M = \delta' - N$.

By the definition of N there is a one-to-one mapping $\varphi: N \to \delta$ such that $|T_\nu \cap A_{\varphi(\nu)}| = \lambda$ for all $\nu \in N$. Put $F = \varphi(N)$. Then $|F| \leqslant \leqslant \delta' < \delta \leqslant \mathrm{cf}(\lambda)$, and $|T_{\varphi^{-1}(\{\xi\})} \cap A_\xi| = \lambda$ for $\xi \in F$. Using the regularity of δ we now find a $\zeta < \delta$ such that $F \subset \zeta$ and $|T_\nu \cap A_\xi| < \lambda$ for $\nu \in M$. After this we consider the set mapping $G(x)$ on the set $\bigcup_{\xi \in F} T_{\varphi^{-1}(\{\xi\})} \cap A_\xi$. Then, as a corollary of Lemma 7.6, there is a set X which is free for the set-mapping $G(x)$ and meets $T_\nu \cap A_{\varphi(\nu)}$ in exactly one point for $\nu \in N$. By the construction, $X \in K_\xi$.

Let

(2) $\qquad g(y) = \rho(y)$ for $y \in X$

where the $\rho(y)$ are defined by (1).

Considering $\delta' < \delta \leqslant \mathrm{cf}(\lambda)$, $|A_\xi \cap \bigcup_{\mu \in M} T_\mu| < \lambda$. Hence there are $x \in \in A_\xi(X, g)$ and $\nu \in N$ such that $x \in T_\nu$. By the choice of X there is $y \in X$, $y \in T_\nu$. Then by the definition (2) of g and by the definition of f

$$\{x, y\} \subset T_\nu, \;\; f(\{x, y\}) = g(y) = \rho(y), \;\; \{x, y\} \in \mathcal{G}$$

and this contradicts (1).

Theorem 8.1 implies that if $2^\kappa = \kappa^+$ then there is a "triangle-free" graph $\mathcal{G}$ on κ^+ satisfying $P^*(\mathcal{G}, \kappa^+, \kappa^+)$. However, if we assume this hypothesis we can do better than that.

First we restate some old results.

Definition 8.2. The generalized Specker graph $GS_n(\kappa)$ is defined for $1 \leq n < \omega$ as follows. The vertices are the increasing $n^2 + n + 1$-tuples $x \in {}^{n^2+n+1}\kappa$. Two vertices x, y are joined if either

$$x_n < y_0 < x_{n+1} < y_1 < \ldots < x_{n^2+n} < y_{n^2}$$

or

$$y_n < x_0 < y_{n+1} < x_1 < \ldots < y_{n^2+n} < x_{n^2}.$$

As to these graphs we offer [13] as a reference where several other generalization and history are dealt with.

We state

Lemma 8.3 (P. Erdős – A. Hajnal [5] p. 76, Theorem 7.4).

(A) $GS_n(\kappa)$ *contains no* C_{2i+1} *for* $1 \leq i \leq n$.

(B) *If* $n < \omega \leq \kappa$ *then* $\mathrm{Chr}(GS_n(\kappa)) = \kappa$.

As a corollary of this for all $\kappa \geq \omega$ and $1 \leq n < \omega$ there is a graph on κ containing no short odd circuits C_{2i+1} $(1 \leq i \leq n)$ and having chromatic number κ.

Since in [5] $GS_n(\kappa)$ was defined differently (using $2n^2 + 1$-tuples instead of $n^2 + n + 1$-tuples) and the statement analogous to (A) was not actually proved we outline a proof of (A).

First we prove an elementary lemma. Assume $1 \leq i \leq n < \omega$, $\{1, \ldots, 2i+1\} = I_0 \cup I_1$, $I_0 \cap I_1 = \phi$, $|I_0| > |I_1|$. Then there exist $k_0, \ldots, k_{2i+1} \in n^2 + n + 1$ such that

$$j \in I_0 \Rightarrow k_j \geq k_{j-1} - n\,,$$

$$j \in I_1 \Rightarrow k_j \geq k_{j-1} + n + 1, \quad k_{2i+1} \leq k_0\,.$$

To see this, without loss of generality we can assume that $|I_0| = i + 1$, $|I_1| = i$. For $1 \leq j \leq 2i+1$ let $s_j = -n$ if $j \in I_0$, $s_j = n + 1$ if $j \in I_1$. Let $\mu = \min\{0, s_1, s_1 + s_2, \ldots, s_1 + \ldots + s_{2i+1}\}$. For

$0 \leqslant j \leqslant 2i+1$, let $k_j = -\mu + s_1 + \ldots + s_j$. It is easy to check that all requirements are fulfilled.

Now suppose $GS_n(\kappa)$ contains a C_{2i+2} for some $1 \leqslant i \leqslant n$. Let $a^{(0)}, \ldots, a^{(2i+1)}$ be the set of vertices of this C_{2i+1}. For $1 \leqslant j \leqslant 2i+1$ put $j \in I_0$ if $a_n^{(j-1)} < a_0^{(j)}$ and $j \in I_1$ if $a_n^{(j)} < a_0^{(j-1)}$. Then $|I_0| \neq$ $\neq |I_1|$, and we may assume $|I_0| > |I_1|$. Obtain $k_0, \ldots, k_{2i+1}$ from the above lemma. By the definition of $GS_n(\kappa)$ we then have

$$a_{k_0}^{(0)} < a_{k_1}^{(1)} < \ldots < a_{k_{2i+1}}^{(2i+1)} = a_{k_{2i+1}}^{(0)} \leqslant a_{k_0}^{(0)}$$

a contradiction.

We now state another old result.

Lemma 8.4 (P. Erdős – A. Hajnal – E.C. Milner [10] p. 222, Lemma 14.1).

Assume $\kappa \geqslant \omega$, $2^\kappa = \kappa^+$. Then there is a mapping $f\colon [\kappa^+]^2 \to \kappa^+$ such that for all $A \in [\kappa^+]^\kappa$, $B \in [\kappa^+]^{\kappa^+}$ there is a $\xi \in A$ such that $\kappa^+ = \{f(\{\xi, \eta\})\colon \eta \in B\}$.

This is a generalization of $\kappa^+ \nrightarrow \left[\binom{\kappa^+}{\kappa}\right]^2_{\kappa^+}$ and the proof is to be carried out using auxiliary functions "g" just as in the proof of the previous theorem.

Theorem 8.5. *Let $\kappa \geqslant \omega$, $2^\kappa = \kappa^+$. Then for any $n < \omega$ there is a graph $\mathcal{G}$ on κ^+ such that $P^*(\mathcal{G}, \kappa^+, \kappa^+)$ and $\mathcal{G}$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$.*

Proof. First we split κ^+ into the union of κ^+ disjoint sets $\kappa^+ = \bigcup_{\xi < \kappa^+} A_\xi$ where $|A_\xi| = \kappa^+$. By Lemma 8.3, we can choose a graph $\hat{\mathcal{G}}$ on κ^+ such that $\mathrm{Chr}(\hat{\mathcal{G}}) = \kappa^+$ and $\hat{\mathcal{G}}$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$.

Let $\mathcal{G} = \{[A_\xi, A_\eta]\colon \{\xi, \eta\} \in \hat{\mathcal{G}}\}$. Let further $\hat{f}$ be a mapping satisfying the requirements of Lemma 8.4 and let $f = \hat{f} \mid \mathcal{G}$. $\hat{\mathcal{G}}$ being homomorphic to $\mathcal{G}$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$ either.

Let now $\kappa^+ = \bigcup_{\nu<\kappa^+} T_\nu$ be a disjoint partition of κ^+. We define $\hat{T}_\nu = \{\xi < \kappa^+ : |T_\nu \cap A_\xi| = \kappa^+\}$. By $\kappa^+ \to (\kappa^+)^1_\kappa$, we have $\kappa^+ = \bigcup_{\nu<\kappa} \hat{T}_\nu$. Using that $\mathrm{Chr}(\hat{\mathcal{G}}) = \kappa^+$ it follows that there are $\nu < \kappa$ and $\xi \neq \eta \in \hat{T}_\nu$, $\{\xi, \eta\} \in \hat{\mathcal{G}}$. Then there are $A \subset T_\nu \cap A_\xi$, $B \subset T_\nu \cap A_\eta$, $|A| = \kappa$, $|B| = \kappa^+$. By 8.4, there is $\rho \in A$ with $\kappa^+ = \{f(\{\rho, \sigma\}) : \sigma \in B\}$.

Without assuming G.C.H. we can prove the existence of a short odd circuitless graph $\mathcal{G}$ satisfying $P^*(\mathcal{G}, \kappa^+, \kappa^+)$ only on a set of cardinality $(2^\kappa)^+$.

Theorem 8.6. *Assume $n < \omega \leqslant \kappa$. There is a graph $\mathcal{G}$ on $(2^\kappa)^+$ such that $P^*(\mathcal{G}, (2^\kappa)^+, \kappa^+)$ and $\mathcal{G}$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$.*

Proof. Let $\lambda = (2^\kappa)^+$. We now split λ into the union of λ disjoint sets $\lambda = \bigcup_{\xi<\lambda} A_\xi$ such that $|A_\xi| = \lambda$ for $\xi < \lambda$. Applying 8.3 we choose a $\hat{\mathcal{G}}$ on λ with $\mathrm{Chr}(\hat{\mathcal{G}}) = \lambda$ and not containing C_{2i+1} for $1 \leqslant i \leqslant n$. Let $\mathcal{G} = \{[A_\xi, A_\eta] : \text{for } \{\xi, \eta\} \in \hat{\mathcal{G}}\}$. Just as in the previous proof $\mathcal{G}$ does not contain C_{2i+1} for $1 \leqslant i \leqslant n$. Let now $\xi < \eta < \lambda$, $\{\xi, \eta\} \in \hat{\mathcal{G}}$ be fixed and define $f \mid [A_\xi, A_\eta]$ as follows. The set

$$B_{\xi,\eta} = \{\langle X, g\rangle : X \in [A_\xi]^{\leqslant \kappa} \wedge g : X \to \lambda\}$$

has cardinality λ. Choose a one-to-one mapping $h_{\xi,\eta}$ of $B_{\xi,\eta}$ into A_η. If $y \in A_\eta$, $y = h_{\xi,\eta}(X, g)$ for some $\langle X, g\rangle \in B_{\xi,\eta}$ and $x \in X$ put $f(\{x, y\}) = g(x)$ and put $f(\{x, y\}) = 0$ in the other cases.

We claim that this f establishes $P^*(\mathcal{G}, \lambda, \kappa^+)$ on $\mathcal{G}$. Let $\lambda = \bigcup_{\nu<\kappa} T_\nu$ be a disjoint partition of λ, the set of vertices, into the union of κ sets.

Assume now indirectly that for all $x \in T_\nu$ and $\nu < \kappa$ there is $\rho(x) < \lambda$ such that

(1) $\quad f(\{x, y\}) \neq \rho(x)$ holds for all $y \in T_\nu$.

Define $R(Y) \subset \lambda$ for all $Y \in P(\kappa)$ as follows:

$$R(Y) = \{\xi < \lambda : Y = \{\nu < \kappa : A_\xi \cap T_\nu \neq \phi\}\}.$$

Obviously $\lambda = \bigcup_{Y \in P(\kappa) - \{\phi\}} R(Y)$. Since $\mathrm{Chr}(\hat{\mathscr{G}}) = \lambda > 2^\kappa$, there are $Y \in P(\kappa) - \{\phi\}$ and $\xi < \eta$, $\xi, \eta \in R(Y)$ such that $\{\xi, \eta\} \in \hat{\mathscr{G}}$. Then there is $X \in [A_\xi]^{\leqslant \kappa}$ such that $|X \cap T_\nu| = 1$ for all $\nu \in Y$. Define $g: X \to \lambda$ as follows:

For $x \in X$ let $g(x) = \rho(x)$ where ρ is the function defined in (1). By the construction there is a $y = h_{\xi, \eta}(X, g) \in A_\eta$ and $f(\{x, y\}) = g(x)$ for this y and $x \in X$. Using that $\eta \in R(Y)$, $\nu \in Y$ for the ν with $y \in T_\nu$. Then for this ν there are $x, y \in T_\nu$, $\{x, y\} \in \mathscr{G}$ with $f(\{x, y\}) = \rho(x)$ a contradiction to (1).

§9. GRAPH CONSTRUCTIONS. SPLITTING KNOWN GRAPHS

First we recall a technical lemma about sets of the form ${}^i\kappa$.

Lemma 9.1. *Let* $\kappa \geqslant \omega$, A *a set of ordinals. Let* $\mathrm{Inc}\,({}^A\kappa) = \{x \in {}^A\kappa\colon x_\alpha < x_\beta$ *for* $\alpha < \beta$; $\alpha, \beta \in A\}$. *Each of these sets has a natural lexicographical ordering* $\prec = \prec_{A,\kappa}$. *Assume now that* $1 \leqslant j < \omega$, $\delta < \mathrm{cf}(\kappa)$ *and* $\mathrm{Inc}\,({}^j\kappa) = \bigcup_{\xi < \delta} T_\xi$ *is a partition of* $\mathrm{Inc}\,({}^j\kappa)$. *For* $i \leqslant j$ *define*

$$T_\xi^i = \{x \in \mathrm{Inc}\,({}^i\kappa)\colon \{y \in \mathrm{Inc}\,({}^{j-i}\kappa)\colon$$
$$\langle x_0 \dots x_{i-1} y_0 \dots y_{j-i-1}\rangle \in T_\xi\}$$
$$\textit{has type } \kappa^{j-i} \textit{ in the lexicographical ordering}\}.$$

The following statements are easy to verify.

(A) $\mathrm{Inc}\,({}^i\kappa) = \bigcup_{\xi < \delta} T_\xi^i$ *for* $i \leqslant j$.

(B) *If* $X \in T_\xi^i$, $i < j$, *then* $|\{y < \kappa\colon \langle x_0, \dots, x_{i-1}, y\rangle \in T_\xi^{i+1}\}| = \kappa$.

(C) *Assume* $i_0, i_1 < j$, $x \in T_\xi^{i_0}$, $y \in T_\xi^{i_1}$; $x_\nu \neq x_\mu$ *for* $\nu < i_0$, $\mu < i_1$. *Then for any ordering condition* $\prec$ *of the set* $\{x_\nu\colon \nu < j\} \cup \{y_\mu\colon \mu < j\}$ *which is an extension of the given ordering of* $\{x_\nu\colon \nu < i_0\} \cup \{y_\mu\colon \mu < i_1\}$ *there are* $x', y' \in T_\xi$, $x' | i_0 = x$, $y' | i_1 = y$ *satisfying the ordering condition.*

We leave the verification of these statements to the reader.

Lemma 9.2. *Let* $\kappa \geqslant \omega$, $\lambda \geqslant 1$. *Suppose there are* λ *"edge-disjoint" graphs* $\mathscr{G}_\xi$, $(\xi < \lambda)$ *on* κ *such that for any partition of* κ *into* $< \delta$ *classes there is a class in which each of the graphs has a vertex with relative valency* κ. *Then* $P(GS_n(\kappa), \lambda, \delta)$ *holds for all* $1 \leqslant n < \omega$.

Proof. Note that the assumption implies $\delta \leqslant \mathrm{cf}(\kappa)$. Let $j = n^2 + n + 1$. Then, by definition 8.2, $\mathrm{Inc}\,({}^j\kappa)$ is the set of vertices V of $GS_n(\kappa)$. We define $f\colon GS_n(\kappa) \to \lambda$ by the following stipulations. Let $\{x, y\} \in GS_n(\kappa)$. Put $f(\{x, y\}) = \xi$ if $\{x_0, y_0\} \in \mathscr{G}_\xi$ and let $f(\{x, y\}) = 0$ if $\{x_0, y_0\}) \notin \bigcup_{\xi<\lambda} \mathscr{G}_\xi$. To see that this f establishes $P(GS_n(\kappa), \lambda, \delta)$ let $V = \bigcup_{\nu<\delta'} T_\nu$ be a partition of the vertices with $\delta' < \delta$. By 9.1, $\kappa = \bigcup_{\nu<\delta'} T_\nu^1$ is a partition of κ into the union of $< \delta$ sets. By the assumption, there is $\nu < \delta'$ such that for all $\xi < \lambda$ there is $x_0 \in T_\nu^1$ with relative valency κ in $\mathscr{G}_\xi$. Let $\xi < \lambda$ be fixed and let x_0 satisfy this requirement. By 9.1 (B) we can choose $x' \in T_\nu^{n+1}$ with $x_0' = x_0$. By the assumption, there is $y_0 \in T_\nu^1$ with $\{x_0, y_0\} \in \mathscr{G}_\xi$, $x_n' < y_0$. By 9.1 (C) there are $x, y \in T_\nu$, $\{x, y\} \in GS_n(\kappa)$ such that $X \mid n+1 = x'$, $y \mid 1 = y_0$. Then $f(\{x, y\}) = \xi$.

We now obtain

Corollary 9.3. *If* $\kappa \geqslant \omega$ *is regular and* $\kappa \nrightarrow [\kappa]^2_\lambda$ *then* $P(GS_n(\kappa), \lambda, \kappa)$ for all $1 \leqslant n < \omega$.

Proof. Let $f\colon [\kappa]^2 \to \lambda$ establish $\kappa \nrightarrow [\kappa]^2_\lambda$. Let $\mathscr{G}_\xi = \{e \in [\kappa]^2 : f(e) = \xi\}$. Assume $\kappa = \bigcup_{\nu<\delta'} T_\nu$ for some $\delta' < \kappa$. Then, by the regularity of κ, there is a $\nu < \delta'$ with $|T_\kappa| = \kappa$. For this T_ν each $\mathscr{G}_\xi$ contains an $x \in T_\nu$ with relative valency κ otherwise there is a free set of $\mathscr{G}_\xi$ with cardinality $|T_\nu| = \kappa$. Hence the result follows from the previous lemma.

Note that by Corollary 9.3 we see that the relation $\kappa \nrightarrow [\kappa]^2_\lambda$ plays an important role in constructing graphs satisfying P and P^* properties. We restate here some old results.

Lemma 9.4 (F. Galvin – S. Shelah [17] pp. 170, 171).

(a) $\omega_1 \nrightarrow [\omega_1]^2_4$,

(b) $\mathrm{cf}(2^{\aleph_0}) \nrightarrow [\mathrm{cf}(2^{\aleph_0})]^2_{\aleph_0} \wedge 2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$.

We get the following corollaries.

Corollary 9.4.1. *For all* $1 \leqslant n < \omega$, $P^*(GS_n(\omega_1), 4, \aleph_1))$.

Proof. By 9.3 and 9.4 (a) we get $P(GS_n(\omega_1), 4, \aleph_1)$. Then, by 7.1, we have $P^*(GS_n(\omega_1), 4, \aleph_1)$ as well.

Corollary 9.5. *If* $\mathrm{cf}(2^{\aleph_0}) = \aleph_1$ *then for all* $1 \leqslant n < \omega$

$$P(GS_n(\omega_1), \aleph_1, \aleph_1).$$

Proof. By 9.4 (b) $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0}$, hence, by 9.3, $P(GS_n(\omega_1), \aleph_0, \aleph_1)$. Considering that $P(\mathscr{G}, \aleph_0, \aleph_2)$ is false for every graph $\mathscr{G}$ on ω_1 the result now follows from lemma 7.3.

Note that by theorem 8.5 the stronger assumption $2^{\aleph_0} = \aleph_1$ implies the stronger conclusion that $P^*(\mathscr{G}, \aleph_1, \aleph_1)$ holds for some "short odd circuitless" $\mathscr{G}$ on ω_1. We want to mention that with the proof of Lemma 7.3 one can obtain the conclusion $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0} \Rightarrow \aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_1}$ as well.

Now we state two other corollaries.

Corollary 9.6. $P^*(GS_n(\mathrm{cf}(2^{\aleph_0}))), \aleph_0, \mathrm{cf}(2^{\aleph_0}))$.

Proof. $\mathrm{cf}(2^{\aleph_0})$ is regular. Hence the corollary follows from 9.3, 9.4 (b) and 7.1.

Corollary 9.7. *Let* λ *be an infinite cardinal and let* δ *be the least cardinal such that* $\lambda^\delta > \lambda$. *Then for all* $1 \leqslant n < \omega$, $P(GS_n(\lambda), \lambda, \delta)$.

Proof. It is easy to see that the graph constructed for the proof of Theorem 8.1, and the coloring f of this graph given there satisfies the requirements of Lemma 9.2 with $\mathscr{G}_\xi = f^{-1}(\{\xi\})$ for $\xi < \lambda$. We omit the details.

It is now time to state some open problems.

Problem 5. Is it a theorem of ZFC that there is a graph $\mathcal{G}$ on ω_1 such that $P(\mathcal{G}, \aleph_0, \aleph_1)$, or in fact does $P([\omega_1]^2, \aleph_0, \aleph_1)$ hold?

Note that this is equivalent to $P(\mathcal{G}, \aleph_1, \aleph_1)$ (by 7.3). The relevant partial results are Corollaries 9.4, 9.5 and 9.20 (a) which will be proved later.

Also note that, by 9.2, a positive answer would follow if $\omega_1 \nrightarrow$ $\nrightarrow$ [stationary subset of $\omega_1]^2_{\aleph_0}$ is provable in ZFC. However, we do not know the answer to this problem either. In view of the fact that the results stated in 9.4 give all we know about the $\rightarrow [\quad]$ symbol in ZFC this latter question is probably even more interesting than Problem 5 itself.

Problem 6. Is the following statement provable in ZFC?

(A) For any $1 \leqslant n < \omega$ there is a graph $\mathcal{G}$ on $2^{\aleph_0}$ such that $P^*(\mathcal{G}, 2^{\aleph_0}, \aleph_1)$ and $\mathcal{G}$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$.

By Theorem 8.5, C.H. $\Rightarrow$ (A). By Theorem 8.1, (A) holds for $n = 1$. By Theorem 8.6, $P^*(\mathcal{G}, (2^{\aleph_0})^+, \aleph_1)$ holds for some $\mathcal{G}$ on $(2^{\aleph_0})^+$. By Corollary 9.7 we have $P(GS_n(2^{\aleph_0}), 2^{\aleph_0}, \aleph_1)$ hence by 7.3 we also have $P^*(GS_n(2^{\aleph_0}), \aleph_0, \aleph_1)$. By Corollary 9.8 we have $P^*(GS_n(\mathrm{cf}\,(2^{\aleph_0})), \aleph_0, \mathrm{cf}\,(2^{\aleph_0}))$.

We now formulate some results and problems of finite character.

Corollary 9.8. *$P(GS_n(\omega), \aleph_0, \aleph_0)$ holds for all $1 \leqslant n < \omega$.*

Proof. By Corollary 9.7.

Corollary 9.9. *For all $n, h, k < \omega$ there is an $m < \omega$ such that $P(GS_n(m), h, k)$ holds.*

Proof. By Corollary 9.8 and by compactness.

Theorem 9.10. *If $m \rightarrow (p)_k^{n^2+n+1}$ and $m \nrightarrow [p]_h^{2n^2+2n+2}$ for some p then $P(GS_n(m), h, k+1)$.*

Proof. By the assumption there is a $g\colon [m]^{2n^2+2n+2} \to h$ such that all $X \subset m$, $|X| = p$ are completely inhomogenous for this g i.e. $g([x]^{2n^2+2n+2}) = h$. Let now $V = \mathrm{Inc}\,({}^{n^2+n+1}m)$ be the set of vertices of $GS_n(m)$. V can be canonically identified to $[m]^{n^2+n+1}$. Let now $x, y \in V$, $\{x, y\} \in GS_n(m)$. Put $f(\{x, y\}) = g$ (Range $(x) \cup$ Range (y)). We claim that f establishes $P(GS_n(m), h, k+1)$. Otherwise there is $\bigcup_{j<k} T_j = V$ contradicting this. By $m \to (p)_k^{n^2+n+1}$, $[X]^{n^2+n+1} \subset T_j$ for some $j < k$, $|X| = p$ and this X is not completely inhomogeneous.

Corollary 9.11. *If* $m \to (2n^2 + 2n + 2)_k^{n^2+n+1}$ *then* $\mathrm{Chr}\,(GS_n(m)) > k$.

This is the general method which is used in [5] to construct short odd circuitless large chromatic graphs.

Finally to conclude this chapter we prove some results about splitting graphs of different type.

Definition 9.12. If φ is an order type $\mathrm{tp}\,(R(\prec) = \varphi$ we consider the edge-graph $E(\varphi)$ the set of vertices of which is $[R]^2$. Two pairs $\{x, y\}$, $\{z, u\}$ are joined in $E(\varphi)$ iff either $x \prec y = z \prec u$ or $z \prec u = = x \prec y$.

Lemma 9.13 (P. Erdős – A. Hajnal [7]). *If* $|\varphi| \geqslant \aleph_0$, *then* $\mathrm{Chr}\,(E(\varphi))$ *equals to the least cardinal* κ *such that* $2^\kappa \geqslant |\varphi|$.

Lemma 9.14. *Suppose* $|\varphi| = |\psi| = \aleph_\alpha$, $\delta \geqslant \omega$. *Then*

$$P(E(\varphi), \lambda, \delta) \Leftrightarrow P(E(\psi), \lambda, \delta)\,,$$

$$P^*(E(\varphi), \lambda, \delta) \Leftrightarrow P^*(E(\psi), \lambda, \delta)\,.$$

Proof. Assume $\mathrm{tp}\,\omega_\alpha(\prec_1) = \varphi$, $\mathrm{tp}\,\omega_\alpha(\prec_2) = \psi$ for some orderings $\prec_1$ and $\prec_2$. It is sufficient to prove the implications from left to right. Let $f\colon [\omega_\alpha]^3 \to \lambda$ establish $P(E(\varphi), \lambda, \delta)$. We claim that f establishes $P(E(\psi), \lambda, \delta)$ as well. Let $[\omega_\alpha]^2 = \bigcup_{\nu<\kappa} T_\nu$ for some $\kappa < \delta$ be given. Put $T_{\nu,0} = \{\{x, y\}\colon x \prec_1 y \wedge x \prec_2 y\} \cap T_\nu$, $T_{\nu,1} = \{\{x, y\}\colon x \prec_1 y \wedge \wedge\, y \prec_2 x\} \cap T_\nu$. Then $2\kappa < \delta$, hence by the assumption, there are $\nu < \kappa$

and $\epsilon < 2$ such that for all $\xi < \lambda$ there are $\alpha_0 \prec_1 \alpha_1 \prec_1 \alpha_2$ with $f(\{\alpha_0, \alpha_1, \alpha_2\}) = \xi$, $\{\alpha_0, \alpha_1\}$, $\{\alpha_1, \alpha_2\} \in T_{\nu, \epsilon}$, $\{\{\alpha_0, \alpha_1\}, \{\alpha_1, \alpha_2\}\} \in$ $\in E(\varphi)$. Then either $\alpha_0 \prec_2 \alpha_1 \prec_2 \alpha_2$ or $\alpha_2 \prec_2 \alpha_1 \prec_2 \alpha_0$ hence $\{\{\alpha_0, \alpha_1\}, \{\alpha_1, \alpha_2\}\} \in E(\psi)$ in both cases and $\{\alpha_0, \alpha_1\}, \{\alpha_1, \alpha_2\} \in T_\nu$.

The same argument works for the proof of the second equivalence.

Lemma 9.15. *Let* φ, ψ *be order types. If* $\varphi \nrightarrow [\psi]^3_\lambda$ *and* $\varphi \to (\psi)^2_\kappa$ *for all* $\kappa < \delta$ *then* $P(E(\varphi), \lambda, \delta)$.

Proof. Let $\operatorname{tp} R(\prec) = \varphi$. Let $g \colon [R]^3 \to \lambda$ establish $\varphi \nrightarrow [\psi]^3_\lambda$. Put $f(\{x, y\}) = g(x \cup y)$ for $\{x, y\} \in E(\varphi)$. Let $\bigcup_{\nu < \kappa} T_\nu = [R]^2$ be a partition of length $\kappa < \delta$ of the set of vertices. By $\varphi \to (\psi)^2_\kappa$ there are $\nu < \kappa$ and $X \subset R$, $\operatorname{tp} X(\prec) = \psi$ and $[X]^2 \subset T_\nu$. Then, by the definition of g, $g([X]^3) = \lambda$ and by the definition of f, $f([[X]^2]^2 \cap E(\varphi)) = \lambda$.

To draw the consequences of this lemma we need

Lemma 9.16. *Let* $\lambda \geq \omega, \gamma, \delta$ *be cardinals. Assume* $\lambda \nrightarrow [\gamma]^2_\delta$. *Then* $\lambda^+ \nrightarrow [\gamma + 1]^2_\delta$.

Proof. For each $\xi < \lambda^+$ let $f_\xi \colon [\xi]^2 \to \delta$ establish $\lambda \nrightarrow [\gamma]^2_\delta$. For $\{\zeta, \eta, \xi\} \in [\lambda^+]^3$, $\zeta < \eta < \xi$, put $f(\{\zeta, \eta, \xi\}) = f_\xi(\{\zeta, \eta, \xi\})$. f obviously establishes $\lambda^+ \nrightarrow [\gamma + 1]^3_\gamma$.

Theorem 9.17.

(a) $\kappa \geq \aleph_0 \Rightarrow P(E((2^\kappa)^+), 2, \kappa^+)$,

(b) $2^{\aleph_0} < 2^{\aleph_1} \Rightarrow P(E((2^{\aleph_0})^+, 4, \aleph_1)$,

(c) $2^\kappa \nrightarrow [\kappa^+]^2_\delta \Rightarrow P(E((2^\kappa)^+, \delta, \kappa^+)$.

Proof. For all proofs note that by the Erdős – Rado theorem $(2^\kappa)^+ \to (\kappa^+ + 1)^2_\kappa$ holds. We always apply Lemma 9.15, we only have to indicate the ψ and the partition relations we use.

(a) $\psi = \kappa^+$, $(2^\kappa)^+ \nrightarrow [\kappa^+]^3_2$ follows from $2^{2^\kappa} \nrightarrow (\kappa^+)^3_2$, see [11] p. 125, Lemma 5/A.

(b) $\psi = \omega_1$. By [9] (p. 13. Problem 17). $2^{\aleph_1} \nrightarrow [\omega_1]^3_4$.

(c) $\psi = \kappa^+ + 1$. By 9.16.

Let $(n)^r_k$ and $[n]^r_k$ denote the minimal m for which $m \to (n)^r_k$ and $m \to [n]^r_k$ hold respectively.

Theorem 9.18. *For each* $1 \leqslant k < \omega$ *there is an* m *such that* $P(E(m), k, k+1)$. *As a corollary of this* $P(E(\omega), \aleph_0, \aleph_0)$ *holds.*

Proof. By 9.15 it is sufficient to see that there is an integer n for which $(n)^2_k < [n]^3_k$.

By known methods (see e.g. [3]).

$(n)^2_k < k^{kn}$ while, for all $\epsilon > 0$ and $n > N(k, \epsilon)$,

$[n]^3_k > \left(\frac{k}{k-1}\right)^{\left(\frac{1}{6} - \epsilon\right)n^2}$ This proves the theorem.

Now we prove a lemma which leads to a strengthening of 9.17 (c) and 9.18.

Lemma 9.19. *Assume* $\tau, \lambda \geqslant \omega$ *and* $\delta^\tau \leqslant \lambda$. *Then* $\forall \kappa < \tau (P(E(\lambda), \kappa, \tau))$ *implies* $P(E(\lambda), \delta, \tau)$.

Proof. By the assumption, $P(E(\lambda), |X|, \tau)$ holds for all $X \in [\delta]^{<\tau}$. By $\delta^\tau \leqslant \lambda$, we can choose a disjoint partition

$$E(\lambda) = \bigcup_{X \in [\delta]^\tau} G_X$$

such that $P(\mathcal{G}_X, |X|, \tau)$. We can choose a mapping $f_X : \mathcal{G}_X \to X$ which establishes this fact. Put $f = \bigcup_{X \in [\delta]^{<\tau}} f_X$. We claim that f establishes $P(E(\lambda), \delta, \tau)$. In fact if $[\lambda]^2 = \bigcup_{\nu < \kappa} T_\nu$ for some $\kappa < \tau$, then for each $X \in [\delta]^{<\tau}$, there is $\nu(X) < \kappa$ such that for all $\xi \in X$ there are $u, v \in T_{\nu(X)}$ with $f(\{u, v\}) = \xi$, $\{u, v\} \in \mathcal{G}_X \subset E(\lambda)$.

Now the statement follows because there is a $\nu < \kappa$ such that $\bigcup \{X \in [\delta]^{<\tau} : \nu(X) = \nu\} = \delta$. This holds because otherwise there is $\eta_\nu \notin \cup \{X \in [\delta]^{<\tau} : \nu(X) = \nu\}$, $\eta_\nu < \delta$ for all $\nu < \kappa$, and

$\hat{X} = \{\eta_\nu : \nu < \kappa\} \in [\delta]^{<\tau}$, a contradiction.

Theorem 9.20.

(a) $P(E(\omega_\alpha), \aleph_\alpha, \aleph_0)$ *for all* α,

(b) $2^\kappa \nrightarrow [\kappa^+]^2_\kappa \Rightarrow P(E((2^\kappa)^+), (2^\kappa)^+, \kappa^+)$,

(c) $2^\kappa = \kappa^+ \Rightarrow P(E(\kappa^{++}), \kappa^{++}, \kappa^+)$.

Proof. Considering $\aleph_\alpha^{\aleph_0} = \aleph_\alpha$ (a) follows from Theorem 9.18 and Lemma 9.19. By Theorem 9.17 (c) $2^\kappa \nrightarrow [\kappa^+]^2_\kappa$ implies $P(E((2^\kappa)^+), \kappa, \kappa^+)$. Applying Lemma 9.19 with $\lambda = (2^\kappa)^+$, $\delta = (2^\kappa)^+$, $\tau = \kappa^+$ we get that $P(E((2^\kappa)^+), (2^\kappa)^+, \kappa^+)$ holds. This proves (b), (c) is a corollary of this since $2^\kappa = \kappa^+$ implies $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$.

Let us remark that for finite graphs 9.8 gives a result which is stronger than 9.18 since $GS_n(\omega)$ does not contain short odd circuits. However, here it is possible to exclude all short circuits.

Theorem 9.21. *For any* $n < \omega$ *there is a graph* $\mathcal{G}$ *on* ω *such that* $P(\mathcal{G}, \aleph_0, \aleph_0)$ *and* $\mathcal{G}$ *contains no* C_i *for* $3 \leq i \leq n + 3$.

The proof can be carried out by using the "probabilistic method". Since this is not in the line of this paper we omit it.

Finally we state a problem left open by the above results.

Problem 7. Given $n < \omega$, is there is short circuitless graph $\mathcal{G}$ on ω such that $P^*(\mathcal{G}, \aleph_0, \aleph_0)$ holds?

Note that the previous theorem gives an affirmative answer P^* is replaced by P, and 7.1 yields then $P^*(\mathcal{G}, k, \aleph_0)$ for all $k < \omega$.

§ 10. CONSTRUCTIONS OF RELATIVELY SMALL n-TUPLE SYSTEMS NOT CONTAINING LARGE FREE SETS

Definition 10.1. Let $\langle R, \prec \rangle$ be an ordered set, $1 \leq n < \omega$, $X, Y \in [R]^n$. X, Y are said to intersect canonically iff $X = \{x_i : i < n\}$, $Y = \{y_i : i < n\}$ where both sequences are increasing in the ordering $\prec$ of

R, and $X \cap Y = \{x_i \colon i \in N\} = \{y_i \colon i \in N\}$ for some $N \subset n$.

Theorem 10.2. *There is a triple system* $\mathcal{S}$ *on* ω_1 *such that*

(1) *there is no independent* $\aleph_1$*-set for* $\mathcal{S}$,

(2) *the induced graph contains no complete* $\aleph_1$ *(in fact for any* $X \in [\omega_1]^{\aleph_1}$ *there exist* $A \in [X]^{\aleph_0}$, $B \in [X]^{\aleph_1}$ *such that no triple meets both* A *and* B*).*

(3) *if two triples of* $\mathcal{S}$ *have an edge in common then they intersect canonically, hence*

(4) *each triple of* $\mathcal{S}$ *has two edges which are contained in at most* $\aleph_0$ *triples of* $\mathcal{S}$ *and*

(5) *any four points contain at most two triples.*

Proof. Let $\prec_1$ be a Specker ordering of ω_1 and let $\prec_2$ be an ordering of ω_1 which is embeddable is the ordering of reals. Let $\mathcal{S}$ be the set of all triples $\{x, y, z\} \in [\omega_1]^3$ such that $x < y < z$, $y \prec_1 z \prec_1 x$ and $z \prec_2 y \prec_2 x$. The first two statements follow from Lemma 7 of [17] while (3), (4) and (5) are easy consequences of the definition.

Definition 10.3. Let $\mathcal{T}_3$ be the triple-system with four vertices and three triangles. Let $\mathcal{T}_4$ be the triple-system having five vertices and three triples having one point in common in such a way that two of the triples meet in at most one point. See-Diagrams 3, 4.

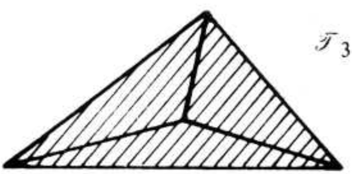

$\mathcal{T}_3$

Diagram 3

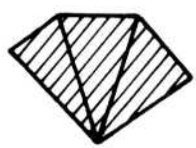

$\mathcal{T}_4$

Diagram 4

Corollary 10.4.

$$\aleph_1 \nrightarrow (\aleph_1, \mathcal{T}_3)^3 .$$

Theorem 10. 2 collects facts we can establish in ZFC for a triple-system on ω_1 not containing a complete $\aleph_1$.

Assuming G.C.H. we can prove the following stronger

Theorem 10.5. *If* $2^\kappa = \kappa^+$ *then there is a triple system* $\mathcal{S}$, $\mathcal{S} = \bigcup_{\mu<\kappa^+} \mathcal{S}_\mu$ *on* κ^+ *such that*

(1) $\mathcal{S}_\mu \cap \mathcal{S}_\nu = \phi$ *for* $\mu < \nu < \kappa^+$,

(2) *there is no independent* κ^+ *for any* $\mathcal{S}_\mu$,

(3) *the graph induced by* $\mathcal{S}$ *contains no complete* κ^+,

(4) *if two triples in* $\mathcal{S}$ *have a common edge it is the first edge of both, hence they intersect canonically and*

(5) *any* n*-set contains at most* $\left[\frac{(n-1)^2}{4}\right]$ *triples of* $\mathcal{S}$.

Let us remark that (5) is best possible by Corollary 3.6. We can not prove (5) even for $\kappa = \aleph_2$ without assuming C.H.

Proof. By $2^\kappa = \kappa^+$, we can choose an $h\colon [\kappa^+]^2 \to \kappa^+$ which satisfies the following requirement slightly stronger than establishing $\kappa^+ \nrightarrow [\kappa^+]^2_{\kappa^+}$.

(i) For each set $B \subset [\kappa^+]^2$, $|B| = \kappa$ of vertex disjoint pairs and for all $\sigma < \kappa^+$ there is a $\xi < \kappa^+$ such that for all $\xi < \eta < \kappa^+$ there are κ different $Z \in B$ with $h(\{u, \eta\}) = \sigma$ for $u \in Z$.

The routine proof of this we leave to the reader. We now define $R_\xi \subset [\xi + 1]^3$ for $\xi < \kappa^+$ with the intention to put $\mathcal{S} = \bigcup_{\xi<\kappa^+} R_\xi$. R_ξ will consist of triples of the form $\{x_0, x_1, \xi\} = X$ such that $\{x_0, x_1\} \cap \{y_0, y_1\} = \phi$ for $X \neq Y \in R_\xi$. Moreover we can choose R_ξ to satisfy the following requirements (ii), (iii):

(ii) $h(\{x_0, x_1\}) = 1$, $2 \leqslant h(\{x_0, \xi\}) = h(\{x_1, \xi\})$ for $X \in R_\xi$.

Let $[\kappa^+]^\kappa = \{A_\xi : \xi < \kappa^+\}$ and $\mathscr{F}_\xi = \{A_\eta : \eta < \xi \wedge A_\eta \subset \xi\}$.

(iii) Let $A_\eta \in \mathscr{F}_\xi$, $2 \leqslant \sigma < \xi$. There are κ triples in R_ξ such that $h(\{x_0, x_1\}) = 1$, $h(\{x_0, \xi\}) = h(\{x_1, \xi\}) = \sigma$ provided it is possible to choose κ triples satisfying the above requirements at all. We now put $X = \{x_0, x_1, \xi\} \in \mathscr{S}_\mu$ iff $X \in R_\xi \wedge h(\{x_0, \xi\}) = \mu + 2$. Obviously then $\mathscr{S} = \bigcup_{\mu < \kappa} \mathscr{S}_\mu$ and (1) holds. (3) holds since $h(e) \neq 0$ for all edges e in the induced graph. To see that (2) holds let $A \in [\kappa^+]^{\kappa+}$, $\mu < \kappa^+$.

By (i), there is a set $B \subset [A]^2$, $|B| = \kappa$ of vertex disjoint pairs with $h(e) = 1$ for $e \in B$. By (i) and (iii) there is a $\xi_0 < \kappa^+$ such that for all $\xi_0 < \xi < \kappa^+$ there is an $\{x_0, x_1\} \in B$ with $\{x_0, x_1, \xi\} \in R_\xi$, $h(\{x_0, \xi\}) = h(\{x_1, \xi\}) = \sigma = \mu + 2$. Then there is a $\xi \in A$ satisfying this requirement, hence $\{x_0, x_1, \xi\} \in [A]^3 \cap \mathscr{S}_\mu$. (4) is obvious from the choice of R_ξ, and (5) is a corollary of (4).

The following are immediate

Corollary 10.6.

$$2^\kappa = \kappa^+ \Rightarrow \kappa^+ \nrightarrow [(\kappa^+)_{\kappa+}, \mathscr{T}_3 \vee \mathscr{T}_4]^3,$$

$$\text{C.H.} \Rightarrow \aleph_1 \nrightarrow (\aleph_1, \mathscr{T}_3 \vee \mathscr{T}_4]^3.$$

This should be compared with Corollary 10.4 obtained in ZFC.

Corollary 10.7. *If* $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, *then*

$$h_3(n, \alpha) = \left[\frac{(n-1)^2}{4}\right].$$

Proof. By Corollary 3.6 and Theorem 10.5 (5).

The first two clauses of the next theorem show that there are arbitrarily large chromatic finite n-tuple systems $\mathscr{T}$ with $\omega_1 \rightarrow (\omega_1, \mathscr{T})^n$.

Theorem 10.8. *For any positive integer* $n \geqslant 2$ *there is an* n*-tuple system* $\mathscr{S}$ *such that*

(i) $\kappa \to (\kappa, \mathscr{S})^n$ *for every regular cardinal* κ.

(ii) $\mathrm{Chr}(\mathscr{S}) = \aleph_0$.

(iii) *for any* $\mathscr{S}' \subset \mathscr{S}$ *with* $\mathrm{Chr}(\mathscr{S}') > 2$ *there exist* $X, Y \in \mathscr{S}'$ *with* $|X \cap Y| = n - 1$.

Proof. Note first that is sufficient to prove that for each $k < \omega$ there are n-tuple systems $\mathscr{S}_k$ satisfying (i), (iii) and $\mathrm{Chr}(\mathscr{S}_k) > k$. In fact if $\kappa \to (\kappa, \mathscr{S}_k)^n$ for $k < \omega$, then $\kappa \to (\kappa, \mathscr{S})^n$ where $\mathscr{S}$ is the union of vertex disjoint copies of the $\mathscr{S}_k$, $(k < \omega)$. We now use induction on n. The statement is true for $n = 2$ and for all $k < \omega$ because of $\kappa \to (\kappa, k)^2$ and since all graphs with valency one have chromatic number at most 1. Let now $n \geqslant 3$, and assume the statement is true for $n - 1$ for all k. We now use induction on k to prove the statement for n and for all $k < \omega$. We may assume $k \geqslant 2$ and that the statement is true for k.

Let $\mathscr{S}_0$ be an $(n-1)$-tuple system with $\mathrm{Chr}(\mathscr{S}_0) \geqslant k + 1$ such that (i) holds for $\mathscr{S}_0$. Let further $\mathscr{S}(Y)$ be an n-tuple system satisfying (i), (iii) and $\mathrm{Chr}(\mathscr{S}(Y)) \geqslant k$ for all $Y \in \mathscr{S}_0$. We may assume $\mathscr{S}_0$, $\mathscr{S}(Y)$ $(Y \in \mathscr{S}_0)$ to be vertex disjoint. For each $Y \in \mathscr{S}_0$ let $\hat{\mathscr{S}}(Y) = \{Y \cup \{x\}: x \in \bigcup \mathscr{S}(Y)\}$. We claim that $\mathscr{S} = \bigcup \{\mathscr{S}(Y) \cup \hat{\mathscr{S}}(Y): Y \in \mathscr{S}_0\}$ satisfies (i), (iii) and $\mathrm{Chr}(\mathscr{S}) \geqslant k + 1$.

To see that (i) holds, let $\mathscr{Z}$ be an n-tuple system on κ such that there is no κ-set free for $\mathscr{Z}$. Let $\hat{\mathscr{Z}} = \{Y \in [\kappa]^{n-1}: |\{x \in \kappa: Y \cup \{x\} \in \mathscr{Z}\}| = \kappa\}$. If there is a κ-set $A \subset \kappa$ free for $\hat{\mathscr{Z}}$, then every maximal subset of A free for $\mathscr{Z}$ has cardinality κ. Hence we may assume that there is no κ-set free for $\hat{\mathscr{Z}}$ either. By the assumption on $\mathscr{S}_0$, $\hat{\mathscr{Z}}$ contains $\mathscr{S}_0$. We can now choose disjoint κ-sets $A(Y) \subset \kappa$ for $Y \in \mathscr{S}_0$ such that $Y \cup \{x\} \in \mathscr{Z}$ for all $x \in A(Y)$. By the choice of $\mathscr{S}(Y)$, each $\mathscr{Z} \mid A(Y)$ contains $\mathscr{S}(Y)$ hence $\mathscr{Z}$ contains $\mathscr{S}$.

We now claim $\mathrm{Chr}(\mathscr{S}) > k$. Let $V = \bigcup \mathscr{S}$ and assume $f: V \to k$. By the choice of $\mathscr{S}_0$, there are $Y \in \mathscr{S}_0$ and $\nu < \kappa$ such that $f(Y) = \{\nu\}$. Then, by the choice of $\mathscr{S}(Y)$ and $\hat{\mathscr{S}}(Y)$ either $f(z) = \{\nu\}$ for some $z \in \mathscr{S}(Y)$ or $f\colon \bigcup \mathscr{S}(Y) \to k - \{\nu\}$ and $f(Z) = \{\mu\}$ for some $Z \in \mathscr{S}(Y)$ and $\mu < \kappa$.

Finally to see that (iii) holds assume $|X \cap Y| < n - 1$ for all $X \neq$ $\neq Y \in \mathscr{S}'$ for some $\mathscr{S}' \subset \mathscr{S}$. Let $V(Y)$, V_0 denote the set of vertices of $\mathscr{S}(Y)$, $\mathscr{S}_0$ respectively. Then, by the assumption, there are $f_Y\colon V(Y) \to 2$ establishing the fact that $\mathscr{S}' \wedge \mathscr{S}(Y)$ is at most two-chromatic. By the construction, and by the choice of $\mathscr{S}'$, $|\mathscr{S}' \wedge \hat{\mathscr{S}}(Y)| \leqslant 1$ for all $Y \in \mathscr{S}_0$. Hence there is at most one vertex $x \in V(Y)$ contained in an element of $\mathscr{S}' \cap \hat{\mathscr{S}}(Y)$ for $Y \in \mathscr{S}_0$. We may assume that $f_Y(x) = 1$ for this x and we can define f by $f(x) = f_Y(x)$ for $x \in V(Y)$, $Y \in \mathscr{S}_0$ and $f(x) = 0$ for $x \in V_0$. This f then establishes $\mathrm{Chr}\,(\mathscr{S}') \leqslant 2$.

We now state two more results which can be proved using similar constructions. The first of these theorems shows that for every infinite cardinal κ there are e.g. triple systems $\mathscr{S}$ with chromatic number $> \kappa$ such that all subsystems not containing two triples with a common edge have chromatic number at most two.

An old problem of Erdős and Hajnal [5] asks if every graph of chromatic number $> \kappa$ contains a subgraph of chromatic number $> \kappa$ not containing triangles or more generally C_{2i+1} for some fixed n and $1 \leqslant i \leqslant n$. This can be reformulated as follows: It is true that if $\mathscr{G}_0$ is a fixed graph such that there are graphs with arbitrary large chromatic number not containing $\mathscr{G}_0$ then every graph $\mathscr{G}$ with chromatic number $> \kappa \geqslant \aleph_0$ contains a subgraph $\mathscr{G}'$ with chromatic number $> \kappa$ and not containing $\mathscr{G}_0$. Theorems 10.9 and 10.10 show that a generalization of this fails to be true for n-tuple systems with $n \geqslant 3$ and in a sense for λ-tuple systems.

Theorem 10.9. *Let $2 \leqslant n < \aleph_0 \leqslant \kappa$. Then there is an n-tuple system $\mathscr{S}$ such that:*

(i) $|\mathscr{S}| = \kappa^{\underset{\smile}{\kappa}} = \sum_{\lambda < \kappa} \kappa^{\lambda}$.

(ii) $\mathrm{Chr}\,(\mathscr{S}) = \kappa$.

(iii) *For any* $\mathcal{S}' \subset \mathcal{S}$ *with* $\mathrm{Chr}(\mathcal{S}') > 2$ *there exist* $X, Y \in \mathcal{S}'$ *with* $|X \cap Y| = n - 1$.

Theorem 10.10. *For any infinite cardinals* κ *and* λ *there is a* λ-*tuple system* $\mathcal{S}$ *such that:*

(i) $X, Y \in \mathcal{S} \wedge X \neq Y \Rightarrow |X - Y| = |Y - X| = \lambda$.

(ii) $\mathrm{Chr}(\mathcal{S}) = \kappa$.

(iii) *For any* $\mathcal{S}' \subset \mathcal{S}$ *with* $\mathrm{Chr}(\mathcal{S}') > 2$ *there exist* $X, Y \in \mathcal{S}'$ *with* $X \neq Y$ *and* $|X \cap Y| = \lambda$.

We omit the proofs.

To carry out the contruction for the proof of the last theorem the following lemma is useful.

Lemma 10.1 (P. Erdős – E.C. Milner [22]). *For any infinite cardinal* $\kappa, [\kappa]^{\leq \aleph_\alpha}$ *is the union of* $2^{\aleph_\alpha}$ *antichains.*

To conclude this chapter we state

Problem 8. Characterize the finite triple systems $\mathcal{T}$ such that $\aleph_1 \to (\aleph_1, \mathcal{T})^3$ holds.

3.3 and 3.6 give classes of finite triple systems for which $\aleph_1 \to$ $\to (\aleph_1, \mathcal{T})^3$ holds.

4.2 shows that $\aleph_1 \to (\aleph_1, \mathcal{T}_2)^3$ holds where $\mathcal{T}_2$ does not belong to the above classes.

Theorems 10.2, 10.5 and Corollaries 10.4, 10.6, 10.7 give negative results.

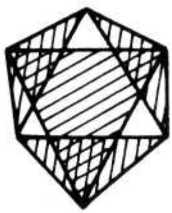
Theorem 10.8 shows that there are finite triple systems $\mathcal{T}$ with $\aleph_1 \to$ $\to (\aleph_1, \mathcal{T})^3$ having preassigned chromatic number $k < \omega$ for all $k < \omega$.

We now call attention to a few instances of the above problem and some related problems.

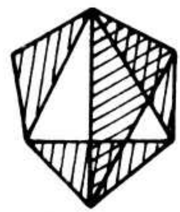
Problem 8/A. Does $\aleph_1 \to (\aleph_1, \mathcal{S}_1 \vee \mathcal{S}_2)^3 \Rightarrow \aleph_1 \to (\aleph_1, \mathcal{S}_1)^3 \vee$ $\vee\ \aleph_1 \to (\aleph_1, \mathcal{S}_2)^3$?

Problem 8/B. For which pairs κ, λ of infinite cardinals is it true that for every finite triple system $\mathcal{S}$, $\kappa \to (\kappa, \mathcal{S})^3 \Rightarrow \lambda \to (\lambda, \mathcal{S})^3$?

Definition 10.12. Let $\mathcal{T}_5$, $\mathcal{T}_6$ be the following triple systems



$\mathcal{T}_5$



$\mathcal{T}_6$

Diagram 5

Problems

8/C. Does $\aleph_1 \to (\aleph_1, \mathcal{T}_5)^3$?

8/D. Does $\aleph_1 \to (\aleph_1, \mathcal{T}_6)^3$?

8/E. Does $\mathrm{ZFC} \vdash \aleph_1 \nrightarrow (\aleph_1, \mathcal{T}_4)^3$?

8/E should be compared with 10.4 and 10.6.

As to 8/C, $\mathcal{T}_5$ is the simplest (3, 1)-system for which we cannot prove an arrow relation. On the other hand we remark that it is not a theorem of ZFC that $\aleph_1 \to (\aleph_1, \mathcal{T})^3$ olds for all finite (3, 1)-systems $\mathcal{T}$ as shown by the following results:

A Steiner triple system $\mathcal{S}$ is a (3, 1)-system such that all pairs of vertices of $\mathcal{S}$ are contained in an element of $\mathcal{S}$.

Theorem 10.13.

(A) *If there is a Suslin* $\mathrm{cf}(\kappa)$*-tree then there is a triple system* $\mathcal{S}$ *on* κ *such that*

(1) $\mathscr{S}$ *has no free set of power* κ,

(2) $\mathscr{S}$ *contains no Steiner triple system consisting of more than one triple.*

(B) *If* κ *is a singular cardinal then there is a triple system* $\mathscr{S}$ *on* κ *such that*

(1) $\mathscr{S}$ *has no free set of power* κ,

(2) *If* $\mathscr{S}'$ *is a finite Steiner triple system contained in* $\mathscr{S}$ *then* $\mathscr{S}'$ *has* $2^m - 1$ *vertices for some* $m < \omega$.

We omit the proofs.

§11. CONSTRUCTIONS OF RELATIVELY SMALL LARGE CHROMATIC TRIPLE SYSTEMS

Quite a few of the constructions to be given will have a common structure. We think it worthwile to introduce the following special conventions. (All constructions described in this chapter give λ chromatic triple systems which assuming G.C.H. have cardinality λ.)

Definition 11.1. Consider a fixed ordinal λ. For $\alpha, \beta < \lambda$ let $F_\alpha = \{\langle\alpha, \beta\rangle : \beta < \lambda\}$ and $R_\beta = \{\langle\alpha, \beta\rangle : \alpha < \lambda\}$. The R_β's are ranks, the F_α's are files. A Δ-*system* (∇-system) on $\lambda \times \lambda$ is a triple system whose elements all have the form $\{x, y, z\}$ where $x, y \in R_\alpha$, $x \neq y$, $z \in R_\beta$ and $\alpha < \beta$, ($\alpha > \beta$). We call $\{x, y, z\}$ a triple with base $\{x, y\}$ and apex z lying on the ranks α and β. We define graphs $\mathscr{G}_1(\mathscr{S})$, $\mathscr{G}_2(\mathscr{S})$ on λ as follows: $\{\alpha, \beta\} \in [\lambda]^2$ is an edge of $\mathscr{G}_1(\mathscr{S})$ iff there is a triple in $\mathscr{S}$ whose base meets F_α and F_β, and edge of $\mathscr{G}_2(\mathscr{S})$ iff there is a triple in $\mathscr{S}$ lying on R_α and R_β.

Lemma 11.2. *Let* $\lambda \geq \aleph_0$ *be a cardinal. If* $\mathscr{G}_1, \mathscr{G}_2$ *are* λ-*chromatic graphs on* λ, *then there is a* λ-*chromatic* Δ-*system* (∇-*system*) $\mathscr{S}$ *on* $\lambda \times \lambda$ *with* $\mathscr{G}_1(\mathscr{S}) = \mathscr{G}_1$, $\mathscr{G}_2(\mathscr{S}) = \mathscr{G}_2$.

Proof. Let $\mathscr{S}$ be the set of all triples of the form $\{\langle\alpha_1, \alpha_2\rangle, \langle\alpha'_1, \alpha_2\rangle, \langle\beta_1, \beta_2\rangle\}$ where $\{\alpha_1, \alpha'_1\} \in \mathscr{G}_1$, $\{\alpha_2, \beta_2\} \in \mathscr{G}_2$ and

$\alpha_2 < \beta_2$, $(\alpha_2 > \beta_2)$. Then $\mathscr{G}_i(\mathscr{S}) = \mathscr{G}_i$ for $i = 1, 2$. Let $f\colon \lambda \times \lambda \to \tau$ for some $\tau < \lambda$. Let $T_\nu = \{\beta < \lambda$: there are α_1, α_1' such that $\{\alpha_1, \alpha_1'\} \in \mathscr{G}_1$ and $f(\langle \alpha_1, \beta \rangle) = f(\langle \alpha_2, \beta \rangle) = \nu\}$ for $\nu < \tau$.

Considering that $\operatorname{Chr}(\mathscr{G}_1) = \lambda$, we have $\lambda = \bigcup_{\nu < \tau} T_\nu$. Using $\operatorname{Chr}(\mathscr{G}_2) = \lambda$ there are $\alpha_2, \beta_2 < \lambda$ and $\nu < \tau$ such that $\{\alpha_1, \beta_2\} \in \mathscr{G}_2$ and $\alpha_2, \beta_2 \in T_\nu$. We can choose α_1, α_1' and β_1 in such a way that $\{\alpha_1, \alpha_1'\} \in \mathscr{G}_1$, and $f(\langle \alpha_1, \alpha_2 \rangle) = f(\langle \beta_1, \beta_2 \rangle) = \nu$. This proves $\operatorname{Chr}(\mathscr{S}) = \lambda$.

Definition 11.3. Let $\mathscr{T}_0$ be the triple system with four points and two triples. Let $\mathscr{T}_7$ be the triple system containing five vertices and four triples such that three of these triples have a common edge and the fourth triple does not meet this edge. Let $\mathscr{T}_8$ be a triple system with ten vertices and five triples and forming a pentagon as shown on the Diagram 6.

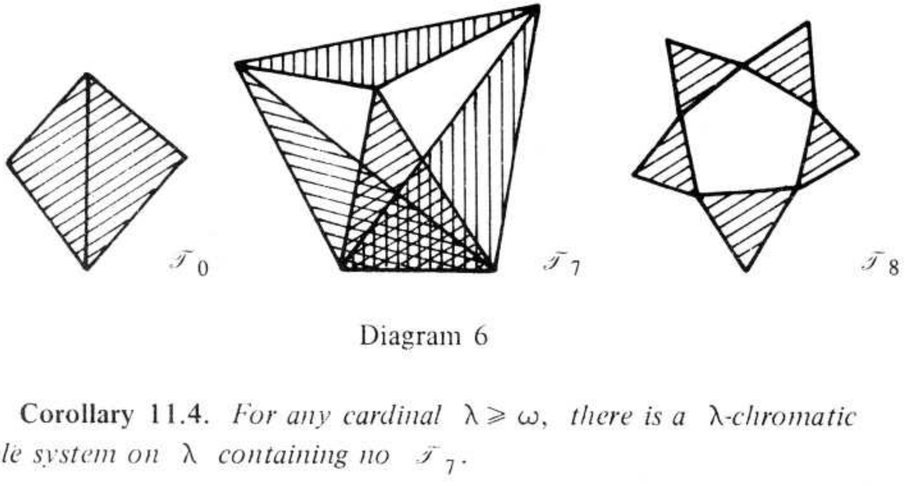


Diagram 6

Corollary 11.4. *For any cardinal $\lambda \geq \omega$, there is a λ-chromatic triple system on λ containing no $\mathscr{T}_7$.*

Proof. In lemma 11.2 take $\mathscr{G}_1$ to be the complete graph on λ and $\mathscr{G}_2$ a λ-chromatic graph on λ containing no triangles.

Note that by the reasoning given in the proof of Theorem 10.8 $\lambda \to (\lambda, \mathscr{T}_7)^3$ holds for all regular λ.

Lemma 11.5. *Let κ be an infinite cardinal, and let $\mathscr{G}_1, \mathscr{G}_2$ be*

κ-*chromatic graphs on* κ. *For each ordinal* $\alpha < \kappa$ *let* $P_\alpha = {}^\alpha\mathscr{G}_1$, *and let* $P = \bigcup_{\alpha<\kappa} P_\alpha$. *Let* $\mathscr{S}$ *be the triple system on* $P \times \kappa$ *consisting of the triples of the form* $\{\langle\sigma, u\rangle, \langle\sigma, v\rangle, \langle\tau, w\rangle\}$ *where* $\sigma \in P_\alpha$, $\tau \in P_\beta$, $\alpha < \beta < \kappa$, $\{\alpha, \beta\} \in \mathscr{G}_2$, $\sigma \subset \tau$, $\tau(\alpha) = \{u, v\}$, *and* $w < \kappa$. *Then* $\mathrm{Chr}(\mathscr{S}) = \kappa$.

Proof. Let $f\colon P \times \kappa \to \delta$ for some $\delta < \kappa$. We now define a $\rho \in {}^{\omega_1}\mathscr{G}_1$ by induction on $\alpha < \omega_1$. Assume $\rho|\alpha$ has already been defined. Then $\rho|\alpha \in P_\alpha$. Considering that $\mathrm{Chr}(\mathscr{G}_1) = \kappa$, we can pick $u_\alpha, v_\alpha \in \kappa$, $\{u_\alpha, v_\alpha\} \in \mathscr{G}_1$ and $\nu_\alpha < \delta$ such that $f(\langle\rho|\alpha, u_\alpha\rangle) = f(\langle\rho|\alpha, v_\alpha\rangle) = \nu_\alpha$. We then put $\rho(\alpha) = \{u_\alpha, v_\alpha\}$. By $\mathrm{Chr}(\mathscr{G}_2) \geqslant \kappa$, there are $\alpha < \beta < \kappa$, $\{\alpha, \beta\} \in \mathscr{G}_2$, and $\nu < \delta$ such that $\nu_\alpha = \nu_\beta = \nu$. Put $\nu = \nu_\alpha = \nu_\beta$, $u = u_\alpha$, $v = v_\alpha$, $w = u_\beta$, $\sigma = \rho|\alpha$, $\tau = \rho|\beta$. Then $X = \{\langle\sigma, u\rangle, \langle\sigma, v\rangle, \langle\tau, w\rangle\} \in \mathscr{S}$ and $f(X) = \{\nu\}$. Hence $\mathrm{Chr}(\mathscr{S}) \geqslant \kappa$. Since $\mathrm{Chr}(\mathscr{S}) \leqslant \kappa$ is obvious this proves the lemma.

Theorem 11.6. *For any infinite cardinal* κ *is a triple system* $\mathscr{S}$ *such that*

(1) $\mathrm{Chr}(\mathscr{S}) = \kappa^+$,

(2) $|\mathscr{S}| = 2^\kappa$,

(3) $\mathscr{S}$ *contains no* $\mathscr{T}_1, \mathscr{T}_3, \mathscr{T}_4, \mathscr{T}_7, \mathscr{T}_8$,

(4) *if* $\mathscr{S}_0 \subset \mathscr{S}$ *and* $\mathscr{S}_0$ *contains no* $\mathscr{T}_0$ *then* $\mathrm{Chr}(\mathscr{S}_0) \leqslant 2$,

(5) *if* $\mathscr{S}_0 \subset \mathscr{S}$ and $|\mathscr{S}_0| < \aleph_0$, *then* $\lambda \to (\lambda, \mathscr{S}_0)^3$ *for every regular* λ.

Proof. We use Lemma 11.5, with κ replaced by κ^+. We take for $\mathscr{G}_1$ a graph with no triangles or pentagons, take for $\mathscr{G}_2$ a graph with no triangles, both of chromatic number κ. (1) follows from the lemma. $|\mathscr{S}| = \sum_{\alpha<\pi^+} |{}^\alpha\kappa| = 2^\kappa$ hence (2) holds.

To prove the rest observe first that if $\langle\tau, w\rangle \in P \times \kappa$ and a triple X of $\mathscr{S}$ contains it, and X has a vertex $\langle\sigma, u\rangle$ with $D(\sigma) < D(\tau)$ then X is uniquely determined by $D(\tau)$; $X = \{\langle\tau|D(\sigma), u\rangle, \langle\tau|D(\sigma), v\rangle, \langle\tau, w\rangle\}$

where $\{u, v\} = \tau(D(\sigma))$.

It follows that the pairs of $P \times \kappa$ contained in more than one triple of $\mathscr{S}$ have the form $\{\langle\sigma, u\rangle, \langle\sigma, v\rangle\}$ and all the other vertices of the triples containing this pair are of the form $\langle\tau, w\rangle$ with $D(\tau) > D(\sigma)$, $\sigma \subset \tau$.

This arrangement implies immediately that $\mathscr{S}$ contains no $\mathscr{T}_3$ and $\mathscr{T}_4$, $\mathscr{T}_7$ are excluded because $\mathscr{G}_2$ does not contain triangles.

In case of $\mathscr{T}_1$ and $\mathscr{T}_8$ there is a unique way to pick an edge from each triple so that these edges form a triangle and a pentagon respectively. Now meditation shows that if $\mathscr{S}$ contained a $\mathscr{T}_1$ or a $\mathscr{T}_8$ then the triangle or the pentagon would be contained in $\mathscr{G}_1$. We omit the cumbersome discussion. We only want to point out that the same argument does not work for circuits of length seven defined analogously. This proves (3).

(4) and (5) follow since all finite subsystems $\mathscr{S}_0$ of $\mathscr{S}$ are contained in systems constructed for the proof of Theorem 10.8.

Lemma 11.7. *Let λ be an infinite cardinal. Suppose $\mathscr{G}_1, \mathscr{G}_2$ are graphs on λ such that $P(\mathscr{G}_1, \lambda, \kappa)$ and $P^*(\mathscr{G}_2, \lambda, \kappa)$. Then there is a V-system $\mathscr{S}$ on $\lambda \times \lambda$ such that:*

(1) $\mathrm{Chr}(\mathscr{S}) \geq \kappa$.

(2) *$\mathscr{G}_1(\mathscr{S})$, $\mathscr{G}_2(\mathscr{S})$ are subgraphs of $\mathscr{G}_1, \mathscr{G}_2$ respectively.*

(3) *any two triples lying on the same two ranks have the same apex.*

(4) *no two triangles have the same base.*

Proof. Choose mappings f_1, f_2^* establishing $P(\mathscr{G}_1, \lambda, \kappa)$ and $P^*(\mathscr{G}_2, \lambda, \kappa)$ respectively.

Let $f_2 = h \cdot f_2^*$ where h is a λ-to-1 mapping of λ onto λ.

Let $\mathscr{S}$ consist of all triples of the form

$$\{\langle\alpha_1, \alpha_2\rangle, \langle\beta_1, \beta_2\rangle, \langle\beta_1, \beta_2\rangle\}$$

such that $\alpha_2 < \beta_2$, $\{\alpha_2, \beta_2\} \in \mathcal{G}_2$, $f_2(\{\alpha_2, \beta_2\}) = \alpha_1$, $\{\beta_1, \beta_1'\} \in \mathcal{G}_1$ and $f_1(\{\beta_1, \beta_1'\}) = \alpha_2$.

(2), (3) and (4) hold by the construction. To see (1) let $g: \lambda \times \lambda \to \delta$ for some $\delta < \kappa$. By the choice of f_1, for each $\beta < \lambda$ there exists $\nu = \hat{g}(\beta)$ such that for all $\alpha < \lambda$ there are $\gamma, \delta < \lambda$ with $\{\gamma, \delta\} \in \mathcal{G}_1$, $f_1(\{\gamma, \delta\}) = \alpha$ and $g(\langle \gamma, \beta \rangle) = g(\langle \delta, \beta \rangle) = \nu$. By the choice of f_2^*, there are $\nu < \delta$ and $\alpha_2 < \lambda$ such that for all $\xi < \lambda$ there is a β with $\{\alpha_2, \beta\} \in \mathcal{G}_2$, $f_2^*(\{\alpha_2, \beta\}) = \xi$ and $\hat{g}(\alpha_2) = \hat{g}(\beta) = \nu$. By $\hat{g}(\alpha_2) = \nu$, there is an α_1 such that $g(\langle \alpha_1, \alpha_2 \rangle) = \nu$. Then there is a $\beta_2 > \alpha_2$ such that $\hat{g}(\beta_2) = \nu$, $\{\alpha_2, \beta_2\} \in \mathcal{G}_2$, and $h(f_2^*(\{\alpha_2, \beta_2\})) = \alpha_1$. By $\hat{g}(\beta_2) = \nu$, there are $\beta_1, \beta_1' < \lambda$ such that $\{\beta_1, \beta_1'\} \in \mathcal{G}_1$, $f_1(\{\beta_1, \beta_1'\}) = \alpha_2$, and $g(\langle \beta_1, \beta_2 \rangle, \langle \beta_1', \beta_2 \rangle) = \nu$. Then $X = \{\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle, \langle \beta_1, \beta_2 \rangle\} \in \mathcal{S}$ and $g(X) = \nu$. This proves $\mathrm{Chr}(\mathcal{S}) \geqslant \kappa$.

Lemma 11.8. *Let λ be an infinite cardinal, κ the least cardinal such that $\lambda^\kappa > \lambda$ and $n < \omega$. Then there is a* V*-system on $\lambda \times \lambda$ such that:*

(1) $\mathrm{Chr}(\mathcal{S}) \geqslant \kappa$.

(2) *$\mathcal{G}_1(\mathcal{S})$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$.*

(3) *$\mathcal{G}_2(\mathcal{S})$ contains no triangles.*

(4) *any two triples lying on the same two ranks have the same apex.*

(5) *no two triples have the same base.*

Proof. By the previous lemma it is sufficient to exhibit graphs $\mathcal{G}_1$, $\mathcal{G}_2$ on λ such that $P(\mathcal{G}_1, \lambda, \kappa)$, $P^*(\mathcal{G}_2, \lambda, \kappa)$, $\mathcal{G}_1$ contains no C_{2i+1} for $1 \leqslant i \leqslant n$ and $\mathcal{G}_2$ contains no triangles. The existence of such graphs follows from Theorems 9.7 and 8.1 respectively.

Lemma 11.9. *Let κ be an infinite cardinal such that $2^\kappa = \kappa^+$ and let $n < \omega$. Then there is a* V*-systems $\mathcal{S}$ on $\kappa^+ \times \kappa^+$ such that:*

(1) $\mathrm{Chr}(\mathcal{S}) = \kappa^+$.

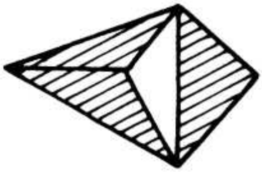
(2) *$\mathcal{G}_1(\mathcal{S})$ and $\mathcal{G}_2(\mathcal{S})$ contain no C_{2i+1} for $1 \leqslant i \leqslant n$.*

(3) *any two triples lying on the same two ranks have the same apex.*

(4) *no two triples have the same base.*

Proof. By Lemma 11.7 and Theorem 8.5.

Definition 11.10. Let $\mathcal{T}_9$ be a triple system with five vertices and three triples. Two of these triples have an edge in common, the third triple does not meet this edge; see Diagram 7.



$\mathcal{T}_9$

Diagram 7

Theorem 11.11. *Let* λ *be an infinite cardinal,* κ *the least cardinal such that* $\lambda^\kappa > \lambda$*. Then there is a triple system* $\mathcal{S}$ *on* λ *such that* $\mathrm{Chr}(\mathcal{S}) \geqslant \kappa$ *and* $\mathcal{S}$ *contains no* $\mathcal{T}_9$.

Proof. Let $\mathcal{S}$ be the triple system constructed in Lemma 11.8 with $n = 1$. Then $\mathrm{Chr}(\mathcal{S}) \geqslant \kappa$. Assume $\mathcal{S}$ contains a $\mathcal{T}_9$. Then the common edge of the two triangles of $\mathcal{T}_9$ meets two ranks, and then $\mathcal{G}$ would have to contain a triangle.

Lemma 11.12. *Let* κ *be an infinite cardinal such that* $2^\kappa = \kappa^+$, *and let* $n < \omega$. *Then there is a* Δ*-system* ($\vee$*-system*) $\mathcal{S}$ *on* $\kappa^+ \times \kappa^+$ *such that:*

(1) $\mathrm{Chr}(\mathcal{S}) = \kappa^+$.

(2) $\mathcal{G}_1(\mathcal{S})$ *and* $\mathcal{G}_2(\mathcal{S})$ *contain no* C_{2i+1} *for* $1 \leqslant i \leqslant n$.

(3) *any two triples with the same base have their apexes on the same rank.*

(4) *any two triples with the same apex have disjoint bases.*

Proof. By Theorem 8.5 we can choose a graph $\mathcal{G}$ on κ^+ which contains no C_{2i+1} for $1 \leq i \leq n$ such that $P(\mathcal{G}, \kappa^+, \kappa^+)$. Let f be a mapping $f\colon \mathcal{G} \to \kappa^+$ which establishes $P(\mathcal{G}, \kappa^+, \kappa^+)$. Define π: $\kappa^+ \times \kappa^+ \to \kappa^+$ by $\pi(\langle \beta_1, \beta_2 \rangle) = \beta_1$. For $\alpha, \beta < \kappa^+$ let $\mathscr{H}(\alpha, \beta)$ be the collection of all X such that:

(i) $X \subset [R_\beta]^2$.

(ii) $\{x, y\} \in X \Rightarrow \{\pi(x), \pi(y)\} \in \mathcal{G}$ and $f(\{\pi(x), \pi(y)\}) = \alpha$.

(iii) $e \neq e' \in X \Rightarrow e \cap e' = \phi$.

(iv) $|X| = \kappa$.

Now, by a routine transfinite induction, one can construct a Δ-system (∇-system) $\mathscr{S}$ satisfying the following conditions:

(a) $\mathcal{G}_1(\mathscr{S}), \mathcal{G}_2(\mathscr{S})$ are subgraphs of $\mathcal{G}$,

(b) if a triple has apex $\langle \alpha_1, \alpha_2 \rangle$ and base $\{\langle \beta_1, \beta_2 \rangle, \langle \beta_1', \beta_2 \rangle\}$ then $f(\{\beta_1, \beta_1'\}) = \alpha_2$,

(c) any two triangles with the same apex have disjoint bases,

(d) if $\{\alpha_2, \beta_2\} \in \mathcal{G}$, $\alpha_2 > \beta_2$ $(\alpha_2 < \beta_2)$ and $X \in \mathscr{H}(\alpha_2, \beta_2)$ then for every sufficiently large $\alpha_1 < \kappa^+$ there is a triple in $\mathscr{S}$ with apex $\langle \alpha_1, \alpha_2 \rangle$ whose base belongs to X.

We omit the details of the construction. Now $\mathscr{S}$ satisfies the requirements (2), (3), (4) by the construction. Let $g\colon \kappa^+ \times \kappa^+ \to \kappa$. By the choice of $\mathcal{G}$ for each $\beta < \kappa^+$ there exists a $\nu = \hat{g}(\beta)$ satisfying the following requirement. For each $\alpha < \kappa^+$ there are κ^+ "vertex disjoint" pairs $\{x, y\} \in R_\beta$, with $\{\pi(x), \pi(y)\} \in \mathcal{G}$, $f(\{\pi(x), \pi(y)\}) = \alpha$ and $g(x) = g(y) = \nu$. Again, by the choice of $\mathcal{G}$ there are $\alpha_2, \beta_2 < \kappa^+$, $\{\alpha_2, \beta_2\} \in \mathcal{G}$, such that $\hat{g}(\alpha_2) = \hat{g}(\beta_2) = \nu$ for some $\nu < \kappa^+$. Say we are constructing a Δ-system and $\alpha_2 < \beta_2$. By $\hat{g}(\beta_2) = \nu$ there is an $X \in$ $\in \mathscr{H}(\alpha_2, \beta_2)$ such that $g(x) = g(y) = \nu$ holds for all pairs $\{x, y\} \in X$. Then, by (d), and by $\hat{g}_2(\alpha) = \nu$ there is an α_1 such that $g(\langle \alpha_1, \alpha_2 \rangle) = \nu$ and there is a triple Y in $\mathscr{S}$ with apex $\langle \alpha_1, \alpha_2 \rangle$ whose base belongs to X. But then $g(X) = \{\nu\}$ for this triple of $\mathscr{S}$. Hence $\mathrm{Chr}(\mathscr{S}) = \kappa^+$.

Theorem 11.13. *Let* κ *be an infinite cardinal such that* $2^\kappa = \kappa^+$ *and let* $N < \omega$. *Then there is a* κ^+*-chromatic triple system* $\mathscr{S}$ *on* κ^+ *such that for each* $n < N$ *any* n *points contain at most* $\left[\frac{n^2}{8}\right]$ *triples.* *

Proof. Take $\mathscr{S}$ for the system given by Lemma 11.12 with some n, $2n+1 \geqslant N$. It is a matter of easy finite computation to verify the statement for this $\mathscr{S}$.

Corollary 11.14. *Assume* G.C.H. *Then* $\hat{g}_3(m, \alpha) = \left[\frac{m^2}{8}\right]$ *for all* α.

Proof. By Corollary 3.3 and Theorem 11.13.

To conclude this chapter we state

Problem 9. Characterize the (finite) triple systems that occur in every $\aleph_1$-chromatic triple system on ω_1.

To have a short notation: Let $G_3(\kappa)$ denote the class of finite triple systems which occur in every κ-chromatic triple system on κ.

Corollary 3.3 shows that $G_3(\omega_1)$ is large e.g. $\mathscr{T}_0 \in G_3(\omega_1)$. By Corollary 11.4, $\mathscr{T}_7 \notin G_3(\omega_1)$ but $\aleph_1 \to (\aleph_1, \mathscr{T}_7)^3$. By Theorem 11.6 C.H. implies $\mathscr{T}_1, \mathscr{T}_3, \mathscr{T}_4, \mathscr{T}_8 \notin G_3(\omega_1)$. By Theorem 11.11 C.H. implies $\mathscr{T}_9 \notin G_3(\omega_1)$. Note that all these examples $\mathscr{T} \notin G_3(\omega_1)$ are such that they satisfy the necessary conditions given by Corollary 11.14.

We now state the simplest unsolved instances and some related problems.

Problems

9/A. Does ZFC $\vdash \mathscr{T}_1, \mathscr{T}_3, \mathscr{T}_4, \mathscr{T}_8, \mathscr{T}_9 \notin G_3(\omega_1)$?

9/B. Is there an $\aleph_1$-chromatic triple system on ω_1 which avoids both $\mathscr{T}_1$, and $\mathscr{T}_9$?

9/C. Is there an $\aleph_1$-chromatic triple system on ω_1 which avoids $\mathcal{T}_1, \mathcal{T}_3, \mathcal{T}_4$ and $\mathcal{T}_9$?

9/D. Let $\mathcal{S}_1, \mathcal{S}_2$ be finite triple systems. Suppose that for each of them there is an $\aleph_1$-chromatic triple system on ω_1, avoiding it. Does it follow that there is an $\aleph_1$-chromatic triple system on ω_1, avoiding both?

9/E. Let $\kappa, \lambda \geqslant \aleph_0$. Is $G_3(\kappa) = G_3(\lambda)$?

§12. CONSTRUCTIONS OF (n, i)-SYSTEMS HAVING LARGE CHROMATIC NUMBER

Put $\beth_0(\kappa) = \kappa$, $\beth_{n+1}(\kappa) = 2^{\beth_n(\kappa)}$ for all κ and $n < \omega$.

The next lemma is our main tool in the constructions to be given here. As we have already mentioned in §6, it concerns P and P^* properties.

Lemma 12.1. *Let* $1 \leqslant r < s < \aleph_0 \leqslant \delta \leqslant \lambda$. *Assume that at least one of the following conditions holds:*

(a) $s - r = 1$ *and* $\lambda^\kappa \leqslant \lambda$ *for* $\kappa < \delta$,

(b) $s - r \leqslant r$ *and* $\lambda^{\kappa^+} \leqslant \lambda$ *for* $\kappa < \delta$,

(c) $2 \leqslant s - r$, *and there are* $(s - r, r)$*-systems* $\mathcal{S}_\kappa$, *such that* $\lambda^{|\mathcal{S}_\kappa|} = \lambda$ *and* $P(\mathcal{S}_\kappa, \kappa, \kappa^+)$ *for* $1 \leqslant \kappa < \delta$.

Then there exists an (s, r)*-system* $\mathcal{S}$, $|\mathcal{S}| = \lambda$ *such that*

$$P^*(\mathcal{S}, \lambda, \delta, r).$$

Proof. If (c) holds let $\mathcal{S}_\kappa = \bigcup_{\nu < \kappa} \mathcal{S}_{\kappa,\nu}$ be a disjoint partition of $\mathcal{S}_\kappa$ establishing $P(\mathcal{S}_\kappa, \kappa, \kappa^+)$ for $1 \leqslant \kappa < \delta$.

If (a) holds let $\mathcal{S}_\kappa$ consist of a single one element set and put $\mathcal{S}_{\kappa,\nu} = \mathcal{S}_\kappa$ for $\nu < \kappa$.

If (b) holds let $\mathcal{S}_\kappa = [\kappa^+]^{s-r}$, $\mathcal{S}_\kappa = \mathcal{S}_{\kappa,\nu}$ for $\nu < \kappa$. Let τ be the smallest cardinal such that $\lambda^\tau > \lambda$. Then $\omega \leqslant \delta \leqslant \tau \leqslant \lambda$ and τ is regular in all cases. We are going to define $\mathscr{S}$ on $\lambda \times \tau$. Put $R_\beta = \lambda \times \{\beta\}$, $T_\beta = \lambda \times \beta$ for $\beta < \tau$.

For each $\beta < \tau$, let $\mathscr{H}(\beta)$ consist of the set of all sequences $X = \langle X_\nu : \nu < \kappa \rangle$ satisfying the following conditions:

(1) $1 \leqslant \kappa < \delta$,

(2) $X_\nu \subset [T_\beta]^r$, $|X_\nu| = |\mathscr{S}_{\kappa,\nu}|$, $X_\nu \cap X_\mu = \phi$ for $\nu, \mu < \delta$, $\nu \neq \mu$,

(3) $A \cap B = \phi$ for $A \in X_\nu$, $B \in X_\mu$, $A \neq B$, $\nu, \mu < \kappa$.

Let further $\mathscr{K}(\beta)$ be the set of all triples $\langle X, g, \xi \rangle$ such that $X \in \mathscr{H}(\beta)$, $D(g) = \bigcup_{\nu < \kappa} X_\nu$, $R(g) \subset \lambda$ and $\xi < \tau$. By the assumption, $|\mathscr{H}(\beta)| = |\mathscr{K}(\beta)| = \lambda$ for all $\beta < \tau$.

We can now assume that for each $\beta < \tau$ and $\Lambda = \langle X, g, \xi \rangle$ there are "vertex-disjoint" copies $\mathscr{S}^\beta_\kappa(\Lambda)$ of $\mathscr{S}_\kappa$ in R_β where $\kappa = D(X)$. We will denote by $\mathscr{S}^\beta_{\kappa,\nu}(\Lambda)$ the subsets of $\mathscr{S}^\beta_\kappa(\Lambda)$ corresponding to the sets $\mathscr{S}_{\kappa,\nu}$ for $\nu < \kappa$ respectively. We now choose a one-to-one mapping $\varphi_{\beta,\nu}(\Lambda)$ of $\mathscr{S}^\beta_{\kappa,\nu}(\Lambda)$ onto X_ν for each $\beta < \tau$, $1 \leqslant \kappa < \delta$, $\nu < \kappa$, $\Lambda = \langle X, g, \xi \rangle \in \mathscr{K}(\beta)$. $\mathscr{S}$ will consist of sets of the form $Y \cup Z$, $Y \in [R_\beta]^{s-r}$, $Z \in [T_\beta]^r$ for $\beta < \tau$ satisfying the following conditions:

(4) There are $\beta < \tau$, $\Lambda = \langle X, g, \xi \rangle \in \mathscr{K}(\beta)$ such that $Y \in \mathscr{S}^\beta_\kappa(\Lambda) \wedge \kappa = D(X)$.

(5) There is a $\nu < \kappa$ such that $Y \in \mathscr{S}^\beta_{\kappa,\nu}(\Lambda)$ and $Z = \varphi_{\beta,\nu}(\Lambda)(Y)$. We put $\beta(Y \cup Z) = \beta$ for all $Y \cup Z \in \mathscr{S}$.

First of all it is obvious that $|\mathscr{S}| = \lambda$. We now check that $\mathscr{S}$ is an (s,r)-system. Let $U = Y \cup Z$, $U' = Y' \cup Z'$; $U, U' \in \mathscr{S}$, $U \neq U'$, $\beta(U) = \beta$, $\beta(U') = \beta'$. If $\beta \neq \beta'$ then obviously $|U \cap U'| \leqslant r$ in all cases. Assume $\beta = \beta'$. Considering that the $\mathscr{S}^\beta_\kappa(\Lambda)$ are "vertex-disjoint" and $|Z| = |Z'| = r$, we can assume that $U, U' \in \mathscr{S}^\beta_\kappa(\Lambda)$ for some $\Lambda = \langle X, g, \xi \rangle$, $D(X) = \kappa$. Then, by (3), (4), (5), and by the definition of $\varphi_{\beta,\nu}(\Lambda)$, Z and Z' and disjoint. If (a) and (b) hold then $|U \cap U'| \leqslant |Y \cap Y'| \leqslant s - r \leqslant r$. If (c) holds then $Y \neq Y'$, hence $|U \cap U'| \leqslant |Y \cap Y'| \leqslant r$ because of the $\mathscr{S}_\kappa$ are $(s-r, r)$-systems.

It remains to see that $P^*(\mathcal{S}, \lambda, \delta, r)$ holds. We define a mapping $f\colon \mathcal{S} \to \lambda$ as follows. Let $\beta(U) = \beta$, $U = Y \cup Z \in \mathcal{S}$. Then $Y \in \mathcal{S}^{\beta}_{\kappa}(\Lambda)$ for some $\Lambda = \langle X, g, \xi \rangle \in \mathcal{K}(\beta)$ with $D(X) = \kappa$ and $Z \in \bigcup_{\nu<\kappa} X_{\nu}$. Let $f(U) = g(Z)$. We claim that this f establishes $P^*(\mathcal{S}, \lambda, \delta, r)$. In fact, assume $\lambda \times \tau = \bigcup_{\mu<\kappa} P_{\mu}$ is a disjoint partition of the set of vertices into $\kappa < \delta$ classes. Let $N = \{\mu < \kappa \colon |P_{\mu}| \geqslant \tau\}$ and $M = \kappa - N$. By the regularity of τ, we can choose a $\beta < \tau$ and an $X \in \mathcal{M}(\beta)$, $D(X) = \kappa$ such that $\bigcup X_{\mu} \subset P_{\mu}$ for $\mu \in N$. Assume indirectly, that for all $\mu < \kappa$, $Z \in$ $\in [P_{\mu}]^r$ there is an $h(Z) < \lambda$ such that $f(Y \cup Z) \neq h(Z)$ for all $Y \cup Z \in$ $\in \mathcal{S}$, $Y \cup Z \subset P_{\mu}$. Let $g = h \restriction \bigcup_{\mu<\kappa} X_{\mu}$. We can now choose a $\xi < \tau$ such that for $\Lambda = \langle X, g, \xi \rangle \in \mathcal{K}(\beta)$, $\bigcup \mathcal{S}^{\beta}_{\kappa}(\Lambda) \subset \bigcup_{\mu \in N} P_{\mu}$. Then either of the assumptions (a), (b), (c) implies that there is a $\mu \in N$, such that for all $\nu < \kappa$ there is a $Y \in \mathcal{S}^{\beta}_{\kappa,\nu}(\Lambda)$, $Y \subset P_{\mu}$. If (a) holds this is trivial, if (b) holds this follows from $\kappa^+ \to (s-r)^1_{\kappa}$ and from the fact that each $Y \in \mathcal{S}^{\beta}_{\kappa}$ belongs to each $\mathcal{S}^{\beta}_{\kappa,\nu}$ for all $\nu < \kappa$. If (c) holds this is true because $\mathcal{S} = \bigcup_{\nu<\kappa} \mathcal{S}_{\kappa,\nu}$, establishes $P(\mathcal{S}_{\kappa}, \kappa, \kappa^+)$. Now pick a $Y \in$ $\in \mathcal{S}^{\beta}_{\kappa,\mu}(\Lambda)$, $Y \subset P_{\alpha}$ and let $Z = \varphi_{\beta,\mu}(\Lambda)(Y)$. Then $Y \cup Z \subset P_{\mu}$, $Y \cup Z \in \mathcal{S}$. By the choice of f, $f(Y \cup Z) = g(Z)$. However this contradicts the definition of h and g.

The next theorem yields the "if" part of Theorem B.

Theorem 12.2. *Let* $1 \leqslant i < n \leqslant mi + 1 < \aleph_0 \leqslant \lambda$. *Let* δ *be the least cardinal such that* $\lambda^{\beth_{m-1}(\delta)} > \lambda$. *Then there is an* (n, i)*-system* $\mathcal{S}$ *such that* $|\mathcal{S}| = \lambda$ *and* $P^*(\mathcal{S}, \lambda, \delta, i)$.

Proof. The assumption implies $m \geqslant 1$. We prove the theorem by induction on m. Assume $m = 1$. Put $r = i$, $s = n$. Then $s - r = 1$, $\beth_{m-1}(\kappa) = \kappa$ hence $\lambda^{\kappa} \leqslant \lambda$ for $\kappa < \delta$, and the statement follows from Lemma 12.1 (a).

Assume $m > 1$ and the theorem is true for $m - 1$. Put $r = i$, $s = n$. Then $s - r \leqslant (m-1)i + 1$. We now apply the induction hypothesis for $\lambda' = \beth_{m-1}(\kappa)$ for each $\kappa < \delta$. Considering that $\lambda'^{\beth_{m-2}(\kappa)} \leqslant \lambda'$ for each κ, it follows that there exist $(s-r, r)$-systems $\mathcal{S}_{\kappa}$, $|\mathcal{S}_{\kappa}| = \beth_{m-1}(\kappa)$

such that $P(\mathscr{S}_\kappa, \kappa, \kappa^+)$ holds for $\kappa < \delta$. Then, by the definition of δ, condition (c) of Lemma 12.1 holds, hence there is an $(s, r) = (n, i)$-system $\mathscr{S}$, $|\mathscr{S}| = \lambda$ such that $P^*(\mathscr{S}, \lambda, \delta, i)$ holds.

Corollary 12.3. *If* $1 \leqslant i < n \leqslant mi + 1 < \aleph_0 \leqslant \kappa$ *than there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_m(\kappa)$ *and* $P^*(\mathscr{S}, \beth_m(\kappa), \kappa^+, i)$ *holds.*

Proof. Since $\lambda^{\beth_{m-1}(\kappa)} = \lambda$ for $\lambda = \beth_m(\kappa)$, $m \geqslant 1$ we can use the above theorem with $\lambda = \beth_m(\kappa)$, $\delta = \kappa^+$.

Corollary 12.4. *Assume* G.C.H. *If* $1 \leqslant i < n \leqslant mi + 1 < \aleph_0$ *then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \aleph_{\alpha+m}$ *and* $P^*(\mathscr{S}, \aleph_{\alpha+m}, \aleph_{\alpha+1}, i)$ *holds and, as a corollary of this,* $\mathrm{Chr}(\mathscr{S}) > \aleph_\alpha$.

We now state a corollary for finite set-systems.

Corollary 12.5. *For any positive integers* i, n, k *with* $1 \leqslant i \leqslant n$ *there is a finite* (n, i)*-system such that* $P^*(\mathscr{S}, k, k+1, i)$ *holds.*

Proof. There is an integer m such that $n \leqslant mi + 1$. By Corollary 12.3, there is an (n, i)-system $\mathscr{S}'$ such that $P^*(\mathscr{S}', k, k+1, i)$ holds. The result now follows by compactness.

In case we do not assume G.C.H. Theorem 12.2 is not the only way to exploit the force of Lemma 12.1. In fact we are going to prove several results which seem to be incomparable with 12.2 in the absence of G.C.H.

Theorem 12.6. *Let* $1 \leqslant i < n \leqslant mi < \aleph_0 \leqslant \lambda$ *and let* δ *be the least cardinal such that* $\lambda^{\beth_{m-2}(\delta^+)} > \lambda$. *Then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \lambda$ *and* $P^*(\mathscr{S}, \lambda, \delta, i)$.

Proof. By the assumptions, $m \geqslant 2$. We prove the theorem by induction on m. Let $m = 2$. Put $r = i$, $s = n$. Then $s - r \leqslant r$, and $\beth_{m-2}(\kappa^+) = \kappa^+$; $\lambda^{\kappa^+} \leqslant \lambda$ holds for $\kappa < \delta$. Hence the result follows from Lemma 12.1 (b). Assume now $m > 2$ and that the statement is true for $m - 1$. Put $r = i$, $s = n$. Then $r - s \leqslant (m-1)i$. Let $\lambda'_\kappa = \beth_{m-2}(\kappa^+)$ for $\kappa < \delta$. Hence $\delta_\kappa'^{\beth_{m-3}(\tau^+)} \leqslant \lambda'_\kappa$ for $\tau < \kappa$. It follows, by the induction hypothesis, that for each $1 \leqslant \kappa < \delta$ there exist $(s-r, r)$-systems

$\mathscr{S}_\kappa$, $|\mathscr{S}_\kappa| = \beth_{m-2}(\kappa^+)$ such that $P(\mathscr{S}, \kappa, \kappa^+)$. Since $\lambda^{\beth_{m-2}(\kappa^+)} \leqslant \lambda$ for $\kappa < \delta$, condition (c) of Lemma 12.1 holds. Hence, by 12.1, there exists an $(s, r) = (n, i)$-system $\mathscr{S}$, $|\mathscr{S}| = \lambda$ with $P^*(\mathscr{S}, \lambda, \delta, i)$.

Corollary 12.7. *If* $1 \leqslant i < n \leqslant mi < \aleph_0 \leqslant \kappa$, *then there is an* (n, i)-*system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{m-1}(\kappa)$ *and* $P^*(\mathscr{S}, \beth_{m-1}(\kappa), \kappa, i)$.

Proof. Let $\lambda = \beth_{m-1}(\kappa)$. Since $m \geqslant 2$, $\lambda^{\beth_{m-2}(\tau^+)} \leqslant \beth_{m-1}(\kappa)$ for all $\tau < \kappa$. Hence, by the above theorem, $P^*(\mathscr{S}, \beth_{m-1}(\kappa), \kappa, i)$ holds.

Lemma 12.8. *Let* $1 \leqslant r < n < \aleph_0 \leqslant \kappa < \lambda$. *If there are order types* φ, ψ *such that* $|\varphi| = \lambda$, $\varphi \to (\psi)^1_\kappa$ *and* $\varphi \nrightarrow [\psi]^{r+1}_\kappa$, *then there is an* n-*tuple system* $\mathscr{S}$, *such that* $|\mathscr{S}| = \lambda$ *and* $P^*(\mathscr{S}, \kappa, \kappa^+, 1)$.

Proof. In fact since there is no other requirement we can take $\mathscr{S} = [\lambda]^n$ and prove that $P^*(\mathscr{S}, \kappa, \kappa^+, 1)$ holds. By 7.1, it is sufficient to see that $P(\mathscr{S}, \kappa, \kappa^+)$ holds. Let $\prec_1$ be an ordering of λ such that $\mathrm{tp}\,\lambda(\prec_1) = \varphi$. Let now $\hat{f}\colon [\lambda]^{r+1} \to \kappa$ establish $\varphi \nrightarrow [\psi]^{r+1}_\kappa$. We write each element $Y \in [\lambda]^n$ in the form $Y = Y_0 \cup Y_1$ where $Y_0 \in [\lambda]^{r+1}$, $Y_1 \in [\lambda]^{n-r-1}$ and $Y_0 \prec_1 Y_1$. Define $f\colon [\lambda]^n \to \kappa$ by $f(Y) = \hat{f}(Y_0)$ for $Y \in [\lambda]^n$. Let now $\lambda = \bigcup_{\mu<\kappa} P_\mu$. Choose $P_\mu = \Theta_\mu \cup R_\mu$ so that $\Theta_\mu \prec_1 R_\mu$, $|R_\mu| \leqslant n - r - 1$ and $|R_\mu| = n - r - 1$ if Θ_μ has a last element for $\mu < \kappa$. By $\kappa \geqslant \omega$, and $\varphi \nrightarrow [\psi]^{r+1}_\kappa$ we have $|\psi| \geqslant \kappa$. Hence, by $\varphi \to (\psi)^1_\kappa$ there is a $\mu < \kappa$ such that $\mathrm{tp}\,\Theta_\mu(\prec_1) \geqslant \psi$. By $\varphi \to [\psi]^{r+1}_\kappa$ for each $\nu < \kappa$ there is $Y_{0,\nu} \in [\Theta_\mu]^{r+1}$ with $\hat{f}(Y_{0,\nu}) = \nu$. There is $Y_{1,\nu} \subset P_\mu$, $Y_{0,\nu} \prec_1 Y_{1,\nu}$, $|Y_{1,\nu}| = n - r - 1$. Hence $f(Y_{0,\nu} \cup Y_{1,\nu}) = \nu$, $Y_{0,\nu} \cup Y_{1,\nu} \subset P_\mu$.

Lemma 12.9. *Let* $1 \leqslant t \leqslant i < n \leqslant mi + 1 < \aleph_0 \leqslant \kappa \leqslant \lambda$ *and* $m \geqslant 2$. *Suppose there are order types* φ, ψ *such that* $|\varphi| = \lambda$, $\varphi \to (\psi)^1_\kappa$, $\varphi \nrightarrow [\psi]^{t+1}_\kappa$. *Then there is an* (n, i)-*system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{m-1}(\lambda)$ *and* $P^*(\mathscr{S}, \beth_{m-1}(\lambda), \kappa^+, i)$.

Proof. By induction on m. Assume $m = 2$. Put $r = i$, $s = n$. Then $s - r \leqslant r + 1$. If $s - r \leqslant r$ then because of $\kappa^+ \leqslant \lambda$, $\beth_1(\lambda)^{\kappa^+} \leqslant \beth_1(\lambda)$ holds. If $s - r = r + 1 \geqslant t + 1$, then by the previous lemma there are $(s - r, r)$-systems $\mathscr{S}_\tau$, $|\mathscr{S}_\tau| \leqslant \lambda$ satisfying $P(\mathscr{S}_\tau, \tau, \tau^+)$ for all $\tau < \kappa^+$.

Hence, by 12.1, there is an $(s, r) = (n, i)$-system $\mathscr{S}$, $|\mathscr{S}| = \beth_{m-1}(\lambda)$ such that $P^*(\mathscr{S}, \beth_{m-1}(\lambda), \kappa^+, i)$. The general step of the induction follows from Lemma 12.1 the same way as in the previous proofs.

Theorem 12.10. *If* $1 \leqslant i < n \leqslant mi + 1 < \aleph_0$ *and* $m \geqslant 2$, *then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{m-1}(\mathrm{cf}(2^{\aleph_0}))$, *and* $P^*(\mathscr{S}, \beth_{m-1}(\mathrm{cf}(2^{\aleph_0})), \aleph_1, i)$.

Proof. Choose $t = 1$, $\varphi = \psi = \mathrm{cf}(2^{\aleph_0})$ in the previous lemma. $\mathrm{cf}(2^{\aleph_0}) \to (\mathrm{cf}(2^{\aleph_0}))^1_{\aleph_0}$ holds because $\mathrm{cf}(2^{\aleph_0})$ is regular and $> \aleph_0$. $\mathrm{cf}(2^{\aleph_0}) \nrightarrow [\mathrm{cf}(2^{\aleph_0})]^2_{\aleph_0}$ is the result of Galvin and Shelah mentioned in 9.4. The statement now follows from Lemma 12.9.

Theorem 12.11. *If* $1 \leqslant k < i < n \leqslant mi + 1 < \aleph_0$ *and* $m \geqslant 2$ *then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{m-1}(\aleph_k)$ *and* $P^*(\mathscr{S}, \beth_{m-1}(\aleph_k), \aleph_k, i)$.

Proof. To apply Lemma 12.9 put $t = k + 1$, $\varphi = \psi = \omega_k$, $\kappa = \aleph_{k-1}$. By a result of F. Galvin and S. Shelah [17] p. 168, $\aleph_k \nrightarrow [\aleph_k]^{k+2}_{\aleph_k}$. Hence the theorem follows from lemma 12.9.

Theorem 12.12. *If* $1 \leqslant r \leqslant i < n \leqslant mi + 1 < \aleph_0$, $m \geqslant 2$ *and* $2^{\aleph_\alpha} \leqslant \aleph_{\alpha+r}$, *then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{m-1}(\aleph_{\alpha+r})$ *and* $P^*(\mathscr{S}, \beth_{m-1}(\aleph_{\alpha+r}), \aleph_{\alpha+r}, i)$.

Proof. To apply Lemma 12.9 put $t = r$, $\varphi = \psi = \aleph_{\alpha+r}$, $\kappa = \aleph_{\alpha+r-1}$. By a result of S. Shelah [23], $2^{\aleph_\alpha} \leqslant \aleph_{\alpha+r} \Rightarrow \aleph_{\alpha+r} \nrightarrow [\aleph_{\alpha+r}]^{r+1}_{\aleph_{\alpha+r}}$.

Finally we prove a consistency result

Theorem 12.13. Con (ZF) $\Rightarrow$ Con (ZFC + $2^{\aleph_0} = 2^{\aleph_1} =$ *anything reasonable + for each integer* $m \geqslant 2$), *there is an* $(m+1, 1)$*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{m-1}(\aleph_0)$ *and* $P^*(\mathscr{S}, \beth_{m-1}(\aleph_0), \aleph_1, 1)$.

Proof. By a result of Baumgartner [1] Con (ZF) $\Rightarrow$ Con (ZFC + + $2^{\aleph_0} = 2^{\aleph_1}$ = anything reasonable + $\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0}$). Assume now that

$\aleph_1 \nrightarrow [\aleph_1]^2_{\aleph_0}$ and $2^{\aleph_0} = 2^{\aleph_1}$. We apply Lemma 12.9 with $t = 1 = i$, $n = m + 1$; $\kappa = \aleph_0$; $\varphi = \psi = \aleph_1$. If follows that there is an $(m + 1, 1)$-system $\mathscr{S}$ with $|\mathscr{S}| = \beth_{m-1}(\aleph_1) = \beth_{m-1}(\aleph_0)$ such that $P^*(\mathscr{S}, \beth_{m-1}(\aleph_0), \aleph_1, i)$ holds.

§13. CONSTRUCTIONS OF 3-CIRCUITLESS n-TUPLE SYSTEMS OF LARGE CHROMATIC NUMBER

In [5] p. 94 a general concept of s-circuitless n-tuple systems was defined. In this paper we are going to consider 3-circuitless set-systems and we give a definition of this special case only.

Definition. A set-system $\mathscr{S}$ is said to be 3-circuitless if no two members intersect in more than one point, and every family of pairwise intersecting members of $\mathscr{S}$ has a nonempty intersection.

Note that a 3-circuitless n-tuple system is an $(n, 1)$-system as well. 3-circuitless graphs are "triangle-free" graphs. Our aim is again to construct 3-circuitless n-tuple systems having large chromatic numbers. This way we are going to generalize the instances concerning $(n, 1)$-systems of the previous results. The proofs of these results follow the same pattern as well. First we prove a lemma corresponding to 12.1.

Lemma 13.1 *Let* $2 \leq n < \aleph_0 \leq \delta \leq \lambda$. *For each cardinal* κ, $1 \leq \kappa < \lambda$ *let* $\mathscr{S}_\kappa$ *be a 3-circuitless* n*-tuple system such that* $P(\mathscr{S}_\kappa, \kappa, \kappa^+)$. *Assume that for* $1 \leq \kappa < \delta$ *we have* $2^\kappa < \lambda = \lambda^{|\mathscr{S}_\kappa|}$. *Then there is a 3-circuitless* $n + 1$*-tuple system* $\mathscr{S}$ *with* $|\mathscr{S}| = \lambda$ *such that* $P^*(\mathscr{S}, \lambda, \delta, 1)$.

We are going to prove the following more general result.

Lemma 13.1/A. *Assume that the conditions of Lemma* 13.1 *hold. Let* $\hat{\mathscr{G}}$ *be a graph on* λ *with* $\mathrm{Chr}(\hat{\mathscr{G}}) = \lambda$. *Then the set-system* $\mathscr{S}$ *defined below on* $\lambda \times \lambda$ *satisfies the requirements of* 13.1.

(Note that for the proof of 13.1 $\hat{\mathscr{G}}$ can be taken to $[\lambda]^2$.)

Proof. Put $R_\beta = \lambda \times \{\beta\}$. For each $\mathscr{S}_\kappa$, $1 \leq \kappa < \delta$ let

$\mathscr{S}_\kappa = \bigcup_{\nu<\delta} \mathscr{S}_{\kappa,\nu}$ be a disjoint partition of $\mathscr{S}_\kappa$ establishing $P(\mathscr{S}_\kappa, \kappa, \kappa^+)$. For each $\beta, \xi < \lambda$ and $1 \leqslant \kappa < \delta$ we choose n-tuple systems $\mathscr{S}_\kappa(\beta, \xi)$ isomorphic to $\mathscr{S}_\kappa$ with set of vertices $V_\kappa(\beta, \xi)$ in such a way that the $V_\kappa(\beta, \xi)$ are disjoint and $R_\beta = \bigcup\{V_\kappa(\beta, \xi)\colon 1 \leqslant \kappa < \delta \wedge \xi < \lambda\}$. We denote the corresponding partitions by $\mathscr{S}_{\kappa,\nu}(\beta, \xi)$ for $\nu < \kappa$.

For each $\alpha < \lambda$ and $1 \leqslant \kappa < \delta$ let $\mathscr{H}(\alpha, \kappa)$ be the set of all sequences X satisfying the following conditions:

(1)(a) $X = \langle X_\nu \colon \nu < \kappa \rangle$,

(b) $X_\nu \cap X_\mu$ for $\nu \neq \mu$; $\nu, \mu < \kappa$,

(c) $|X_\nu| = |\mathscr{S}_{\kappa,\nu}|$ for $\nu < \kappa$,

(d) $\hat{X} = \bigcup_{\nu<\kappa} X_\nu$, $\hat{X} \subset R_\alpha$,

(e) $|\hat{X} \cap V_\kappa(\alpha, \xi)| \leqslant 1$ for $1 \leqslant \kappa < \delta$, $\xi < \lambda$.

For each $\beta < \lambda$, $1 \leqslant \kappa < \delta$ let $\mathscr{K}(\beta, \kappa)$ be the set of triples $\Lambda = \langle X, g, \eta \rangle$ satisfying the following conditions:

(2)(a) There is $\alpha < \beta$ such that $\{\alpha, \beta\} \in \hat{\mathscr{G}}$ and $X \in \mathscr{H}(\alpha, \kappa)$,

(b) $g\colon \hat{X} \to \lambda$,

(c) $\eta < \lambda$.

Let $L = \{\beta$: There is an $\alpha < \beta$ with $\{\alpha, \beta\} \in \hat{\mathscr{G}}\}$. It is obvious from the assumptions that $|\mathscr{H}(\alpha, \kappa)| = |\mathscr{K}(\beta, \kappa)| = \lambda$ for $\alpha < \lambda$, $\beta \in L$, $1 \leqslant \kappa < \delta$. For each $\beta \in L$ and $1 \leqslant \kappa < \delta$ we choose a one-to-one mapping $\varphi_{\beta,\kappa}$ of λ onto $\mathscr{K}(\beta, \kappa)$. For each $\beta \in L$, $1 \leqslant \kappa < \delta$, and $\xi < \lambda$ we choose a one-to-one mapping $\varphi_{\beta,\kappa,\xi}$ of $\mathscr{S}_\kappa(\beta, \xi)$ onto $\hat{X}$ where $\varphi_{\beta,\kappa}(\xi) = \Lambda$ and $\Lambda = \langle X, g, \xi \rangle$ in such a way that $\varphi_{\beta,\kappa,\xi}$ maps $\mathscr{S}_{\kappa,\nu}(\beta, \xi)$ onto X_ν. This is possible by (1) (a)-(d) and (2) (a).

We are now in a position to define the $n+1$-tuple system $\mathscr{S}$ satisfying the requirements of the theorem. $\mathscr{S}$ will consist of $n+1$-tuples having the form $Z = Y \cup \{x\}$ satisfying the following conditions:

(3)(a) There are $\beta \in L$, $1 \leqslant \kappa < \delta$, $\xi < \lambda$ such that $Y \in \mathscr{S}_\kappa(\beta, \xi)$,

(b) $x = \varphi_{\beta, \kappa, \xi}(Y)$.

We now define an $f\colon \mathscr{S} \to \lambda$ which we claim establishing $P^*(\mathscr{S}, \lambda, \delta, 1)$. If $Y \cup \{x\}$ satisfies the above requirements, then $\varphi_{\beta, \kappa}(\xi) = \Lambda = \langle X, g, \eta \rangle$ for some $\Lambda \in \mathscr{K}(\beta, \kappa)$ and $x \in \hat{X}$. Put

$$f(Y \cup \{x\}) = g(x) .$$

It is obvious that $\mathscr{S}$ is an $n + 1$-tuple system with $|\mathscr{S}| = \lambda$.

Note now that for an $Y \cup \{x\} \in \mathscr{S}$, the numbers β, κ, ξ are uniquely determined and depend only on Y. We denote them by $\beta(Y)$, $\kappa(Y)$, $\xi(Y)$. Moreover there is a unique $\nu < \kappa$ for which $Y \in \mathscr{S}_{\kappa, \nu}(\beta, \xi)$ and $x \in X_\nu$ for the corresponding x. Denote this ν by $\nu(Y)$. There is also a unique $\alpha(Y) = \alpha$ such that $x \in R_\alpha$, $\{\alpha, \beta\} \in \hat{\mathscr{G}}$.

First we are going to check that $\mathscr{S}$ is 3-circuitless. Just as in the proof of Lemma 12.1 it is easy to see that $\mathscr{S}$ is an $(n + 1, 1)$-system. To see that $\mathscr{S}$ is 3-circuitless it is now obviously sufficient to see that if $Z_i = Y_i \cup \{x_i\}$, $i < 2$ are three different members of $\mathscr{S}$ having pairwise non-empty intersections then $\bigcap\limits_{i<2} Z_i \neq \phi$. Assume now that the Z_i have pairwise non-empty intersections.

Put $\alpha_i = \alpha(Y_i)$, $\beta_i = \beta(Y_i)$, $\kappa_i = \kappa(Y_i)$, $\xi_i = \xi(Y_i)$, $\nu_i = \nu(Y_i)$ for $i < 2$. Note that $|Y_i| = n \geqslant 2$ for $i < 2$ and the Y_i and different as well. We may assume $\beta_0 \geqslant \beta_1 \geqslant \beta_2$. We now distinguish several cases to see that $\bigcap\limits_{i<2} Z_i \neq \phi$.

Case a. $\beta_0 = \beta_1 = \beta_2$.

a/1. $Y_0 \cap Y_1 \neq \phi$. Then $\kappa_0 = \kappa_1$, $\xi_0 = \xi_1$, $x_0 \neq x_1$, hence Y_2 must meet say Y_0. Then $\kappa_0 = \kappa_2$, $\xi_0 = \xi_2$, $x_1 \neq x_2$, $x_0 \neq x_2$. Hence Y_2 must meet Y_1 as well. Then $\bigcap\limits_{i<2} Y_i \neq \phi$ since $\mathscr{S}_{\kappa_0}(\beta_0, \xi_0)$ is 3-circuitless.

a/2. $Y_i \cap Y_j = \phi$ for $i, j < 2$, $i \neq j$. Then $\bigcap\limits_{i<2} Z_i = \{x_0\} = \{x_1\} = \{x_2\}$.

Case b. $\beta_0 = \beta_1 > \beta_2$.

b/1. $Y_0 \cap Y_1 = \phi$. Then $\bigcap_{i<2} Z_i = \{x_0\} = \{x_1\}$.

b/2. $Y_0 \cap Y_1 \neq \phi$. Then $\kappa_0 = \kappa_1$, $\xi_0 = \xi_1$, $x_0 \neq x_1$. By the assumption, $x_0, x_1 \in Z_2$. By (1)(d) and (2)(a) we have $\alpha_0 = \alpha_1$. Then $\alpha_0 = \beta_2$, $x_0, x_1 \in V_{\kappa_2}(\beta_2, \xi_2)$. This contradicts (1)(e).

Case c. $\beta_0 > \beta_1, \beta_2$. In this case $\bigcap_{i<2} Z_i = \{x_0\}$.

It remains to prove that the f defined above establishes $P^*(\mathscr{S}, \lambda, \delta, 1)$. Let $\lambda \times \lambda = \bigcup_{\mu<\kappa} P_\mu$ be a disjoint partition of $\lambda \times \lambda$ for some $\kappa < \delta$. Assume now indirectly, that for each $x \in \lambda \times \lambda$, $x \in P_\nu$ there is a $\rho(x) < \lambda$ such that

(4) $f(Z) \neq \rho(x)$ holds for all $x \in Z \subset P_\nu$, $Z \in \mathscr{S}$. For each $\beta < \lambda$ let

$$\Theta(\beta) = \{\mu < \kappa \colon |P_\mu \cap R_\beta| = \lambda\}.$$

Considering, that $\kappa < \mathrm{cf}(\lambda)$, $\Theta(\beta) \neq \phi$ for $\beta < \lambda$. Considering that $\Theta(\beta) \subset P(\kappa)$, $2^\kappa < \lambda$ and the fact that $\mathrm{Chr}(\hat{\mathscr{G}}) = \lambda$ it now follows that there are $\alpha < \beta < \lambda$, $\{\alpha, \beta\} \in \hat{\mathscr{G}}$ such that $\Theta(\alpha) = \Theta(\beta) = \Theta \neq \phi$. Note that then $\beta \in L$. Using the fact that $|P_\mu \cap R_\alpha| = \lambda$ for $\mu \in \Theta$, and that $|V_{\kappa'}(\alpha, \xi)| < \lambda$ we can choose pairwise disjoint sets $X_\mu \subset R_\alpha$ such that $|X_\mu| = |\mathscr{S}_{\kappa,\mu}|$ for $\mu < \kappa$, $X_\mu \subset P_\mu$ for $\mu \in \Theta$ and that $|X_\mu \cap V_{\kappa'}(\alpha, \beta)| \leq 1$ for $\mu < \kappa$, $1 \leq \kappa' < \delta$, $\xi < \lambda$. Then $X = \langle X_\mu \colon \mu < \kappa \rangle \in \mathscr{H}(\alpha, \kappa)$, by (1). Let $\hat{X} = \bigcup_{\mu<\kappa} X_\mu$ and $g = \rho \mid \hat{X}$.

Now using the fact that $|P_\mu \cap R_\beta| < \lambda$ for $\mu \notin \Theta$ and $\kappa < \mathrm{cf}(\lambda)$ we can find numbers $\xi, \eta < \lambda$ such that $V_\kappa(\beta, \xi) \subset \bigcup_{\mu \in \Theta} P_\mu$ and $\varphi_{\beta,\kappa}(\xi) = \langle X, g, \eta \rangle = \Lambda$. Using the fact that $P(\mathscr{S}_\kappa, \kappa, \kappa^+)$ is established by $\mathscr{S}_{\kappa,\nu}$ $(\nu < \kappa)$ we now find a $\mu \in \Theta$ such that for all $\nu < \kappa$ there is a $Y \in \mathscr{S}_{\kappa,\nu}(\beta, \xi)$, $Y \subset P_\mu$. Pick a $Y \subset P_\mu$, $Y \in \mathscr{S}_{\kappa,\mu}(\beta, \xi)$. Let $x = \varphi_{\beta,\kappa,\xi}(Y)$. Then by the choice of this function $x \in X_\mu \subset P_\mu$, $Y \cup \{x\} \in \mathscr{S}$ and by the definition of f, $f(Y \cup \{x\}) = g(x)$.

Then, by the definition of g, $f(Y \cup \{x\}) = \rho(x)$, $x \in Y \cup \{x\} \in \mathscr{S}$,

$Y \cup \{x\} \subset P_\mu$ and this contradicts the definition (4) of ρ.

Theorem 13.2. *Let* $2 \leqslant n < \aleph_0 \leqslant \lambda$ *and let* δ *be the least cardinal such that* $\lambda^{\beth_{n-2}(\delta)} > \lambda$. *Then there is a 3-circuitless n-tuple system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \lambda$ *and* $P^*(\mathscr{S}, \lambda, \delta, 1)$.

Proof. By induction on n. For $n = 2$ this is Theorem 8.1. Assume the theorem is true for some $n \geqslant 2$. Let δ' be the least cardinal such that $\lambda^{\beth_{n-1}(\delta')} > \lambda$. Assume $\kappa < \delta$. Then $\lambda^{\beth_{n-1}(\kappa)} = \lambda$ and $2^\kappa < \lambda^{\beth_{n-1}(\kappa)}$ for $\kappa < \delta$. On the other hand the least cardinal τ for which $\beth_{n-1}(\kappa)^{\beth_{n-2}(\tau)} > \beth_{n-1}(\kappa)$ is not less than κ^+. Hence applying the induction hypothesis we get that there are 3-circuitless n-tuple systems $\mathscr{S}_\kappa$ with $|\mathscr{S}_\kappa| = \beth_{n-1}(\kappa)$ satisfying $P(\mathscr{S}_\kappa, \kappa, \kappa^+)$ for $1 \leqslant \kappa < \delta'$. Then, by the previous lemma, there is a 3-circuitless $n+1$-tuple system $\mathscr{S}$ with $|\mathscr{S}| = \lambda$ such that $P^*(\mathscr{S}, \lambda, \delta, 1)$ holds.

Corollary 13.3. *If* $2 \leqslant n < \aleph_0 \leqslant \kappa$, *then there is a 3-circuitless n-tuple system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{n-1}(\kappa)$ *and* $P^*(\mathscr{S}, \beth_{n-1}(\kappa), \kappa^+, 1)$.

Proof. Let $\lambda = \beth_{n-1}(\kappa)$. Then $\lambda^{\beth_{n-2}(\tau)} = \lambda$ for all $\tau < \kappa^+$. Hence the result follows from Theorem 13.2.

Corollary 13.4. *For any positive integers* n, k *with* $n \geqslant 2$ *there is a finite 3-circuitless n-tuple system* $\mathscr{S}$ *such that* $P^*(\mathscr{S}, k, k+1, 1)$

Proof. As a corollary of the previous result there is an $\mathscr{S}'$ satisfying all the requirements but the finiteness. The result then follows by compactness. The following is an improvement of Theorem 13.2 for $\kappa = \aleph_0$ and $n \geqslant 3$.

Theorem 13.5. *If* $3 \leqslant n < \aleph_0$, *then there is a 3-circuitless n-tuple system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{n-2}(\mathrm{cf}\,(2^{\aleph_0}))$ *and* $P^*(\mathscr{S}, \beth_{n-2}(\mathrm{cf}\,(2^{\aleph_0})), \aleph_1, 1)$.

Proof. By induction on n. Assume $n = 3$. Let $\lambda = 2^{\mathrm{cf}(2^{\aleph_0})}$, $\delta = \aleph_1$. By our Theorem 9.6, we have $P^*(GS_n(\mathrm{cf}\,(2^{\aleph_0}), \aleph_0, \mathrm{cf}\,(2^{\aleph_0}))$ and by 8.3, $GS_n(\tau)$ is "triangle-free". Considering $\mathrm{cf}\,(2^{\aleph_0}) \geqslant \aleph_1$, it follows that for $1 \leq \kappa < \aleph_1$ there are "triangle-free" graphs $\mathcal{G}_\kappa$ satisfying $P(\mathcal{G}_\kappa, \kappa, \kappa^+)$, $\lambda^{|\mathcal{G}_\kappa|} \leq \lambda^{\mathrm{cf}(2^{\aleph_0})} = \lambda$. Moreover if $\kappa < \aleph_1$, then $2^\kappa \leq 2^{\aleph_0} < 2^{\mathrm{cf}(2^{\aleph_0})}$ hence the result follows from Lemma 13.1.

The general step of the induction is to be carried out the same way as in the proof of Theorem 3.2. We omit it.

Corollary 13.6. *For any cardinal* κ, *there is a graph* $\mathcal{G}$ *such that:*

(1) $\mathcal{G}$ *contains no quadrilateral with a diagonal,*

(2) $\mathcal{G} \to (3)^1_\kappa$,

(3) $\kappa > \aleph_0$ *implies* $|\mathcal{G}| = 2^{2^\kappa}$,

(4) $\kappa = \aleph_0$ *implies* $|\mathcal{G}| = 2^{\mathrm{cf}(2^{\aleph_0})}$,

(5) $\kappa < \aleph_0$ *implies* $|\mathcal{G}| < \aleph_0$,

Proof. The result follows from Corollaries 13.3, 13.4 and Theorem 13.5 considering the fact that if $\mathcal{S}$ is a 3-circuitless triple-system then $\mathcal{S}$ is the set of triangles contained in the graph induced by $\mathcal{S}$.

§14. THE "SMALLEST TRIPLE SYSTEMS" OF LARGE CARDINALITY. THE UPPER ESTIMATES FOR $g_n(t, \alpha)$

The induction method described in Lemma 13.1 does not work for "*s*-circuitless" set systems. However we can get triple systems with some specific properties if in the construction given in Lemma 3.1 we start from graphs containing no short odd circuits. The word "smallest" is used here in an intuitive sense. We do not have a proof that all finite triple systems which occur in the systems constructed below do occur e.g. in the special triple-systems constructed in §15 and [12] for the corresponding values of parameters.

Theorem 14.1. *Let* $n < \aleph_0 \leq \kappa$ *and let* $\lambda = 2^{2^\kappa}$. *Let* $R_\beta = \lambda \cup \{\beta\}$ *and assume* $R_\beta = \bigcup \{V(\beta, \xi)\colon \beta < \lambda \wedge \xi < \lambda\}$ *where the* $V(\beta, \xi)$ *are pairwise disjoint and have cardinality* 2^κ. *There exists a* 3*-circuitless triple system* $\mathcal{S}$ *on* $\lambda \times \lambda$ *satisfying the following conditions*

(1) $P^*(\mathscr{S}, 2^{2^\kappa}, \kappa^+, 1)$

(2) *If* $X \in \mathscr{S}$ *then* $|X \cap V(\beta, \xi)| = 2$, $|X \cap V(\alpha, \xi)) = 1$ *for some* $\alpha, \beta, \xi, \eta;\ \alpha < \beta$.

(3) *Suppose* $x, y, x', y' \in V(\beta, \xi)$, $\{x, y\} \neq \{x', y'\}$ *and* $\{x, y, z\}$, $\{x', y', z'\} \in \mathscr{S}$. *Then* $z \in V(\alpha, \eta)$, $z' \in V(\alpha, \xi)$ *for some* $\alpha < \beta$, $\eta \neq \xi$.

(4) *If* $s = \{x, y, z\} \in \mathscr{S}$ *where* $x, y \in V(\beta, \xi)$, $z \in V(\alpha, \eta)$ *let* $e_1(s) = \{\alpha, \beta\}$, $e_2(s) = \{x, y\}$;

$$\mathscr{G}_1 = \{e_1(s)\colon s \in \mathscr{S}\},\ \mathscr{G}_2 = \{e_2(s)\colon s \in \mathscr{S}\}.$$

Then the graphs $\mathscr{G}_1, \mathscr{G}_2$ *do not contain* C_{2i+1} *for* $1 \leqslant i \leqslant n$.

Proof. We apply Lemma 13.1/A. By 8.3 we can choose $\hat{\mathscr{G}}$ on λ wlth $\mathrm{Chr}(\hat{\mathscr{G}}) = \lambda$ and not containing C_{2i+1} for $1 \leqslant i \leqslant n$. We apply 13/1/A with $\delta = \kappa^+$. For all $\mathscr{S}_\tau$ $(\tau < \delta)$ we can choose a graph $\mathscr{G}$ on 2^κ not containing C_{2i+1} for $1 \leqslant i \leqslant n$ and satisfying $P(\mathscr{G}, \kappa, \kappa^+)$. By our theorems 8.3 and 9.7 $GS_n(2^\kappa)$ is such a graph. It is easy to see that the construction described in 13.1/A gives a triple-system satisfying the requirements.

Theorem 14.2. *Let* $n < \aleph_0$. *Put* $\lambda = 2^{\mathrm{cf}(2^{\aleph_0})}$. *Let* R_β *and* $V(\beta, \xi)$ *have the same meaning as in* 13.6. *Then there exists a* 3*-circuitless triple-system* $\mathscr{S}$ *on* $\lambda \times \lambda$ *such that*

$$P^*(\mathscr{S}, 2^{\mathrm{cf}(2^{\aleph_0})}, \aleph_1, 1)$$

and $\mathscr{S}$ *satisfies the requirements* (2)-(4) *of* 13.6.

Proof. We do the same as in the previous proof except for that we choose $\mathscr{G}$ to be graph on $\mathrm{cf}(2^{\aleph_0})$ not containing C_{2i+1} for $1 \leqslant i \leqslant n$ and satisfying $P(\mathscr{G}, \aleph_0, \aleph_1)$. By Theorems 3.8 and 9.6 $GS_n(\mathrm{cf}(2^{\aleph_0}))$ is such a graph. Theorems 14.1 and 14.2 give the "smallest" triple systems of chromatic number $> \kappa$ and $> \aleph_0$ we can construct. Note also that these triple systems do not contain $\mathscr{T}_0, \ldots, \mathscr{T}_7$ and $\mathscr{T}_9$; Let $g_n(t, \alpha)$ be the function defined in the introduction.

As we have already mentioned $g_2(t,\alpha) = \left[\frac{t^2}{4}\right]$ for all α and $t < \omega$.

Our next theorem collects the information we have about $g_3(t,\alpha)$. This is one of the main results of our paper. First we give

Definition 14.3. For $t < \omega$, let

$$\check{g}_3(t) = \max_{\substack{k,t_0,\dots,t_k \\ t_0+\dots+t_k=t}} \sum_{i=1}^{k} \min\left(t_0, \left[\frac{t_i^2}{4}\right]\right)$$

Theorem 14.4.

(1) *If* $\mathscr{S}$ *is a triple system with* $\operatorname{Chr}(\mathscr{S}) > \aleph_0$ *then, for each* $t < \omega$, *there is* $X \in [\bigcup \mathscr{S}]^t$ *suct that* $|\mathscr{S} \cap [X]^3| \geqslant \check{g}_3(t)$.

(2) *For any infinite cardinal* κ *there is a triple system* $\mathscr{S}$ *such that:*

(a) $\operatorname{Chr}(\mathscr{S}) > \kappa$,

(b) $|\mathscr{S}| = 2^{2^{\kappa}}$ *if* $\kappa > \aleph_0$, $|\mathscr{S}| = 2^{\operatorname{cf}(2^{\aleph_0})}$ *if* $\kappa = \aleph_0$,

(c) *for each* $t < \omega$ *if* $X \in [\bigcup \mathscr{S}]^t$, *then* $|\mathscr{S} \cap [X]^3| \leqslant \check{g}_3(t)$,

(3) $\left(\frac{t}{3}\right)^{\frac{3}{2}} - t \leqslant \check{g}_3(t) \leqslant \left(\frac{t}{3}\right)^{\frac{3}{2}}$ *for all* $t < \omega$,

(4) $\check{g}(3t^2) = t^3$ *for all* $t < \omega$,

(5) *The first few values of* $\check{g}_3(t)$ *are given by the following table:*

0	1	2	3	4	5	6	7	8	9	10	11	12	13
0	0	0	1	1	2	2	3	4	4	5	6	8	8

14	15	16	17	18	19	20	21	22	23	24	25	26	27
9	10	12	12	13	14	16	18	18	19	20	22	24	27

Proof. (1) follows from Theorem 3.8, since by this theorem if $\mathrm{Chr}(\mathscr{S}) > \aleph_0$ and $k, s < \omega$ we can choose k "vertex disjoint" $K(s, s)$ and an s^2-subset F of the vertices such that every edge of a $K(s, s)$ is joined from a point of F by a triple of $\mathscr{S}$. If $s > t$ and $t_0 + \ldots + t_k = t$ we can then easily choose a t-subset containing $> \check{g}_3(t)$ triples of $\mathscr{S}$.

(2) follows from Theorems 14.1 and 14.2 respectively.

These theorems give us triple systems satisfying the requirements (a) and (b) of (2). We only have to show that (2) (c) holds for triple-systems satisfying requirements (2)-(4) of Theorem 14.1.

From (4) we only need that $\mathscr{G}_2$ does not contain triangles. Put $f(t_0, \ldots, t_m) = \sum_{i=1}^{m} \min\left(t_0, \left[\frac{t_i^2}{4}\right]\right)$. We need the following

Sublemma. *Assume* $X_0, \ldots, X_m \geqslant 1$. *Then there are* $m, y_0, \ldots \ldots, y_m \geqslant 1$ *such that*

(i) $f(y_0, \ldots, y_m) \geqslant f(x_0, \ldots, x_n)$ *and*

(ii) $y_0 + \ldots + y_m = x_0 + \ldots + x_n$

(iii) $y_0 \geqslant n$, $\min(n, x_0) \leqslant m \leqslant n$, $y_i \geqslant x_i$ *for* $i \leqslant m$.

Proof. We may assume $x_1 \geqslant \ldots \geqslant x_n$. The claim is obviously true with $y_i = x_i$ if $n \leqslant x_0$. Assume $x_0 < n$ and that the claim is true for $n-1$. By the induction hypothesis we can choose $z_0, \ldots, z_k$ with $f(z_0, \ldots, z_k) \geqslant \geqslant f(x_0, \ldots, x_{n-1})$, $z_0 + \ldots + z_k = x_0 + \ldots + x_{n-1}$, $z_i \geqslant x_i$, $(i \leqslant k)$, $z_0 \geqslant n-1$, $x_0 \leqslant k \leqslant n-1$. We may assume $x_n \geqslant 2 \wedge z_0 = n-1$, otherwise $m = k+1$, $y_i = z_i$, $(i \leqslant k)$, $y_{k+1} = x_n$ satisfies the requirements. Now we choose $m = k$, $y_0 = z_0 + x_n - \left[\frac{x_n}{2}\right]$. We can define y_i, $1 \leqslant i \leqslant k$ in such a way that $y_i \geqslant z_i$, $\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} z_i + \left[\frac{x_n}{2}\right]$ and if $I = \{1 \leqslant i \leqslant k \colon y_i > z_i\}$ then $|I| \geqslant \min\left(x_0, \left[\frac{x_n}{2}\right]\right)$. It is easy to check that (ii) and (iii) hold. To see that (i) holds it is sufficient to see that

$f(y_0, \ldots, y_k) - f(z_0, \ldots, z_k) \geq \min\left(x_0, \left[\frac{x_n^2}{4}\right]\right)$. Let $i \in I$. Then $\left[\frac{y_i^2}{4}\right] - \left[\frac{z_i^2}{4}\right] \geq z_i - \left[\frac{z_i}{2}\right] \geq x_n - \left[\frac{x_n}{2}\right]$. Considering that $y_0 - z_0 = x_n - \left[\frac{x_n}{2}\right]$ it follows that $f(y_0, \ldots, y_k) - f(z_0, \ldots, z_k) \geq$ $\geq \sum_{i \in I} \min\left(y_0, \left[\frac{y_i^2}{4}\right]\right) - \min\left(z_0, \left[\frac{z_i^2}{4}\right]\right) \geq \left(x_n - \left[\frac{x_n}{2}\right]\right) \cdot$ $\cdot \min\left(x_0, \left[\frac{x_n}{2}\right]\right) \geq \min\left(x_0, \left[\frac{x_n^2}{4}\right]\right)$ and this proves the sublemma.

Let now $X \in [\lambda \times \lambda]^{<\omega}$. Let $\alpha_0 < \ldots < \alpha_t$ be an enumeration of all ordinals $\alpha < \lambda$ for which $X \cap R_\alpha \neq \phi$. Put $t = t(X)$.

For $1 \leq j \leq t$ let X_s^j, $1 \leq s \leq n_j$ be an enumeration of all now empty sets $X \cap V(\alpha_j, \xi)$. Put $w(X) = \max\{n_j\colon 1 \leq j \leq t(X)\}$. We are going to prove by induction on $t(X)$ that there is a sequence $y_0, \ldots, y_n$ such that $|[X]^3 \cap \mathcal{S}| \leq f(y_0, \ldots, y_n)$, $y_0 + \ldots + y_n = |X|$ and $y_0 \geq$ $\geq \max(w(X), x_0)$. This is trivial for $t = 0$. Let X be as above with $t = t(X) > 0$ and assume that the statement is true for $t - 1$.

Put $e(Y) = |[Y]^3 \cap \mathcal{S}|$ for $Y \in [\lambda \times \lambda]^{<\omega}$. Put $X' = \bigcup\{X \cap R_{\alpha_i}\colon i < t\}$. By the assumption there are $z_0, \ldots, z_m$ so that $z_0 \geq x_0$, $z_0 + \ldots + z_m =$ $= |X'|$, $e(X') \leq f(z_0, \ldots, z_m)$ and $z_0 \geq w(X') = \max\{n_j\colon 1 \leq j < t\}$.

Now the arrangement of the triples in $\mathcal{S}$ described in (2) (3) and (4) implies that

$$e(X) \leq e(X') + \sum_{s=1}^{n_t} \min\left(\max(x_0, w(X')), \left[\frac{(x_s^t)^2}{4}\right]\right)$$

where $x_s^t = |X_s^t|$ for $1 \leq s \leq n_t$. Considering that $\max(x_0, w(X')) \leq z_0$ we have $e(X) \leq f(z_0, \ldots, z_m, x_1^t, \ldots, x_{n_t}^t)$. By the sublemma there are $y_0, \ldots, y_k$ such that $y_0 \geq m + n_t \geq n_t$, $y_0 \geq z_0$ and $e(X) \leq f(y_0, \ldots$ $\ldots, y_k)$. Then $y_0 \geq x_0$ and $y_0 \geq w(X)$ and this proves the claim. We omit the easy finite computations showing (3), (4) and (5).

Definition 14.5. We define functions $\check{g}_n(t)$ by induction on n for $n \geqslant 3$. $\check{g}_3$ is defined in 14.3. Assume $\check{g}_n(t)$ is defined for some $n \geqslant 3$. Put

$$\check{g}_{n+1}(t) = \max_{\substack{k, t_0, \ldots, t_k \\ t = t_0 + \ldots + t_k}} \sum_{i=1}^{k} \min(t_0, \check{g}_n(t_i)) .$$

Theorem 14.6.

(1) *If* $\mathscr{S}$ *is an* n*-tuple system,* $n \geqslant 3$, *with* $\mathrm{Chr}(\mathscr{S}) > \aleph_0$, *then, for each* $t < \omega$ *there is* $X \in [\bigcup \mathscr{S}]^t$ *such that*

$$|\mathscr{S} \cap [X]^n| \leqslant \check{g}_n(t) .$$

(2) *For each infinite cardinal* κ *there is an* n*-tuple system* $\mathscr{S}$ *such that*

(a) $\mathrm{Chr}(\mathscr{S}) > \kappa$,

(b) $|\mathscr{S}| = \beth_{n-1}(\kappa)$ if $\kappa > \aleph_0$,

$|\mathscr{S}| = \beth_{n-2}(\mathrm{cf}(2^{\aleph_0}))$ *if* $\kappa = \aleph_0$,

(c) *for each* $t < \omega$ *if* $X \in [\bigcup \mathscr{S}]^t$ *then*

$$|\mathscr{S} \cap [X]^n| \leqslant \check{g}_n(t),$$

(3) $\check{g}_n(t) \leqslant \left(\frac{t}{n}\right)^{\frac{n}{n-1}}$,

(4) $\check{g}_n(nt^{n-1}) = t^n$.

Proof in outline. By induction on n. We proved this for $n = 3$ in Theorem 14.4. To carry out the induction one uses the construction described in Lemma 13.1 and the ideas of the previous proof. (4) follows using corollary 3.10.

To conclude this chapter we state

Problem 10. Characterize the finite triple systems that occur in every triple system with chromatic number $> \aleph_0$.

A positive result is given by Theorem 3.8. Negative results are given in Theorem 14.2 and in the results listed at the end of §11. We now state the simplest unsolved instances and some related problems.

10.A. Does either $\mathcal{T}_0$ or $\mathcal{T}_8$ occur in all triple systems of chromatic number $> \aleph_0$?

10.B. Let $\mathcal{S}_1, \mathcal{S}_2$ be finite triple systems. Suppose there are $> \aleph_0$-chromatic triple systems $\mathcal{S}'_1, \mathcal{S}'_2$ not containing $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. Does it follow that there is a $> \aleph_0$-chromatic triple system not containing either of $\mathcal{S}_1$ and $\mathcal{S}_2$?

10.C. Let $\mathcal{S}$ be a finite triple system. If every triple system with chromatic number $> \aleph_1$ contains $\mathcal{S}$, does it follow that every triple system with chromatic number $> \aleph_0$ contains $\mathcal{S}$?

10.D. Given a triple system $\mathcal{S}$ with $\operatorname{Chr}(\mathcal{S}) > \aleph_0$ does there exists a triple system $\mathcal{S}_0$ on $2^{2^{\aleph_0}}$ such that $\operatorname{Chr}(\mathcal{S}_0) > \aleph_0$ and every finite subsystem of $\mathcal{S}_0$ is embeddable in $\mathcal{S}$?

§15. SPECIAL CONSTRUCTIONS

The construction described in this § give (n, i)-systems of large chromatic number and sometimes of smaller size than the general constructions described previously. Most constructions use ideas which can be found in [12]. We need some preliminaries.

Definition 15.1. If $1 \leq r < \omega$, we put $F_r(\kappa) = \beth_{r-1}(\kappa)^+$ for all infinite cardinals κ.

Lemma 15.2. *Let $n, r_1, \ldots, r_n$ be positive integers and let κ be an infinite cardinal. Then*

$$\begin{pmatrix} F_{r_1}(\kappa) \\ F_{r_2} F_{r_1}(\kappa) \\ \cdots \\ F_{r_n} \ldots F_{r_2} F_{r_1}(\kappa) \end{pmatrix} \to \begin{pmatrix} m \\ m \\ \cdots \\ m \end{pmatrix}^{r_1, \ldots, r_n}$$

for all $m < \omega$.

This is a corollary of the results of Erdős and Rado for polarized relations, see e.g. [11].

Definition 15.3. Let $n, r_1, \ldots, r_n$ be positive integers. An $(r_1, \ldots \ldots, r_n)$-hypergraph is an $r_1 + \ldots + r_n$-tuple system $\mathscr{H}$ whose vertices are partitioned into disjoint classes $V_1, \ldots, V_n$ so that $|H \cap V_i| = r_i$ for all $H \in \mathscr{H}$, $1 \leqslant i \leqslant n$.

Theorem 15.4. *Suppose there is finite* $(r_1, \ldots, r_n)$*-hypergraph* $\mathscr{H}$ *such that* $|\mathscr{H}| = h$ *and* $x \in \bigcup \mathscr{H} \Rightarrow |\{H \in \mathscr{H} : x \in H\}| \geqslant v$. *Assume that* $\begin{pmatrix} \lambda \\ \cdots \\ \lambda \end{pmatrix} \to \begin{pmatrix} m \\ \cdots \\ m \end{pmatrix}_\kappa^{r_1, \ldots, r_n}$ *for every* $m < \omega$, *for some* $\lambda, \kappa \geqslant \aleph_0$. *There is an* $(h, h - v)$*-system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and* $|\mathscr{S}| = \lambda$.

Proof. Let V denote the set of vertices of $\mathscr{H}$ and assume $V = V_1 \cup \ldots \cup V_n$ is a disjoint partition of the vertices establishing the fact that $\mathscr{H}$ is an $(r_1, \ldots, r_n)$-hypergraph. Assume that $|V_i| = s_i$, $|V| = s$ for $1 \leqslant i \leqslant n$. Fix a well-ordering $\prec_1$ of V such that $V_i \prec_1 V_j$ for $i < j$. We now choose disjoint sets $\hat{V}_i$, $|\hat{V}_i| = \lambda$. We fix a well-ordering $\prec_2$ of $\hat{V} = \bigcup_{1 \leqslant i \leqslant n} \hat{V}_i$ such that $\hat{V}_i \prec_2 \hat{V}_j$ for $1 \leqslant i < j \leqslant n$.

Define $W = [\hat{V}_1, \ldots, \hat{V}_n]^{r_1, \ldots, r_n}$. W will be the set of vertices of $\mathscr{S}$

Put $\mathscr{S} = \{X \in [W]^h$. There is a set $Z \in [\hat{V}_1, \ldots, \hat{V}_n]^{s_1, \ldots, s_n}$ such that the unique monotone map of $V, \prec_1$ onto $Z, \prec_2$ maps $\mathscr{H}$ onto $X\}$.

$\mathscr{S}$ is obviously an h-tuple system, $|\mathscr{S}| = \lambda$. Assume $X_1, X_2 \in \mathscr{S}$, $|X_1 \cap X_2| > h - v$. Let Z_1, Z_2 be the unique sets which make X_1 and X_2 belong to $\mathscr{S}$, respectively. If $Z_1 \neq Z_2$ then either $X_1 - X_2$ or $X_2 - X_1$ has $\geqslant v$ elements. Hence $Z_1 = Z_2$ and thus $X_1 = X_2$. This shows that $\mathscr{S}$ is an $(h, h - v)$-system.

Assume that $W = \bigcup_{\nu < \kappa} P_\nu$ is a partition of W into κ classes. Then, by the assumption, there is a set $Z \in [\hat{V}_1, \ldots, \hat{V}_n]^{s_1,\ldots,s_n}$ homogeneous for this partition i.e. there is a $\nu < \kappa$ such that

$$[Z \cap \hat{V}_1, \ldots, Z \cap \hat{V}_n]^{r_1,\ldots,r_n} \subset P_\nu .$$

Hence for the $X \in \mathscr{S}$ determined by this Z we have $X \subset P_\nu$. It follows that $\mathrm{Chr}(\mathscr{S}) > \kappa$.

Corollary 15.5. *Assume* G.C.H. *Suppose there is a finite* $(r_1, \ldots, r_n)$-*hypergraph* $\mathscr{H}$ *such that* $|\mathscr{H}| = h$ *and* $|\{H \in \mathscr{H} : x \in H\}| \geqslant u$. *Then for any ordinal* α, *there is an* $(h, h-u)$-*system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \aleph_\alpha$ *and* $|\mathscr{S}| = \aleph_{\alpha + r_1 + \ldots + r_n}$.

Proof. By Theorem 15.4 and Lemma 15.2.

Theorem 15.6. *Let* n, i, r *be positive integers,* $i < n$ *and let* κ *be an infinite cardinal. If*

$$\binom{\left[\frac{nr}{n-i}\right]}{r} \geqslant n$$

then there is an (n, i)-*system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and* $|\mathscr{S}| = \beth_{r-1}(\kappa)^+$.

Proof. Let $k = \left[\frac{nr}{n-i}\right]$. To prove the theorem, by Theorem 15.4, we only have to define an (r)-hypergraph i.e. an r-tuple system on k vertices such that $\mathscr{H}$ has n-elements and every vertex is contained in at least $n - i$ elements of $\mathscr{H}$. To do this let us first remark that by $\binom{k}{r} \geqslant n$ there exists an r-tuple system $\mathscr{H}_0$ on k with $|\mathscr{H}_0| = n$. If $X \in \mathscr{H}_0$, $u \in X$, $v \notin X$ and $\mathscr{H}'_0 = \mathscr{H}_0 - \{X\} \cup \{X - \{u\} \cup \{v\}\}$, then $\mathscr{H}'_0 = n$. The valency of u in $\mathscr{H}'_0$ is one less, the valency of v in $\mathscr{H}'_0$ is one more than in $\mathscr{H}_0$ the other valencies remain unchanged. Repeating this procedure a finite number of time, applying it for u with maximal valency and v with the minimal one we obtain an $\mathscr{H}$, $|\mathscr{H}| = n$ such that $|\{H \in \mathscr{H} : u \in H\}| - |\{H \in \mathscr{H} : v \in H\}|$, $u, v \in k$ has absolute value at most one. This $\mathscr{H}$ obviously satisfies the requirements.

The following two corollaries are Theorems 2 and 3 of [12] respectively.

Corollary 15.7. *Let* $1 \leqslant i < n < \aleph_0 \leqslant \kappa$, *then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and* $|\mathscr{S}| = \beth_{n-i-1}(\kappa)^+$.

Proof. Choose $r = n - i$ in the previous theorem.

Corollary 15.8. *If* $0 < r < t < \aleph_0 \leqslant \kappa$ *then there is a* $\left(\binom{t}{r}, \binom{t-1}{r}\right)$*-system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and* $|\mathscr{S}| = \beth_{r-1}(\kappa)$.

Proof. Choose $n = \binom{t}{r}$, $i = \binom{t-1}{r}$ in the previous theorem. It is easy to check that the requirements of the theorem hold.

Lemma 15.9. *Let* $1 \leqslant i < n < \aleph_0 \leqslant \delta \leqslant \kappa$, $\lambda \geqslant 1$, $t < \aleph_0$. *If there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \kappa$ *and* $P(\mathscr{S}, \lambda, \delta)$, *then there is an* $(n + t, i + t)$*-system* $\hat{\mathscr{S}}$ *such that* $\hat{\mathscr{S}} = \kappa$ *and* $P(\hat{\mathscr{S}}, \lambda, \delta)$.

Proof. We may assume that the set of vertices is κ. Let $f\colon \mathscr{S} \to \lambda$ establish $P(\mathscr{S}, \lambda, \delta)$. Define

$$\hat{\mathscr{S}} = \{X \cup Y \colon X \in \mathscr{S} \wedge Y \in [\kappa]^t \wedge X < Y\}.$$

Let $\hat{f}(X \cup Y) = f(X)$ for $X \cup Y \in \hat{\mathscr{S}}$. We claim that $\hat{f}$ establishes $P(\hat{\mathscr{S}}, \lambda, \delta)$.

First let $X_1 \cup Y_1$, $X_2 \cup Y_2$ be two elements of $\hat{\mathscr{S}}$. We may assume $\max X_1 \leqslant \max X_2$. Then $X_1 \cap X_2 = \phi$ and $(X_1 \cup Y_1) \cap \cap (X_2 \cup Y_2) \subset (X_1 \cap X_2) \cup Y_1$, hence $\hat{\mathscr{S}}$ is an $(n + t, i + t)$-system. Assume now that $\hat{f}$ fails to establish $P(\hat{\mathscr{S}}, \lambda, \delta)$. Then there is a partition $\kappa = \bigcup_{\nu < \tau} P_\nu$ for $\tau < \delta$ such that for each $\nu < \lambda$ there is a $\rho(\nu) < \lambda$ for which $X \cup Y \in \hat{\mathscr{S}} \wedge X \cup Y \subset P_\nu$ implies $\hat{f}(X \cup Y) \neq \rho(\nu)$.

We now choose $P'_\nu \subset P_\nu$ such that $|P_\nu - P'_\nu| < \aleph_0$ and either $\mathrm{cf}(\mathrm{tp}\, P'_\nu) \geqslant \omega$ or $P'_\nu = \phi$. We can choose $\sigma < \delta$ in such a way that $\kappa = \bigcup_{\nu < \tau} P'_\nu \cup \bigcup_{\nu \subset \sigma} P''_\nu$ where the P''_ν are one element sets. This is a partition of length $< \delta$ of κ. By the definition of $\hat{\mathscr{S}}$ and $\hat{f}(X) \in \hat{\mathscr{S}} \wedge \wedge X \subset P'_\nu$ implies $f(X) \neq \rho(\nu)$. This contradicts the fact that f establishes $P(\mathscr{S}, \lambda, \delta)$.

Theorem 15.10. *If* $1 \leq \binom{m}{2} \leq i < \aleph_0 \leq \kappa$, *then there is an* $(i+m, i)$*-system* $\mathscr{S}$ *such that*

(1) $|\mathscr{S}| = (2^\kappa)^+$ *and* $\mathrm{Chr}(\mathscr{S}) > \kappa$,

(2) $2^\kappa = \kappa^+ \Rightarrow P(\mathscr{S}, \kappa^{++}, \kappa^+)$.

Proof. For the proof of (1) first apply Corollary 15.8 with $t = m + 1$, $r = 2$. We get that there is an $\left(\binom{m+1}{2}, \binom{m}{2}\right)$-system $\mathscr{S}_0$ of cardinality $(2^\kappa)^+$ with $\mathrm{Chr}(\mathscr{S}_0) > \kappa$. Now apply Lemma 15.9 with $t = i - \binom{m}{2}$. Considering $\binom{m+1}{2} = \binom{m}{2} + m$ we get (1).

To prove (2) note first that the $\left(\binom{m+1}{2}, \binom{m}{2}\right)$-system given by Corollary 15.8 can actually be chosen so that the set of vertices of $\mathscr{S}_0$ is $[(2^\kappa)^+]^2$ and $\mathscr{S} = \{[X]^2 : X \subset (2^\kappa)^+ \wedge |X| = m+1\}$. Now it can be seen just as in the proof of Theorem 9.20 (c) that $2^\kappa = \kappa^+$ implies $P(\mathscr{S}_0, \kappa^{++}, \kappa^+)$. We omit the details of this proof. (2) now follows from Lemma 15.9.

Theorem 15.11. *Let* $1 \leq \binom{m}{2} \leq i < n \leq ri + m < \aleph_0 \leq \kappa$, $r \geq 2$. *If* $2^\kappa = \kappa^+$, *then there is an* (n, i)*-system* $\mathscr{S}$ *such that* $|\mathscr{S}| = \beth_{r-1}(\kappa^{++})$ *and* $P^*(\mathscr{S}, \beth_{r-1}(\kappa^{++}), \kappa^+, i)$.

Proof. By induction on r. For $r = 2$ the result follows from Theorem 15.10 and from Lemma 12.1. For $r > 2$ the result follows from the induction hypothesis, and from Lemma 12.1.

Note that Theorem 15.11 is of interest only if $2^\kappa = \kappa^+$ and $2^{\kappa^+} > \kappa^{++}$, otherwise it gives a weaker result than Corollary 12.3.

Lemma 15.12. *Let* $V = V_1 \cup \ldots \cup V_t$, $V_i \cap V_j = \phi$ *for* $i \neq j$. *Suppose* $\mathscr{S} \subset [V]^3$ *satisfies the following conditions:*

(1) *If* $X \in \mathscr{S}$, *then* $|X \cap V_i| = 2$ *and* $|X \cap V_j| = 1$ *for some* i, j *with* $j < i$,

(2) *each element of* $[V_i]^2$ *is contained in at most one element of* $\mathscr{S}$.

(3) *if* $\{a, b, c\}$, $\{a', b', c'\}$ *are distinct elements of* $\mathcal{S}$ *with* $a, b, a', b' \in V_i$ *then* $c \neq c'$.

If $|V| = \binom{m}{2} + k$ *where* $k < m < \aleph_0$, *then* $|\mathcal{S}| \leqslant \binom{m}{3} + \binom{k}{2}$.

Proof. We may assume that the V_i are non-empty. For nonnegative integers define $f(r_1, \dots, r_t) = \sum_{i=1}^{t} \min\left(r_1 + \dots + r_{i-1}, \binom{r_i}{2}\right)$. Then $|\mathcal{S}| \leqslant f(|V_1|, \dots, |V_t|)$. We prove that if $r_1 + \dots + r_t = \binom{m}{2} + k$ then $f(r_1, \dots, r_t) \leqslant \binom{m}{3} + \binom{k}{2}$. Let $m_n = \max\left\{m : \binom{m}{2} \leqslant r\right\}$, $k_r = r - \binom{m_r}{2} < m_r$. It is sufficient to see that if $r = r_1 + \dots + r_t$ then $f(r_1, \dots, r_t) \leqslant \binom{m_r}{2} + \binom{k_r}{2}$ i.e. $f(r_1, \dots, r_t) \leqslant f(1, 2, \dots, m_r - 1, k_r)$. We prove this by induction r. Let $r_1 + \dots + r_{t-1} = \binom{m}{2} + k$, $k < m$.

By the induction hypothesis, $f(r_1, \dots, r_{t-1}) \leqslant f(1, 2, \dots, m-1, k)$ so

$$f(r_1, \dots, r_t) \leqslant f(1, 2, \dots, m-1, k, r_t) = \binom{m}{3} + \binom{k}{2} + \min\left(\binom{m}{2} + k, \binom{r_t}{2}\right).$$

Case 1. $m_r = m$; then $k_r = k + r_t < m$, and $f(r_1, \dots, r_t) \leqslant \binom{m}{3} + \binom{k + r_t}{2} \leqslant \binom{m}{3} + \binom{k}{2} + \binom{r_t}{2} \leqslant \binom{m_r}{3} + \binom{k_r}{2}$.

Case 2. $m_r \geqslant m + 2$; then $f(r_1, \dots, r_t) \leqslant \binom{m}{3} + \binom{k}{2} + \binom{m}{2} + k < \binom{m}{3} + 2\binom{m}{2} + m = \binom{m+2}{3} \leqslant \binom{m_r}{3} \leqslant \binom{m_r}{3} + \binom{k_r}{2}$.

Case 3. $m_r = m + 1$ and $k_r \geqslant k + 1$.

Then $f(r_1, \dots, r_t) \leqslant \binom{m}{3} + \binom{k}{2} + \binom{m}{2} + k = \binom{m+1}{3} + \binom{k+1}{3} \leqslant \binom{m_r}{3} + \binom{k_r}{2}$.

Case 4. $m_r = m+1$ and $k_r \leq k$. Then $k + r_t = m + k_r$ and $k_r \leq k \leq m$, so $\binom{k}{2} + \binom{r_t}{2} \leq \binom{m}{2} + \binom{k_r}{2}$. Now $f(r_1, \ldots, r_k) \leq$ $\leq \binom{m}{3} + \binom{k}{2} + \binom{r_t}{2} \leq \binom{m}{3} + \binom{m}{2} + \binom{k_r}{2} = \binom{m+1}{3} + \binom{k_r}{2} =$ $= \binom{m_r}{3} + \binom{k_r}{2}$.

Theorem 15.13. *For any* $\kappa \geq \aleph_0$, *the triple system* $\mathscr{S} = \{\{\{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}\}: \alpha < \beta < \gamma < (2^\kappa)^+\}$ *has the following properties:*

(1) $|\mathscr{S}| = (2^\kappa)^+$,

(2) $\mathrm{Chr}(\mathscr{S}) > \kappa$,

(3) *If* $n = \binom{m}{2} + k$, $k \leq m < \aleph_0$ *then* $\max\limits_{X \in [\bigcup \mathscr{S}]^n} |\mathscr{S} \cap [X]^3| = \binom{m}{3} + \binom{k}{2} \sim \frac{\sqrt{2}}{3} n^{\frac{3}{2}}$,

(4) *If* $2^\kappa = \kappa^+$, *then* $P(\mathscr{S}, \kappa^{++}, \kappa^+)$.

Proof. (1), (2) and (4) follow from Theorem 15.10 or else from Theorem 9.20, (3) follows from Lemma 15.12. To apply Lemma 15.12, if $X \in [\bigcup \mathscr{S}]^n$ put $T = \bigcup X = \{\alpha_0, \ldots, \alpha_t\}$ and $V_i = \{\{\alpha_j, \alpha_i\}: j < i\}$ for $1 \leq i \leq t$. Note that though Theorem 15.13 gives a weaker upper estimate for $g_3(t, \alpha)$ then Theorem 14.4, the underlying set has "smaller" cardinality in this theorem.

Theorem 15.14. *If* $2 \leq n < \aleph_0 \leq \kappa$ *then there is an* $(n, n - [\sqrt{n}])$-*system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and* $|\mathscr{S}| = \kappa^{++}$.

Proof. Let $\mathscr{H} = K([\sqrt{n}], [\sqrt{n}])$. Then $\mathscr{H}$ is a (1, 1)-hypergraph such that $|\mathscr{H}| = m \leq n$. For each $x \in \bigcup \mathscr{H}$

$$|\{H \in \mathscr{H} : x \in H\}| \geq [\sqrt{n}] .$$

On the other hand it is well-known that $\begin{pmatrix}\kappa^{++}\\ \kappa^{+}\end{pmatrix} \to \begin{pmatrix}\kappa^{++}\\ t\end{pmatrix}^{1,1}_{\kappa}$ holds for all $t < \omega$. As a corollary of Theorem 15.4 there exists an $(m, m - [\sqrt{n}])$-system $\mathscr{S}$, $|\mathscr{S}| = \kappa^{++}$ with $\mathrm{Chr}(\mathscr{S}) = \kappa^{++}$. The result then follows from Lemma 15.9.

Theorem 15.15. *Let* $1 \leqslant n \leqslant 2^r < \aleph_0$. *For any ordinal* α *there is a* $(2n, n)$*-system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \aleph_\alpha$ *and* $|\mathscr{S}| = \aleph_{\alpha+r+1}$.

Proof. Let $V_1, \ldots, V_{r+1}$, be disjoint sets of two elements $V = \bigcup_{i=1}^{r+1} V_i$. Let $\mathscr{K} = [V_1, \ldots, V_{r+1}]^{1,\ldots,1}$. Then $|\mathscr{K}| = 2^{r+1}$. For each n, $1 \leqslant n \leqslant 2^r$ there is an $\mathscr{H} \subset \mathscr{K}$ such that $|\mathscr{H}| = 2n$ and $H \in \mathscr{H} \Rightarrow V - H \in \mathscr{H}$. Then $|\{H \in \mathscr{H} : x \in H\}| = n$ for each $x \in V$. On the other hand $\begin{pmatrix}\aleph_{\alpha+r+1}\\ \cdots\\ \aleph_{\alpha+r+1}\end{pmatrix} \to \begin{pmatrix}m\\ \cdots\\ m\end{pmatrix}^{1,\ldots,1}_{\aleph_\alpha}$ holds for all $m < \omega$. Hence the result follows from Theorem 15.4.

Note that, by Theorem 5.6, this result is best possible in the sense that for any "fixed $\aleph_\beta$" it is consistent to assume that all $(2n, n+1)$-systems of cardinality $\aleph_\beta$ have chromatic number $\leqslant \aleph_0$.

On the other hand there is no counterexample to the following

Problem 11. Is the following statement provable in ZFC? For all n, $3 \leqslant n < \omega$ there is a $(2n, n)$-system of cardinality $\aleph_2$ and of chromatic number $> \aleph_0$. (See the remarks in § 16 for more information.)

Note that 15.15 yields a (6.3)-system of cardinality $\aleph_3$ and chromatic number $> \aleph_0$ and a (12,6)-system of cardinality $\aleph_4$ and of chromatic number $> \aleph_0$. In view of this last remark the following result gives some new information.

Theorem 15.16. *For any* $\kappa \geqslant \aleph_0$, *there is a* $(12, 6)$*-system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and* $|\mathscr{S}| = (2^\kappa)^{++}$.

Proof. Let V_1, V_2 be disjoint sets, $V = V_1 \cup V_2$, $|V_1| = 4$, $|V_2| = 2$. Let $\mathscr{H} = [V_1, V_2]^{2,1}$. Then $\mathscr{H}$ is a $(2, 1)$-hypergraph,

$|\mathscr{H}| = 12$. It is easy to see that $|\{H \in \mathscr{H} : x \in H\}| = 6$ holds for all $x \in V$. On the other hand $\begin{pmatrix} (2^\kappa)^+ \\ (2^\kappa)^{++} \end{pmatrix} \to \begin{pmatrix} m \\ (2^\kappa)^{++} \end{pmatrix}^{2,1}_\kappa$ holds for all $m < \omega$ as an easy consequence of the fact that $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ holds. Hence the theorem follows from Theorem 15.4.

Note, that by Corollary 12.7, we know that there is a $(12, 6)$-system $\mathscr{S}$ with $\mathrm{Chr}(\mathscr{S}) > \kappa$ and cardinality 2^{κ^+}.

Another theorem of similar type is

Theorem 15.17. *For any* $\kappa \geqslant \aleph_0$ *there is a* $(6, 2)$*-system* $\mathscr{S}$ *such that* $\mathrm{Chr}(\mathscr{S}) > \kappa$ *and*

$$|\mathscr{S}| = \min\left(2^{2^{\kappa^+}}, \left(2^{(2^\kappa)^+}\right)^+\right).$$

Proof. The existence of an $\mathscr{S}_1$ with $\mathrm{Chr}(\mathscr{S}_1) > \kappa$, $|\mathscr{S}_1| = 2^{2^{\kappa^+}}$ follows from Corollary 12.7 if we apply it with $i = 2$, $m = 3$, $\kappa = \kappa^+$.

Let now $V = V_1 \cup V_2$, $|V_1| = |V_2| = 3$, $|V| = 6$, $V_1 = \{a_{11}, a_{12}, a_{13}\}$, $V_2 = \{a_{21}, a_{22}, a_{23}\}$. Let $\mathscr{H} = \{[V_1, V_2] - \{a_{1,j}, a_{2,j'}\} : j \neq j'\}$.

Then $|\mathscr{H}| = 6$, and $|\{H \in \mathscr{H} : x \in H\}| \geqslant 4$ for all $x \in V$. $\mathscr{H}$ is a $(2, 2)$-hypergraph. On the other hand it is easy to see that

$$\begin{pmatrix} \left(2^{(2^\kappa)^+}\right)^+ \\ (2^\kappa)^+ \end{pmatrix} \to \begin{pmatrix} m \\ m \end{pmatrix}^{2,2}_\kappa$$

holds for $\omega \leqslant \omega$ because of $(2^\lambda)^+ \to (\lambda^+)^2$, for $\lambda \geqslant \aleph_0$. It follows from Theorem 15.4 that there is a (6, 2)-system $\mathscr{S}_2$ with $|\mathscr{S}_2| = \left(2^{(2^\kappa)^+}\right)^+$ and $\mathrm{Chr}(\mathscr{S}_2) > \kappa$.

Note, that, Corollary 2.2 if $\kappa \geqslant \aleph_0$ and $\mathscr{S}$ is a $(6, 2)$-system with $\mathrm{Chr}(\mathscr{S}) > \kappa$, then $|\mathscr{S}| \geqslant \kappa^{+++}$. If G.C.H. is true, then $2^{2^{\kappa^+}} = \kappa^{+++}$, $\left(2^{(2^\kappa)^+}\right)^+ = \kappa^{++++}$, but it is easy to see that G.C.H. can be violated so that the second number is smaller than the first one.

To close this chapter we prove one more general result which can yield partial results similar to the ones obtained before.

Theorem 15.18. *Let* $1 \leq i_1 < n_1 \leq \aleph_0$, $1 \leq i_2 < n_2 \leq \aleph_0$. Suppose *there exist an* (n_1, i_1)*-system* $\mathcal{S}_1$ *with* $\mathrm{Chr}(\mathcal{S}_1) > \kappa$ *and* $|\mathcal{S}_1| = \alpha$ *and an* (n_2, i_2)*-system* $\mathcal{S}_2$ *with* $\mathrm{Chr}(\mathcal{S}_2) > \alpha$ *and* $|\mathcal{S}_2| = \beta$. *Let* $n = n_1, n_2$, $i = \max(n_1 i_2, n_2 i_1)$. *Then there is an* (n, i)*-system* $\mathcal{S}$ *with* $\mathrm{Chr}(\mathcal{S}) > \kappa$ *and* $|\mathcal{S}| = \beta$.

Proof. $\mathcal{S} = \{X_1 \times X_2 : X_1 \in \mathcal{S}_1 \wedge X_2 \in \mathcal{S}_2\}$ satisfies the requirements. We omit the details.

§16. DISCUSSION OF SOME RESULTS CONCERNING THE SIZES OF (n, i)-SYSTEMS WITH LARGE CHROMATIC NUMBER. PROBLEMS

Definition 16.1. Let $1 \leq i \leq n < \aleph_0$ and let κ, λ be cardinals. Let $\Theta(n, i, \kappa, \lambda)$ hold iff there is an (n, i)-system $\mathcal{S}$ such that $\mathrm{Chr}(\mathcal{S}) \geq \kappa$ and $|\mathcal{S}| = \lambda$.

We avoided the uses of this symbol up to now for two reasons. First we wanted the paper to be easy to read an second most of the results concerning the relation Θ contain additional information which we wanted to state. Note that using this symbol Theorem 15.8 says

$$\Theta(n_1, i_1, \kappa^+, \lambda) \wedge \Theta(n_2, i_2, \lambda^+, \tau) \Rightarrow$$
$$\Rightarrow \Theta(n_1 \cdot n_2, \max\{n_1 i_2, n_2 i_1\}, \kappa^+, \tau).$$

The simplest instance of the problems which remain unsolved is if it is a theorem of ZFC that $\Theta(6, 3, \aleph_1, \aleph_2)$ holds. This was already stated in Problem 11. on p. 154.

Here is a list of informations concerning this type of problems.

(a) $\neg\Theta(6, 2, \aleph_1, \aleph_2)$, $\neg\Theta(6, 3, \aleph_2, \aleph_2)$ and $\neg\Theta(6, 3, \aleph_1, \aleph_1)$ follow from Corollary 2.2.

(b) ZFC $\not\vdash$ $\Theta(7, 3, \aleph_1, \aleph_2)$ by Theorem 5.6.

(c) $\Theta(5, 3, \aleph_1, \aleph_2)$ and $\Theta(6, 4, \aleph_1, \aleph_2)$ hold by Theorem 15.14.

(d) $\Theta(6, 3, \aleph_0, \aleph_0)$ holds by Theorem 12.2.

(e) $\Theta(6, 3, \aleph_1, \aleph_3)$ holds by Theorem 15.15.

(f) $\Theta(6, 3, \aleph_1, (2^{\aleph_0})^+)$ holds by Corollary 15.8 (because $\binom{4}{2} = 6$, $\binom{3}{2} = 3$).

(g) $\Theta(6, 3, \aleph_1, 2^{\aleph_1})$ holds by Corollary 12.7.

(h) By Lemma 15.9 we have $\Theta(5, 2, \aleph_1, \aleph_2) \Rightarrow \Theta(6, 3, \aleph_1, \aleph_2) \Rightarrow \Rightarrow \Theta(7, 4, \aleph_1, \aleph_2) \Rightarrow \Theta(8, 5, \aleph_1, \aleph_2) \Rightarrow \Theta(9, 6, \aleph_1, \aleph_2)$.

Note that $ZFC \not\vdash \Theta(5, 2, \aleph_1, \aleph_2)$ by Theorem 5.6, and $\Theta(9, 6, \aleph_1, \aleph_2)$ by Theorem 15.14. The following remains open:

Problem 12. Does $ZFC \vdash \Theta(8, 5, \aleph_1, \aleph_2)$?

Or $ZFC \vdash \Theta(7, 4, \aleph_1, \aleph_2)$?

Let us now recapitulate our knowledge about $\Theta(3, 1, \aleph_1, \lambda)$.

(a) $\Theta(3, 1, \aleph_1, \lambda) \Rightarrow \lambda \geqslant \aleph_2$ by Corollary 2.2.

(b) If MA holds, then $\neg\Theta(3, 1, \aleph_1, \lambda)$ holds for $\lambda < 2^{\aleph_0}$, by Theorem 5.6.

(c) $\Theta(3, 1, \aleph_1, (2^{\aleph_0})^+)$ holds by Corollary 15.8.

(d) Con (ZF) $\Rightarrow$ Con (ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \Theta(3, 1, \aleph_1, 2^{\aleph_0})$) by Theorem 12.3.

Problem 13. Is $\neg\Theta(3, 1, \aleph_1, 2^{\aleph_0})$ consistent with $2^{\aleph_0} > \aleph_1$? or

Is $\neg\Theta(3, 1, \aleph_1, 2^{\aleph_1})$ consistent with ZFC? Note that $\neg\Theta(3, 1, \aleph_1, 2^{\aleph_1}) \Rightarrow 2^{\aleph_0} = 2^{\aleph_1} \cap \omega_1 \to [\text{stationary subset of } \omega_1]^2_{\aleph_0}$.

Let us now turn to the problem of $\Theta(4, 1, \aleph_1, \lambda)$.

(a) $\Theta(4, 1, \aleph_1, \lambda) \Rightarrow \lambda \geqslant \aleph_3$ by Corollary 2.2.

(b) $MA \Rightarrow \neg\Theta(4, 1, \aleph_1, 2^{\aleph_0})$ by Theorem 5.7.

(c) $\Theta(4, 1, \aleph_1, (2^{2^{\aleph_0}})^+)$ holds by Corollary 15.7.

(d) $2^\kappa = \kappa^+ \Rightarrow \Theta(4, 1, \kappa^+, 2^{\kappa^{++}})$ by Theorem 15.11 hence C.H. $\Rightarrow$ $\Rightarrow \Theta(4, 1, \aleph_1, 2^{\aleph_2})$.

(e) $\Theta(4, 1, \aleph_1, 2^{2^{\mathrm{cf}(2^{\aleph_0})}})$ holds by Theorem 12.10.

(f) Con (ZF) $\Rightarrow$ Con (ZFC + $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2 + \Theta(4, 1, \aleph_1, 2^{2^{\aleph_0}}) + 2^{2^{\aleph_0}} = \aleph_3$). This follows from Theorem 12.13 since there we may assume $2^{\aleph_2} = \aleph_3$ as well.

Of course there are many problems not answered by these results. Here is one of them.

Problem 14. Does ZFC $\vdash \Theta(4, 1, \aleph_1, (2^{\aleph_0})^{++})$?

The case of $\Theta(4, 2, \aleph_1, \lambda)$ is completely settled since $\neg\Theta(4, 2, \aleph_1, \aleph_1)$ by Corollary 2.2 and by Theorem 15.15, $\Theta(4, 2, \aleph_1, \aleph_2)$ holds.

Finally we say a few words about $\Theta(5, 2, \aleph_1, \lambda)$. It was one of the problems of [12] (see p. 7) if G.C.H. $\Rightarrow \Theta(5, 2, \aleph_1, \aleph_2)$. We now know this

(a) $\Theta(5, 2, \aleph_1, \lambda) \Rightarrow \lambda \geqslant \aleph_2$ by Corollary 2.2.

(b) $MA \Rightarrow \neg\Theta(5, 2, \aleph_1, \lambda)$ for all $\lambda < 2^{\aleph_0}$.

(c) $\Theta(5, 2, \aleph_1, 2^{\aleph_1})$ holds by Theorem 12.11.

Problem 15. Does ZFC $\vdash \Theta(5, 2, \aleph_1, (2^{\aleph_0})^+)$?

***Added in proof.** Theorem 11.13 can be sharpened as follows. Under the assumptions of 11.13 there exists a κ^+-chromatic triple system $\mathcal{S}$ on κ^+ such that any n points contain at most $\left[\frac{n^2}{4}\right]$ triples for all $n < \omega$. To prove this take S to be the triple system given by Lemma 11.12 such that $\mathcal{G}_1(\mathcal{S})$, $\mathcal{G}_2(\mathcal{S})$ do not contain triangles. To compute that n points contain at most $\left[\frac{n^2}{8}\right]$ triples of $\mathcal{S}$ one can use the following lemma. *Assume* $\mathcal{G}$ *is a graph with set of vertices* n *and* x_i; $i < n$ *are nonnegative real numbers with* $\sum_{i<n} x_i = n$, *then* $\sum_{\{x_i, x_j\} \in \mathcal{G}} x_i x_j \leqslant \left[\frac{n^2}{4}\right]$.
See P.S. Motzkin, E.G. Straus, Maxima for graphs and a new proof of a Theorem of Turán, Canadian Journal of Math., 17 (1965), 533-540.

LIST OF SPECIAL TRIPLE SYSTEMS DEFINED IN THE PAPER

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