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ON MAXIMAL ALMOST-DISJOINT FAMILIES OVER SINGULAR CARDINALS

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1. INTRODUCTION

For any infinite cardinal κ define a family $\mathscr{F}$ of sets to be a *κ-almost-disjoint family* (κ-ADF) iff $\bigcup \mathscr{F}$ has cardinality κ, each member of $\mathscr{F}$ has cardinality κ, and the intersection of any two distinct members of $\mathscr{F}$ has cardinality strictly less than κ. Define such a family to be a *κ-maximal almost-disjoint family* (κ-MADF) iff for every set $S \subset \bigcup \mathscr{F}$ of cardinality κ there exists a set $F \in \mathscr{F}$ whose intersection with S has cardinality κ. It is well-known (and easily seen) that if κ has cofinality $\lambda \leqslant \kappa$, then any family of fewer than λ disjoint sets each of cardinality κ is a κ-MADF while *no* family of cardinality λ can be a κ-MADF. Thus for regular cardinals κ there do not exist κ-MADFs of cardinality κ. In a private communication W. Wistar Comfort asked if, however, for singular cardinals κ there exist κ-MADFs of cardinality κ. We shall show that under certain conditions the answer is yes, but we do not know if the answer is ever no.

Since finite sets and families present no real problems to us, we shall always understand the term *cardinal* to mean *infinite* cardinal. As usual, the *cofinality* of a cardinal κ is defined to be the smallest cardinal λ such that there exists a family of cardinality λ each of whose members has cardinality less than κ and whose union has cardinality κ. A cardinal κ is *regular* if it has cofinality κ and *singular* otherwise. Finally, if κ has cofinality λ, then we call a sequence $\{\kappa_\alpha : \alpha < \lambda\}$ of cardinals a *κ-sequence* iff it is strictly increasing and it has supremum κ. We shall denote the cardinality of a set S by $|S|$, the set of functions from sets S into T by ${}^S T$, and the smallest cardinal greater than a given cardinal κ by κ^+.

We shall assume the axioms of Zermelo – Fraenkel set theory including choice throughout, and when we deal with consistency proofs, we shall denote this system by ZFC.

2. THEOREMS

We begin with an easy lemma whose proof is left to the reader.

2.1. Lemma. *Let κ be any infinite cardinal with cofinality $\lambda < \kappa$, let $\{\kappa_\alpha : \alpha < \lambda\}$ be any κ-sequence, and let $\mathcal{S} = \{S_\alpha : \alpha < \lambda\}$ be any family of disjoint sets such that $|S_\alpha| = \kappa_\alpha$ for all $\alpha < \lambda$. Then for any set $T \subseteq \bigcup \mathcal{S}$ of cardinality κ there exists a strictly increasing function $f \in {}^\lambda\lambda$ such that for all $\alpha < \lambda$*

$$|T \cap S_{f(\alpha)}| \geqslant \kappa_\alpha ,$$

We shall also need a lemma which appears as Theorem 3.1 in [2] in a slightly less general form. The proofs are identical.

2.2. Lemma. *Let δ be any cardinal, let κ be any singular cardinal of cofinality λ, and let $\{\kappa_\alpha : \alpha < \lambda\}$ be any κ-sequence. Then if for each $\alpha < \lambda$ there exists a δ-*MADF *of cardinality κ_α, there exists a δ-*MADF *of cardinality κ.*

To obtain κ-MADFs of cardinality κ where κ has cofinality λ

we use two different constructions (2.3 and 2.8) depending upon whether 2^λ is less than or greater than κ. We note that in both cases our constructions will give us some positive information independent of the size of 2^λ. We begin with the construction we will use when 2^λ is less than κ.

2.3. Theorem. *For every singular cardinal κ of cofinality λ and every cardinal $\mu < \kappa$ there exists a κ-MADF of cardinality δ where $\mu \leqslant \delta \leqslant \mu^\lambda$.*

Proof. Let $\{\kappa_\alpha : \alpha < \lambda\}$ be any κ-sequence of regular cardinals each greater than μ, let

$$\mathscr{S} = \{S^\alpha_\beta : \alpha < \lambda,\ \beta < \mu\}$$

be any family of disjoint sets such that $|S^\alpha_\beta| = \kappa_\alpha$, and let

$$\mathscr{G} = \bigcup \{{}^L\mu : L \subseteq \lambda,\ |L| = \lambda\} .$$

Then we may choose a family $\mathscr{F} \subseteq \mathscr{G}$ such that:

1. $f, g \in \mathscr{F} \to [\,|\{\alpha : f(\alpha) = g(\alpha)\}| < \lambda \vee f = g]$,

2. $g \in \mathscr{G} \to \exists f \in \mathscr{F}\,(|\{\alpha : f(\alpha) = g(\alpha)\}| = \lambda)$.

Clearly, $\mathscr{G}$ has cardinality μ^λ, and $\mathscr{F}$ cannot have cardinality less than μ. Set $\delta = |\mathscr{F}|$, and for each $f \in \mathscr{F}$ set

$$S_f = \bigcup \{S^\alpha_{f(\alpha)} : \alpha \text{ is in the domain of } f\} .$$

Then it is not hard to see that the family

$$\mathscr{E} = \{S_f : f \in \mathscr{F}\}$$

is a κ-ADF of cardinality δ. In fact, the only problem is to show that $\mathscr{E}$ is maximal. Thus let G be any subset of $\bigcup \mathscr{S}$ of cardinality κ. Then for each $\alpha < \lambda$ let $T^\alpha = \bigcup_{\beta < \mu} S^\alpha_\beta$ and let $\mathscr{T} = \{T^\alpha : \alpha < \lambda\}$. Now apply 2.1 to $\mathscr{T}$ and G. Then there exists a strictly increasing function $g \in$ $\in {}^\lambda\lambda$ such that

$$|G \cap T^{g(\alpha)}| \geqslant \kappa_\alpha .$$

But each κ_α is regular and greater than μ, so there must exist a function $i \in {}^\lambda\mu$ such that

$$|G \cap S^{g(\alpha)}_{i(\alpha)}| \geqslant \kappa_\alpha .$$

Let L be the range of g. We "normalize" i by looking at it as a function from L into μ as follows. For each $\alpha \in L$ set

$$j(\alpha) = i(g^{-1}(\alpha)) \quad \text{and} \quad h(\alpha) = g^{-1}(\alpha) .$$

The function h is strictly increasing with limit κ, L has cardinality λ, and for each $\alpha \in L$ we now have

$$|G \cap S^{\alpha}_{j(\alpha)}| \geqslant \kappa_{h(\alpha)} .$$

But condition 2 on $\mathscr{F}$ assures us of the existence of a function $f \in \mathscr{F}$ such that

$$|\{\alpha\colon f(\alpha) = i(\alpha)\}| = \lambda ,$$

and from this it follows that $|S_f \cap G| = \lambda$.

From this we obtain:

2.4. Theorem. *If κ is any singular cardinal of cofinality λ, and $\mu^\lambda < \kappa$ for every cardinal $\mu < \kappa$, then there exists a κ-MADF of cardinality κ.*

Proof. First apply 2.3 to obtain a κ-sequence $\{\delta_\alpha\colon \alpha < \lambda\}$ such that for each $\alpha < \lambda$ there exists a κ-MADF of cardinality δ_α, and then use 2.2.

A cardinal κ is defined to be a *strong limit cardinal* iff $\mu < \kappa$ *implies* $2^\mu < \kappa$. We see immediately:

2.5. Corollary. *If κ is any singular strong limit cardinal, then there exist κ-MADFs of cardinality κ.*

Thus, since the generalized continuum hypothesis implies that every singular cardinal is a strong limit cardinal, we have:

2.6. Corollary. *The generalized continuum hypothesis implies that for any cardinal* κ *there exists a* κ-MADF *of cardinality* κ *iff* κ *is singular.*

Finally, in another direction we have:

2.7. Corollary. *If the continuum has cardinality less than* $\aleph_\omega$, *then for every cardinal* δ *such that* $2^{\aleph_0} \leqslant \delta \leqslant \aleph_\omega$ *there exists an* $\aleph_\omega$-MADF *of cardinality* δ.

Proof. It is well-known that any cardinal δ satisfying $2^{\aleph_0} \leqslant \delta < \aleph_\omega$ also satisfies $\delta^{\aleph_0} = \delta$.

This last corollary shows us that 2.5 is strictly weaker than 2.4. In particular, we see that the continuum hypothesis implies the existence of an $\aleph_\omega$-MADF of cardinality $\aleph_\omega$. On the other hand, Solovay [1], [4] has shown that it is consistent with the continuum hypothesis that $2^{\aleph_1} = \aleph_{\omega+1}$ i.e. that $\aleph_\omega$ not be a strong limit cardinal.

We now go to the construction which we shall need to deal with the case $2^\lambda > \kappa$.

2.8. Theorem. *If* κ *is any singular cardinal of cofinality* λ, *and there exists a* λ-MADF *of cardinality* μ, *then there exists a* κ-MADF *of cardinality* μ.

Proof. Let $\{\kappa_\alpha : \alpha < \lambda\}$ be an κ-sequence, let $\mathscr{S} = \{S_\alpha : \alpha < \lambda\}$ be any sequence of disjoint sets such that each S_α has cardinality κ_α, and let $\mathscr{F} = \{F_\alpha \subseteq \lambda : \alpha < \mu\}$ be any λ-MADF. Next, for each $\alpha < \mu$ define

$$G_\alpha = \bigcup \{S_\beta : \beta \in F_\alpha\}, \quad \text{and}$$

$$\mathscr{G} = \{G_\alpha : \alpha < \mu\}.$$

It is easily seen that $\mathscr{G}$ is a κ-ADF, so let T be any subset of $\bigcup \mathscr{G}$ of cardinality κ. Then T is, of course, a subset of $\bigcup \mathscr{S}$, and we may use 2.1 to obtain a strictly increasing function $f \in {}^\lambda\lambda$ such that for ıt for every $\alpha < \mu$

$$|T \cap S_{f(\alpha)}| \geq \kappa_\alpha .$$

Let $F \subseteq \lambda$ be the range of f. Then since $\mathscr{F}$ is maximal, there exists a set $F_\alpha \in \mathscr{F}$ such that $|F \cap F_\alpha| = \lambda$. This, however, implies that $f^{-1}[F_\alpha \cap F]$ has cardinality λ, and, therefore, that $|T \cap G_\alpha| = \kappa$.

Applying this, we have:

2.9. Theorem. *If* κ *is any singular cardinal of cofinality* λ, *then it is consistent with* ZFC *that there exist* κ-MADFs *of every cardinality* $\mu \leq 2^\lambda$ *except* $\mu = \lambda$.

Proof. For $\mu \leq \lambda$ the result follows immediately from our introductory remarks. For $\lambda < \mu$ it is sufficient by 2.2 to consider only the case μ regular, and by 2.8 we may consider λ-MADFs rather than κ-MADFs. But λ is, of course, regular, and it is easily seen that the construction (using Cohen forcing) used in the proof of Theorem 3.2 in [2] can be used with only the most obvious modifications to handle this case.

2.10. Corollary. *If* κ *is any singular cardinal of cofinality* λ, *then it is consistent with* ZFC *that* κ *be less than* 2^λ *and that there exist* κ-MADFs *of cardinality* κ.

Proof. We first use Solovay's [1], [4] construction to obtain a model in which $\kappa < 2^\lambda$, and then we apply 2.9 noting that the construction involved does not affect the size of 2^λ.

Finally, for each singular cardinal κ of cofinality λ consider the hypothesis:

$\mathscr{H}_\kappa$. There exist κ-MADFs of every cardinality $\mu \leq \kappa$ except $\mu = \lambda$. We note that it follows immediately from 2.2 and 2.3 that the generalized continuum hypothesis implies that $\mathscr{H}_\kappa$ holds for every singular cardinal κ. In fact, for a given cardinal κ of cofinality λ it is sufficient that for every cardinal μ such that $\lambda \leq \mu < \kappa$ we have $2^\mu = \mu^+$. At the other extreme, it follows from 2.9 that it is at least consistent with ZFC that $\mathscr{H}_\kappa$ hold when 2^λ is greater than κ. The intermediate case is more delicate. If we have $\kappa > 2^\lambda > \lambda^+$, and for all cardinals μ such that

$2^\lambda < \mu < \kappa$ we have $2^\mu = \mu^+$, then we can use the construction mentioned in the proof of 2.9 to obtain the desired κ-MADFs of cardinality $\mu \leqslant 2^\lambda$. Furthermore, since this construction does not disturb the cardinalities of the power sets of λ or cardinals greater than λ, we will continue to have $2^\mu = \mu^+$ for cardinals $2^\lambda < \mu < \kappa$, and, therefore, by 2.2 and 2.3, the remaining desired κ-MADFs.

3. OPEN PROBLEMS

The major open problem is, of course, to determine if it is consistent with ZFC that there exist a singular cardinal κ for which there do not exist κ-MADFs of cardinality κ. However, there are also some related problems which are also of some interest.

1. Does $\mathscr{H}_\kappa$ hold for every singular cardinal κ? If not, can anything be said about the smallest κ-MADF or any of the "missing" κ-MADFs? In particular, does any partial converse of 2.2 hold?

2. It is known [3] that Martin's Axiom implies that every infinite $\aleph_0$-MADF has cardinality $2^{\aleph_0}$. Does Martin's Axiom also imply that every $\aleph_\omega$-MADF have cardinality no less than $2^{\aleph_0}$? Since Martin's Axiom is known [5] to be consistent with ZFC + $2^{\aleph_0} > \aleph_\omega$, an affirmative answer would settle our main problem.

3. Is there any analogue of 2.7 for cardinals of cofinality greater than ω?

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