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Factorizing the Complete Graph into Factors with Large Star Number

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The graph G has star number n if any n vertices of G belong to a subgraph which is a star. Let $f(n, k)$ be the smallest number m such that the complete graph on m vertices can be factorized into k factors with star number n. In the present paper we prove that $c_1^n k \leqslant f(n, k) < c_2^n k$.

INTRODUCTION

If G is a graph such that for any set S of n vertices of G there exists a subgraph H of G which is a star and $S \subset V(H)$ then G has *star number* (st.) n. The set of graphs $\{F_1, F_2, ..., F_k\}$ is a *decomposition* of G into the *factors* $F_1, F_2, ..., F_k$ if $V(F_i) = V(G)$ $(1 \leqslant i \leqslant k)$, $E(F_i) \cap E(F_j) = \varnothing$ for all $i \neq j$, and $\bigcup_{i=1}^{k} E(F_i) = E(G)$. Let $f(n, k)$ be the smallest number m such that the complete graph on m vertices can be decomposed into k factors of st. n. This problem or various specializations of it have so far been investigated in [1–6]. The best results are

$f(2, 2) = 5, f(2, 3) = 12$ or 13;

$6k - 52 \leqslant f(2, k) \leqslant 6k$ and various better lower bounds for $f(2, k)$ if $k \leqslant 370$ [6];

because $f(n, k) \geqslant f(2, k)$ $(n \geqslant 2)$ we get $f(n, k) \geqslant 6k - 52$; $f(n, k) \leqslant \binom{n^2+1}{n}k$ [5].

In the present paper we prove that

$$\tfrac{1}{2}(\tfrac{3}{2})^n\, k \leqslant f(n,k) \leqslant cn^2 2^n k < c_1(2+\epsilon)^n\, k. \qquad (1)$$

We observe that the upper bound in (1) is better than the upper bound in [5].

The Upper Bound

We will show that if $m > cn^2 2^n k$ then the complete graph on m vertices can be factorized into k factors with st. n using the following

LEMMA. *Given n, the edges of a complete bipartite graph with vertex-set $A \cup B$, $A = \{a_1, a_2, ..., a_\nu\}$, $B = \{b_1, b_2, ..., b_\nu\}$ can be colored with two colors α, β in such a way that for any choice of $C \subset A$ and $D \subset B$, $|C| + |D| \leqslant n$ there exists an index t such that a_t is joined to all vertices of D by edges of color α, and b_t is joined to all vertices of C by edges of color β, provided ν is large enough; in fact $\nu > cn^2 2^n$ is sufficient.*

In [5] this lemma has been proved with $\nu = \binom{n^2+1}{n}$ by an elaborate construction. It is remarkable that we can prove a better result with simple probabilistic methods.

Proof. Let us color all the edges of the bipartite graph with color α with probability $\frac{1}{2}$, and with color β otherwise. Such a coloring is *bad*, if it does not satisfy the conditions of the lemma. The probability p of having a bad coloring satisfies

$$p < \binom{2\nu}{n}\left(1 - \left(\frac{1}{2}\right)^n\right)^\nu$$

since there would exist an n-tuple in $A \cup B$ containing $C \cup D$ that for no index t would be properly joined to a_t and b_t. So $p < (2\nu)^n\, e^{-\nu/2^n}$ and

$$p < 1 \qquad \text{for} \quad 2^n \nu^n < e^{\nu/2^n}.$$

This means there exists a good coloring if $\nu/2^n > n(\log \nu + \log 2)$: But for this

$$\nu > cn^2 2^n$$

is sufficient, c being a sufficiently large constant.

In order to establish the proposed upper bound for $f(n, k)$ we partition the $m = k\nu$ vertices of a complete graph into k parts $K_1, K_2, ..., K_K$ with ν vertices each, where $\nu > cn^2 2^n$. By the choice of ν, we have a complete bipartite graph colored according to the lemma. Denote the vertices of K_i

by v_{ir} ($r = 1, 2, \ldots, \nu$). Now we color the edges of the complete graph on m vertices with k colors as follows:

(1) All edges between vertices of K_i are colored with color i ($i = 1, 2, \ldots, k$).

(2) An edge connecting the vertex v_{ir} in K_i with the vertex v_{js} in K_j ($i < j$) is colored with color i (resp. j) iff the edge between a_r and b_s is colored α (resp. β).

As in [5], this coloring defines a factorization of the complete graph on $m = k\nu > cn^2 2^n k$ vertices into k factors with star number n. Indeed: Given any color i and any n-tuple S of vertices v_{jr}, let $C = \{a_r; v_{jr} \in S, j < i\}$ and $D = \{b_s; v_{js} \in S, j > i\}$. Noting that $|C| + |D| \leqslant n$ we apply the lemma to obtain an index t such that, according to our coloring, v_{it} is joined to all vertices of both, $\{v_{jr}; v_{jr} \in S, j < i\}$ and $\{v_{js}; v_{js} \in S, j > i\}$ by edges of color i. That v_{it} is joined to v_{ir} ($v_{ir} \in S$) by an edge of color i is immediate.

The Lower Bound

We want to prove that

$$\tfrac{1}{2}(\tfrac{3}{2})^n\, k \leqslant f(n, k). \tag{2}$$

Suppose that the edges of the complete graph on m vertices are colored with k colors such that st. n holds for all colors. For a proof of (2) we distinguish 2 cases.

(I) There exist $k/2 = t$ among the k colors, say the colors $1, 2, 3, \ldots, t$, such that none of the vertices is adjacent with more than $\frac{2}{3}m$ edges having one of the colors 1 or 2 or 3 or... or t. If x is a vertex then we denote by $v_{i_1, i_2, \ldots, i_s}(x)$ the number of edges adjacent to x and having one of the colors $i_1, i_2, \ldots, i_s$. Case I therefore means that $v(x) := v_{1,2,\ldots,t}(x) < \frac{2}{3}m$. Using st. n for the colors $1, 2, \ldots, t$ we evidently have

$$\frac{k}{2}\binom{m}{n} \leqslant \sum_{i=1}^{m}\binom{v(x_i)}{n} < m\binom{\frac{2}{3}m}{n} < m\left(\frac{2}{3}\right)^n\binom{m}{n}.$$

Hence $t < m(\frac{3}{2})^n$, i.e.: $m > \frac{1}{2}(\frac{3}{2})^n\, k$.

(II) Let $v_{t+1, t+2, \ldots, k}(x) = w(x)$.

Case II means, then, that there exists a point x_1 such that $v(x_1) \geqslant \frac{2}{3}m$ and a point x_2 such that $w(x_2) \geqslant \frac{2}{3}m$. Denote the set of points which are connected to x_1 with edges of one of the colors $1, 2, \ldots, t$ by A and the set of points which are connected to x_2 with edges of one of the colors

$t+1, t+2, \ldots, k$ by B. ($|A| \geqslant \frac{2}{3}m - 2$, $|B| \geqslant \frac{2}{3}m - 2$ and therefore $|A \cup B - A \cap B| \leqslant \frac{2}{3}(m-2)$). Let $S = (A \cup B - A \cap B) \cup \{x_1\} \cup \{x_2\}$.

We will prove that the coloring of the edges in S is such that each factor has st.$(n-1)$. Let $y_1, y_2, \ldots, y_{n-1}$ be any $(n-1)$-tupel of points in S and i some color. Assume without restriction of generality that $1 \leqslant i \leqslant t$. Consider the n-tupel $y_1, y_2, \ldots, y_{n-1}, x_2$. There exists some point z which is connected to all of them with edges of color i. z cannot be a point in $A \cap B$ because no point in B is connected to x_2 by an edge with color i for $i \leqslant t$. Hence $z \in S$ and connected to all the points $y_1, y_2, \ldots, y_{n-1}$ with edges having color i. Which means that the graph of color i with vertex set S has st.$(n-1)$.

Therefore

$$f(n-1, k) \leqslant |S| \leqslant \tfrac{2}{3}(m-2) + 2$$

and

$$f(n, k) \geqslant \tfrac{3}{2} f(n-1, k) - 1.$$

Since $f(2, k) \geqslant 2k + 1 > \frac{1}{2}(\frac{3}{2})^2 k$, we have by induction $m \geqslant \frac{1}{2}(\frac{3}{2})^n k$ in this case also.

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