# EDGE DECOMPOSITIONS OF THE COMPLETE GRAPH INTO COPIES OF A CONNECTED SUBGRAPH

by

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ABSTRACT

Necessary conditions on $(n,k,\ell)$ are given for the possibility to partition the edges of the complete graph $K_n$ into isomorphic copies of a connected subgraph $G_{k,\ell}$ with $k$ vertices and $\ell$ edges. It is conjectured that these conditions are sufficient if $n$ is large. This conjecture is proved for $k = 2$, 3 and 4.

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## INTRODUCTION

Let $n$ be a natural integer and let $K$ be a set of naturals.

One of the most central problems of combinatorial theory is to determine those parameters $(n,K)$ for which there exist a set $E$ of $n$ elements and a collection $B$ of subsets of $E$, having cardinalities which are elements of $K$, and such that every pair-subset of $E$ is a subset of exactly one member of $B$. If the mentioned configuration exists it will be denoted, following Hanani [5], by $n \in B[K]$ and if $K$ consists of a single value $k$ then $B[\{k\}]$ will be shortened to $B[k]$.

It is known [8] that $n \in B[K]$ only if

(1) $n - 1 \equiv 0 \pmod{\alpha(K)}$, $n(n-1) \equiv 0 \pmod{\beta(K)}$ where $\alpha(K) = gcd\{k-1 | k \in K\}, \beta(K) = gcd\{k(k-1) | k \in k\}$. For $K = \{k\}$ these reduce to

(2) $n - 1 \equiv 0 \pmod{k}$, $n(n-1) \equiv 0 \pmod{k(k-1)}$.

It is also well known that conditions (1) are not sufficient in general but it is contained in a recent remarkable Existence Theorem of Wilson [8], that conditions (1) are sufficient for $n$ sufficien large.

The above combinatorial problem translated in terms of graphs is the problem to determine those parameters $(n,k)$ for which the edges of the complete graph $K_n$ may be partitioned into edge disjoint $K_k$ subgraphs of $K_n$.

The more general problem to partition the edges of $K_n$ into subgraphs being isomorphic copies of some connected graph $G_{k,\ell}$ having $k$ vertices and $\ell$ edges has been approached for $G_{k,\ell}$ being a $k$-cycle by Kotzig [6] and separately by Rosa [7] , for bipartite graphs by Beineke [2] and in general very recently by Wilson [9] .

If a partition as above is possible we will denote it by $n \in B^*[G_{k,\ell}]$.

Then the analogous of the necessary condition (2) becomes

(3) $n - 1 \equiv 0 \pmod{d}$, $n(n-1) \equiv 0 \pmod{2\ell}$ where $d$ is the *gcd* of the degrees of the vertices of $G_{k,\ell}$.

Clearly conditions (3) are not sufficient in general. The following conjecture generalizes an earlier similar conjecture, since a theorem, of Wilson [8]:

Generalized Existence Conjecture.

Given $G_{k,\ell}$ there exists a constant $C = C(G_{k,\ell})$ such that for every $n$ satisfying congruences (3) and $n \geq C$ it holds $n \in B^*(G_{k,\ell})$.

This conjecture in other words says that the necessary conditions (3) are also sufficient if $n$ is large enough. The intuitive meaning of it is that if no obvious reason contradicts it, for ex. divisibility conditions, then for sufficiently large $n$ the edges of $K_n$ may be partitioned into copies of $G_{k,\ell}$ subgraphs of $K_n$.

In section 2 we will prove the above conjecture for $k = 2$, 3 and 4.

The effort to obtain this result involving only small values of $\ell$ seems to be justified, although in [9] a sufficient condition for large $n$ and every $\ell$ is given, because on the other hand the condition in [9] is not necessary.

## 2. Proof of the Generalized Existence Conjecture for $2 \le k \le 4$.

In the proof of the theorem below the following two lemmas are used.

Lemma 1. If $w \in B[K]$ and if $v \in B^*[G]$ whenever $v \in K$ then $w \in B^*[G]$.

The proof of Lemma 1 is so simple that it can be ommited.

Lemma 2. If

(4) $s \in B^*[G]$, $t \in B^*[G]$

and if

(5) $gcd(s-1,t-1) = 1$, $gcd(s(s-1),t(t-1)) = 2\ell$

then there exist a constant $C = C(G)$ such that $n \in B^*[G]$ provided $n \ge C$ and $n$ satisfies

(6) $n(n-1) \equiv 0 \pmod{2\ell}$.

Proof of Lemma 2.

By (5) and the Extended Existence theorem of Wilson, condition (6) is sufficient to have $n \in B[\{s,t\}]$. The statement follows then from (4) and Lemma 1.

THEOREM. The Generalized Existence Conjecture is valid for $2 \le k \le 4$.

Proof.

In five out of the total of nine cases the theorem is true since in each of them a stronger theorem holds than the Generalized Existence Theorem. Namely,

Case 1. $k = 2$, then $\ell = 1$ and clearly for every $n \ge 2$, $n \in B^*(G_{2,1})$.

Case 2. $k = 3$, $\ell = 3$. Condition (3) becomes $n-1 \equiv 0 \pmod 2$, $n(n-1) \equiv 0 \pmod 6$. These conditions are known to be sufficient for $n \in B^*(G_{3,3})$. This is the Steiner triple case.

Case 3. $k = 3$, $\ell = 2$. Condition (3) reduces to $n(n-1) \equiv 0 \pmod 4$ i.e. $n \equiv 0$ or $1 \pmod 4$. $n \equiv 0 \pmod 4$ turns out to be sufficient since $4 \in B^*(G_{3,2})$ and if $n = 4t$ the induction on $t$ completes the proof. $n \equiv 1 \pmod 4$ is also sufficient since then $K_n$ is an Euler graph, the circuit containing an even number of edges and hence may be broken into paths of length two, so $n \in B^*(G_{3,2})$.

Case 4. $k = 4$, $\ell = 6$. Condition (3) becomes $n-1 \equiv 0 \pmod 3$, $n(n-1) \equiv 0 \pmod{12}$. These conditions have been proved by Hanani [4] to be sufficient for $n \in B^*(G_{4,6})$. In this case $G_{4,6} = K_4$.

Case 5. $k = 4$, $\ell = 4$, $G_{4,4}$ being a circle. Condition (3) is then $n(n-1) \equiv 0 \pmod 8$ $n-1 \equiv 0 \pmod 2$ being sufficient for $n \in B^*(G_{4,4})$ as shown by Kotzig [6].

In each of the remaining cases we will provide values $s$ and $t$ satisfying (4) and (5) with the corresponding $G_{k,\ell}$ instead of $G$, hence Lemma 2 applies and the theorem follows.

Case 6. $k = 4$, $\ell = 5$, then $G_{k,\ell}$ is a quadrilateral with a diagonal. In this case $s = 10$ and $t = 11$. Indeed, $10 \in B^*(G_{4,5})$ since, denoting $\overset{\frown}{abcd}$ the quadrilateral $abcd$ with diagonal $bd$ , the required partitioning of the edges of $K_{10}$ is as follows:

$$\begin{array}{ll} 1\overset{\frown}{32}5 & 0\overset{\frown}{29}1 \\ 1\overset{\frown}{62}4 & 0\overset{\frown}{49}3 \\ 8\overset{\frown}{57}4 & 0\overset{\frown}{69}5 \\ 7\overset{\frown}{38}6 & 7\overset{\frown}{98}0 \ . \\ 1\overset{\frown}{82}7 & \end{array}$$

Similarly eleven shifts (mod 11) of $4\overset{\frown}{01}9$ obtained adding 1 (mod 11) to each digit in each step form a required partition of the edges of $K_{11}$. It is easy to check conditions (5).

**Case 7.** $k = 4$, $\ell = 4$, $G_{4,4}$ being a triangle with attached edge. Denoting by $ab\overline{cd}$ a triangle $abc$ with attached edge $cd$, the following two partitions show that $s = 8$ and $t = 9$ are as required in (4)

| | |
|---|---|
| $12\overline{48}$ | $47\overline{12}$ |
| $23\overline{58}$ | $58\overline{23}$ |
| $34\overline{68}$ | $36\overline{97}$ |
| $45\overline{78}$ | $59\overline{13}$ |
| $56\overline{18}$ | $38\overline{46}$ |
| $67\overline{28}$ | $26\overline{78}$ |
| $71\overline{38}$ | $37\overline{56}$ |
| | $29\overline{45}$ |
| | $61\overline{89}$ |

Condition (5) is again easily verified.

Case 8. $k = 4$, $\ell = 3$, $G_{4,3}$ a star. Then $s = 6$ and $t = 7$ verify (5). Denoting by $\hat{a}bcd$ a star with central vertex $a$, the following partition shows that $s = 6$ is as required in (4):

$$\hat{1}236\ ,\quad \hat{2}346\ ,\quad \hat{3}456\ ,\quad \hat{4}516\ ,\quad \hat{5}126$$

The former stars together with the stars $\hat{0}123$, $\hat{0}456$ provide a partition showing that also $t = 7$ is as required.

**Case 9.** $k = 4$, $\ell = 3$, $G_{4,3}$ a path. Then $s = 6$ $t = 13$ verify (5). The paths 1234 , 1362 , 1425 , 1546 , 1653 show that $s = 6$ is as required in (4). For $t = 13$ observe that $4 \in B^*(G_{4,3})$ namely 1234 , 2413 is a partition as claimed and $13 \in B[4]$ as well known [5].

Since the established theorem is of asymptotical nature, and the constant $C$ may be very large, it would be of interest for $G_{4,\ell}$ as in cases 6, 7, 8 and 9, namely when $G_{4,\ell}$ is quadrilateral with diagonal, an attached triangle, a star or a path to determine as large sets for $n \in B^*(G_{4,\ell})$ as possible.

Remark added after the seminar.

Mention here that the sets $B^*(G_{4,\ell})$, excepted in case 6, are at present completely determined. Namely case 8 was solved by P. Cain (Bull. Austral. Math. Soc. 10 (1974), 23-30). In case 9 the partial solution of S. Hung and N. Mendelsohn (Notices A.M.S. 20 (1973), 254-255) has been completed by C. Huang and case 7 has been solved jointly by C. Huang and J. Schönheim.

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