If b is the term with no prime factor exceeding 3, there are also six possibilities.

7. $2^2 \cdot 3 | b \Rightarrow c = p^x$, $d = 2q^y \Rightarrow (c, d)$ satisfies $[p, q, 2]$.
8. $b = 2 \cdot 3^y \Rightarrow a = p^x \Rightarrow (a, b)$ satisfies $[p, 3, 2]$.
9. $b = 2^x$, $3 \nmid a \Rightarrow a = q^y \Rightarrow (b, a)$ satisfies $[2, q, 1]$.
10. $b = 2^x$, $3 | a \Rightarrow c = q^y \Rightarrow (b, c)$ satisfies $[2, q, 1]$.
11. $b = 3^x$, $2^1 \| a \Rightarrow a = 2q^y \Rightarrow (b\ a)$ satisfies $[3, q, 2]$.
12. $b = 3^x$, $2^2 | a \Rightarrow c = 2q^y \Rightarrow (b, c)$ satisfies $[3, q, 2]$.

By section 4, every possibility requires S to include a term with a prime factor exceeding 11, which is forbidden. Thus S does not exist. ∎

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AN EXTREMAL PROBLEM OF GRAPHS WITH DIAMETER 2

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Let $1 \leqq k < p$. We say that a graph has property $P(p, k)$ if it has p points and every two of its points are joined by at least k paths of length $\leqq 2$. The aim of this note is to discuss the following problem. At least how many edges are in a graph with property $P(p, k)$? Denote this minimum by $m(p, k)$.

Construct a graph $G_0(p, k)$ with property $P(p, k)$ as follows. Take two classes of points, k in the first class and $p - k$ in the second, and take all the edges incident with at least one point in the first class. Thus $G_0(p, k)$ has $\binom{p}{2} - \binom{p-k}{2}$ edges.

Murty [2] proved that if $p \geqq \frac{1}{2}(3 + \sqrt{5})k$ then $m(p, k) = \binom{p}{2} - \binom{p-k}{2}$ and $G_0(p, k)$ is the only graph with property $P(p, k)$ that has $m(p, k)$ edges. He also suspected that the same result holds already for $p > 2k$. We shall show that this is not so, in fact $p \geqq \frac{1}{2}(3 + \sqrt{5})k$ is almost necessary for $G_0(p, k)$ to be an extremal graph, and we determine the asymptotic value of $m([ck], k)$ for every constant $1 < c < \frac{1}{2}(3 + \sqrt{5})$, where $[x]$ denotes the integer part of x.

THEOREM. *Let* $1 < c < \frac{1}{2}(3+\sqrt{5})$, $p = [ck]$. *Then* $m(p,k) = c^{3/2}k^2/2 + o(k^2)$.

Proof. Exactly as in [2] (or by a simple counting argument) one can show that

$$m(p,k) \geqq c^{3/2}k^2/2 + O(k).$$

Therefore the problem is to prove an upper bound for $m(p,k)$, i.e., to construct graphs with property $P(p,k)$ that have few edges.

Let $\varepsilon > 0$. Take $p = [ck]$ points and choose each edge with probability $d = c^{-\frac{1}{2}} + \varepsilon$. The law of large numbers implies that, as $k \to \infty$, with probability tending to 1, this graph $G_1(p,k)$ has $\binom{p}{2}(d + o(1))$ edges. Also, by another simple application of the law of large numbers, we obtain that with probability tending to 1 for every two of the points there are $(d^2 + o(1))p$ points joined to both of them. Thus as $p \to \infty$ with probability tending to 1 this graph $G_1(p,k)$ has property $P(p,k)$ and it has $\leqq (d^2 + \varepsilon)\binom{p}{2}$ edges, proving the required inequality.

If the reader is not familiar with the probabilistic terminology or does not like it, we suggest the following combinatorial translation.

Consider all graphs on a set V of p labelled points having $\binom{p}{2} d = q$ edges. The number of these graphs is $\binom{Q}{q}$, where $Q = \binom{p}{2}$. Let a, b be two arbitrary points and let $x < k$ be an integer. Let us compute the number of graphs in which there are exactly x points joined to both a and b. If there are x points joined to both a and b, there are y points in $V - \{a, b\}$ joined to a and there are z points in $V - \{a, b\}$ joined to b; then the edges incident with exactly one of the points a, b can be chosen in

$$\binom{p-2}{x}\binom{p-2-x}{y}\binom{p-2-x-y}{z}$$

different ways. The remaining edges of the graph can be chosen in $\binom{Q'}{q-e}$ ways, where $Q' = \binom{p-2}{2} + 1$ and $e = 2x + y + z$. Consequently the number of graphs in question is

$$\sum_{x+y+z \leqq p-2} \binom{p-2}{x}\binom{p-2-x}{y}\binom{p-2-x-y}{z}\binom{Q'}{q-e},$$

where the summation goes over all pairs of nonnegative integers (y, z) satisfying $x + y + z \leqq p - 2$. Thus there are at most

$$\binom{n}{2} \sum_{x+y+z \leqq p-2} \binom{p-2}{x}\binom{p-2-x}{y}\binom{p-2-x-y}{z}\binom{Q'}{q-e}$$

graphs not having property $P(p,k)$. By a simple but laborious estimation one can prove that if k is sufficiently large then this is less than $\binom{Q}{q}$ (in fact the sum divided by $\binom{Q}{q}$ tends to zero as $k \to \infty$). This proves that if k is sufficiently large there exists a graph with q edges that has property $P(p, k)$.

REMARKS. 1. With a slight improvement of the same method one can prove that if

$$[ck] = p < \frac{3+\sqrt{5}}{2}\, k - k^{\frac{1}{2}}(\log k)^{\alpha}$$

(α sufficiently large) then $m(p, k) = c^{3/2}k^2/2 + o(k^2)$ and *the graph* $G_0(p, k)$ *is not extremal.*

A problem similar to the one discussed here and in [2] was solved in [1]. By the method applied there one could improve the result in [2] slightly. One could show that $G_0(p, k)$ is extremal in a larger range than $p \geqq \frac{1}{2}(3+\sqrt{5})k$, but the method would not bring the lower bound on p down to $\frac{1}{2}(3+\sqrt{5})k - k^{\frac{1}{2}}(\log k)^{\alpha}$.

It would be of interest to determine as accurately as possible the smallest value $p = p(k)$ for which the graph $G_0(p, k)$ is extremal. Furthermore in the range where $G_0(p, k)$ is not extremal determine (again as accurately as possible) $m(p, k)$ and characterize the extremal graphs.

2. One can also give a nonprobabilistic proof of the theorem. As before, let

$$1 < c < \tfrac{1}{2}(3+\sqrt{5}), \qquad \varepsilon > 0, \qquad d = c^{-\frac{1}{2}} + \varepsilon.$$

Furthermore, let p be a natural number and $\alpha = \alpha(p)$ a real number. Denote by $G_1(p, \alpha, d)$ the following graph. The points are $\{1, 2, \cdots, p\}$, and i is joined to j if

$$(i-j)^2\alpha - [(i-j)^2\alpha] < d.$$

It suffices to show that $\alpha = \alpha(p)$ can be chosen in a such a way that if p is sufficiently large $G_1(p, \alpha, d)$ has property $P(p, k)$ and has $\frac{1}{2}dn^2 + o(n^2)$ edges. It indeed follows from well-known theorems on diophantine approximation that $G_1(p, \alpha, d)$ has $\frac{1}{2}dp^2 + o(p^2)$ edges, provided α is irrational. The graph has property $P(p, k)$ if whenever $1 \leqq i < j \leqq p$, the number of integers t, $1 \leqq t \leqq p$, for which

$$(t-i)^2\alpha - [(t-i)^2\alpha] < d \quad \text{and} \quad (t-j)^2\alpha - [(t-j)^2\alpha] < d,$$

is $d^2p + o(p)$ uniformly in i and j. (For sufficiently large p clearly $d^2p + o(p) > k$.) We could not prove this but Cassels showed that this holds if we choose $\alpha = \alpha(p) = 1/q$, where q is the smallest prime not less than p. The proof uses analytic number theory and will not be given here. The same choice of α also ensures that G_1 has $\frac{1}{2}dp^2 + o(p^2)$ edges. This result completes the proof of the theorem.

It would still be of interest to prove the result for every irrational α.

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