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Paul Erdös, S. K. Zaremba

THE ARITHMETIC FUNCTION $\sum_{d|n} \frac{\log d}{d}$

The need to examine the asymptotic behaviour of the arithmetic function

$$\sum_{d|n} \frac{\log d}{d},$$

which we shall denote by $\phi(n)$, arose in connection with work on good lattice points modulo composite numbers (see [2]). Obviously,

$$\liminf_{n \to \infty} \phi(n) = 0 ;$$

the purpose of the present note is to prove the following:

Theorem.

$$\limsup_{n \to \infty} (\log \log n)^{-2} \phi(n) = e^{\gamma},$$

where γ is the Euler constant.

Proof. Let

$$n = \prod_{i=1}^{r} p_i^{\alpha(i)} ,$$

where $p_1, \dots, p_r$ are distinct primes, and $\alpha(1), \dots, \alpha(r)$ are positive integers. Then

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$$(1) \qquad \$(n) = \sum_{i=1}^{r} \sum_{\nu=1}^{\alpha(i)} \frac{\nu \log p_i}{p_i^{\nu}} \sum_{d|(p_i^{-\nu} n)} \frac{1}{d} .$$

Hence

$$(2) \qquad \$(n) < \left(\sum_{i=1}^{r} \frac{\log p_i}{p_i} + \sum_{i=1}^{r} \log p_i \sum_{\nu=2}^{\infty} \frac{\nu}{p_i^{\nu}} \right) \sum_{d|n} \frac{1}{d} .$$

But if, as usual, $\sigma(n)$ denotes the sum of all the divisors of n, we have (see, for instance, Theorem 323 in [1])

$$(3) \quad \limsup_{n \to \infty} (\log \log n)^{-1} \sum_{d|n} \frac{1}{d} = \limsup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^{\gamma}.$$

On the other hand,

$$(4) \qquad \sum_{\nu=2}^{\infty} \frac{\nu}{p_i^{\nu}} = \frac{1}{p_i^2} \left(\frac{p_i}{p_i - 1} + \frac{p_i^2}{(p_i - 1)^2} \right) \leqslant \frac{6}{p_i^2} ,$$

and consequently

$$(5) \qquad \sum_{i=1}^{r} \log p_i \sum_{\nu=2}^{\infty} \frac{\nu}{p_i^{\nu}} \leqslant 6 \sum_{i=1}^{r} \frac{\log p_i}{p_i^2} = O(1).$$

Except for the trivial case when $i = 1$, $p_i = 3$, if r is given, an upper bound for the sum

$$\sum_{i=1}^{r} \frac{\log p_i}{p_i}$$

is obtained by assuming that $p_1, \ldots, p_r$ are the first r consecutive primes. But any n bigger than 6 has fewer than $\log n$ distinct prime factors. Thus $r < \log n$, and by the prime number theorem, $p_r \sim r \log r$. Hence there exists a constant A such that if $r > 1$,

$$p_r \leq Ar \log r < A \log n \log \log n \ .$$

On the other hand (see, for instance, Theorem 425 in [1]),

$$(6) \qquad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Here and in what follows, p is a generic symbol for a prime. In particular,

$$\sum_{i=1}^{r} \frac{\log p_i}{p_i} < \log \log n + \log \log \log n + \log A + O(1),$$

and therefore

$$(7) \qquad \limsup_{n \to \infty} (\log \log n)^{-1} \sum_{i=1}^{r} \frac{\log p_i}{p_i} \leq 1.$$

In view of (2), according to (3), (5), and (7), we find

$$(8) \qquad \limsup_{n \to \infty} (\log \log n)^{-2} \ s(n) \leq e^{\gamma} \ .$$

In order to prove the inverse inequality, note that for the sequence

$$n_j = \prod_{p \leq e^j} p^j$$

(see, for instance, the proof of Theorem 323 in [1]), we have

$$(9) \qquad \lim_{j \to \infty} (\log \log n_j)^{-1} \sum_{d | n_j} \frac{1}{d} = e^{\gamma} .$$

But, according to (1), if the prime factors of n_j are $p_1, \ldots, p_r$,

$$s(n_j) \geqslant \sum_{i=1}^{r} \frac{\log p_i}{p_i} \sum_{d|(p_i^{-1}n_j)} \frac{1}{d},$$

and since

$$\sum_{d|(p_i^{-1}n_j)} \frac{1}{d} \geqslant \sum_{d|n_j} \frac{1}{d} - \frac{1}{p_i} \sum_{d|n_j} \frac{1}{d},$$

we have

$$\text{(10)} \quad \frac{s(n_j)}{(\log\log n_j)^2} \geqslant \frac{\sum_{i=1}^{r} \frac{\log p_i}{p_i} - \sum_{i=1}^{r} \frac{\log p_i}{p_i^2}}{\log\log n_j} \cdot \frac{\sum_{d|n_j} \frac{1}{d}}{\log\log n_j}.$$

The limit of the second fraction in the right-hand side of this inequality is given by (9). To determine the limit of the first fraction, we note that $\log n_j = j\,\vartheta(e^j)$, where

$$\vartheta(x) = \sum_{p \leqslant x} \log p.$$

But (see, for instance, Theorem 414 in [1]), $\vartheta(x)$ is exactly of the order of x. Consequently,

$$\log\log n_j = j + \log j + O(1),$$

and further

$$\frac{1}{\log\log n_j} \sum_{i=1}^{r} \frac{\log p_i}{p_i} = \frac{1}{j} \sum_{p \leqslant e^j} \frac{\log p}{p} \cdot \frac{j}{j + \log j + O(1)}.$$

The first factor in the last expression tends to 1 according to (6), and so obviously does the second factor. Hence, owing to (5),

$$\lim_{j \to \infty} (\log \log n_j)^{-1} \left(\sum_{i=1}^{r} \frac{\log p_i}{p_i} - \sum_{i=1}^{r} \frac{\log p_i}{p_i^2} \right) = 1.$$

In view of (10), combining the last result with (9), we find

$$\liminf_{n \to \infty} (\log \log n_j)^{-2} \, s(n_j) \geq e^{\gamma} .$$

Together with (8), this concludes the proof.

It can be proved that $s(n)$ has a continuous purely singular distribution function. In other terms, the density $\psi(c)$ of integers for which $s(n) > c$ exists and is a continuous strictly increasing purely singular function (see a forthcoming paper by P. Erdős and R. R. Hall in the Journal of Number Theory).

REFERENCES

[1] G.H. H a r d y, E.M. W r i g h t: An introduction to the theory of numbers. Fourth Edition. Oxford 1960.

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MATHEMATICS RESEARCH CENTER, UNIVERSITY OF WISCONSIN-MADISON, MADISON Wis. 53706, U.S.A.