The Annals of Probability
1974, Vol. 2, No. 5, 828-839

SOME PROBLEMS ON RANDOM INTERVALS AND ANNIHILATING PARTICLES[1]

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Particles perform independent random walks on the integers, and are annihilated if they cross paths or land at the same point. The problem is to determine whether the origin is hit infinitely often. The answer is shown to depend on the initial distribution of particles in accordance with a "log log law." Several equivalent models are mentioned.

1. An annihilating particle model. Let us start with a particle at each integer point on the line except 0. Let the particles perform independent simple random walks, moving a unit step to the right or left with probability $\frac{1}{2}$ at each unit of time. If two particles cross each other's paths, or if they are about to land at a common point they "annihilate" each other (i.e., are removed from the game before landing).

QUESTION 1. *Is*

$$P\{\text{the origin is ever hit}\} = 1\,?$$

We conjecture that the answer is yes, but do not have a proof.

One can consider a number of variants of the above model. The particles can move to the left or right with probabilities $p \neq \frac{1}{2}$ and $1 - p$. The walk can be *one sided*, with particles taking a step to the left with probability p and remaining in place with probability $1 - p$. In this case annihilation occurs if a particle moves toward an occupied point, in which case both the occupying and moving particle are removed prior to the landing. The initial distribution of particles can also be varied. The question 1 is open in all these cases, but the model suggests another problem about which we can say something.

2. Two types of particles. Suppose we start with *two* types of particles, say black and white (b and w), arranged on some subset of the integers according to some initial distribution so that the colors alternate. Everything is as in the previous model, but now the probabilities of movement of the b and w particles are different. Thus, for example, in the one sided case white (resp. black) particles would move independently one step to the left with probability p (resp. q), and remain in place with probability $1 - p$ (resp. $1 - q$). Annihilation of particles occurs exactly as before, but observe that it always takes place btween

Received May 11, 1973; revised January 4, 1974.

[1] Research supported in part by the N.S.F. under Grant Number GP-33991X.

AMS 1970 *subject classifications.* 60K35, 60C05.

Key words and phrases. Random walk, interacting particle processes, random intervals, combinatorial probability.

neighboring pairs of particles of *opposite* colors. Hence the colors of particles will continue to alternate. (If $p = q$ this model is the same as the one-type particle case, so we here only consider $p \neq q$.)

QUESTION 2. *How many particles reach the origin*?

We will see that the answer depends on the initial distribution of particles, which is most easily described in terms of yet a third model.

3. Random intervals. Consider the unit segments $\{[n, n+1); n = 0, \pm 1, \cdots\}$, and color the nth white or black with probabilities p_n, and $1 - p_n$ respectively. The line is thus partitioned into black (B) and white (W) intervals. At each unit of time, a white interval may grow by one unit (at the expense of its neighbor) by extending its left end point by one unit with probability p, and similarly a black interval may extend its left end with probability q. (They remain unchanged with probabilities $1 - p$ and $1 - q$ respectively.)

If we think of a white (black) particle as placed at the left end of a white (black) interval, and let the particle move as in the one sided model in paragraph 2 above, then the two models are identical. A white (black) interval stretches from a white (black) particle to the next particle on its *right* (necessarily of the opposite color). An annihilation of two particles corresponds to an interval shrinking to length 0, and its neighbors fusing into one long interval of the same color. Thus we have a picture of intervals growing or shrinking, moving to the left, possibly vanishing or fusing together. (A similar analog to the two sided walk, with interval growing in either direction, can easily be defined.)

QUESTION 2′. *How many intervals cross the origin*?

4. Markov lattice models. The above models bear some resemblance to special cases of Markov lattice models with local transition probabilities, which have attracted some attention, recently. Suppose each unit interval (or locus) on the line starts at time zero in one of two states—say 0 or 1. Each locus remains in its present state for an exponentially distributed random time (these are independent), and then makes a transition to 0 or 1 with probabilities depending on the current states of the locus in question and of its neighbors. Make the special assumption that a 0-state with two 0-neighbors, will with probability one make a transition into 0; and the same for 1's. Thus changes of state cannot occur at the interior of a run of 0's or 1's, but only at the boundary. Will a particular locus change state infinitely often, or will it ultimately remain in a single state?

This is a continuous time analog of the interval model of Section 3, with a new interval reaching the origin corresponding to a change of state there. Although we have not worked out the details of this continuous time model, it seems quite clear that the conclusions of the theorem below also hold for this case.

5. Conclusions. Consider the *one sided*, 2 particle type model of paragraph 2, or the equivalent random interval model; and let the initial distribution be determined by $\{p_j = P([j, j+1) \text{ is white}), n = 0, 1, \cdots\}$. (In this model we need consider only $j \geqq 0$, since no particles from the left can reach 0.) Suppose, for definiteness, that $p > q$.

THEOREM. (i) *If*

$$p_j \leqq \frac{a \log \log j}{j}, \qquad j \geqq 2, \text{ any } a < \frac{q}{p}, \tag{1a}$$

but $\sum p_n = \infty$, *then with probability* 1 *infinitely many particles (intervals) reach the origin.*

(ii) *There exists a* $b < \infty$ *such that if*

$$p_j \geqq \frac{b \log \log j}{j}, \qquad j \geqq 2, \tag{1b}$$

then with probability 1, *at most finitely many particles (intervals) reach the origin.*

In the *two sided* case, black and white particles take independent steps of size $+1, 0, -1$ with probabilities q^+, q^0, q^- and p^+, p^0, p^- respectively. Denote the means by $\bar{b} = q^+ - q^-$, $\bar{w} = p^+ - p^-$. If $\bar{b}$ and $\bar{w}$ are of opposite sign, then it is easy to show that only finitely many particles will cross 0. If $\bar{w} < \bar{b} < 0$ then the same conclusion holds as for the one sided case in the theorem above.[2] If $0 < \bar{b} < \bar{w}$ then the same statement holds, but with p_j replaced by $1 - p_j$. Analogous conclusions also hold for the remaining cases $0 < \bar{w} < \bar{b}$ and $\bar{b} < \bar{w} < 0$.

The proofs for the above cases are similar to the one sided case, so we shall only treat the latter. In fact it seems clear that the proof extends to the case when black and white particles perform arbitrary random walks with *different* means. In the case of equal means one encounters similar difficulties to those of the problem in Section 1.

6. Proofs for the one sided case.[3]

PROOF OF (i). Sometimes in the proof it will be convenient to talk in terms of moving black or white intervals; and sometimes it will be easier to argue in terms of the black or white particles which sit at the boundaries of the intervals. We will feel free to shift back and forth between these vocabularies, as it is convenient.

Consider the intervals

$$J_{k,\delta}^{(n)} = [(1+\delta)^k n, (1+\delta)^{k+1} n), \qquad k = 0, 1, 2, \cdots,$$

where

$$p/q < \delta < 1/a,$$

[2] In the two sided case replace j by $|j|$ on the right sides of (1).

[3] K's and c's with or without subscripts or primes denote constants, not necessarily the same ones each time they appear.

and where n is fixed but will ultimately be chosen large; and let $B_k^{(n)}$ = the event that all unit segments in $J_k^{(n)}$ are black. (Adopt the convention that $[c, d)$ means the interval $[\langle c\rangle, \langle d\rangle)$, where $\langle x\rangle$ = the largest integer in x.)

Now if

$$p_j \leqq \frac{a \log \log j}{j}, \qquad j = 0, 1, 2, \cdots$$

then

$$\begin{aligned} P(B_0^{(n)}) &= \textstyle\prod_{j=n}^{(1+\delta)n} (1 - p_j) \\ &\geqq \left\{1 - \frac{a \log \log n}{n}\right\}^{\delta n} \\ &\geqq \frac{K}{(\log n)^{a\delta}}. \end{aligned}$$

Hence, substituting $(1 + \delta)^k n$ for n in the above

$$P(B_k^{(n)}) \geqq \frac{K}{k^{a\delta}[\log (1 + \delta)n]^{a\delta}}.$$

Since $0 < a\delta < 1$ (by choice of δ) we have

$$\textstyle\sum_k P(B_k^{(n)}) = \infty,$$

and since the events are independent we conclude by the Borel–Cantelli lemma that infinitely many $B_k^{(n)}$'s occur. Furthermore, since $\sum p_j = \infty$, there will be white intervals somewhere to the right of *every* black interval. Hence, with probability 1, there will be infinitely many disjoint black intervals $J_k^{(n)*}$, where k is taken from some infinite subsequence of $\{1, 2, \cdots\}$, such that $J_k^{(n)} \subseteqq J_k^{(n)*}$ (by disjoint we mean separated by white intervals.)

Consider one of these intervals $J_k^{(n)*}$, with a black particle b^* at its left end, and a white particle w^* at its right end; and let x_{b^*} and x_{w^*} denote the locations of these end points. Since $J_k^{(n)} \subseteqq J_k^{(n)*}$, we must have

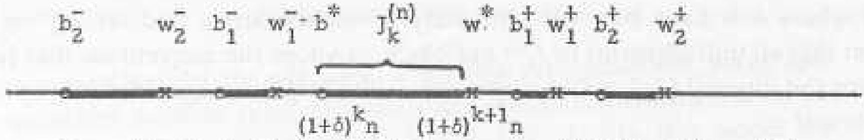
$$0 < x_{b^*} \leqq (1 + \delta)^k n < (1 + \delta)^{k+1} n \leqq x_{w^*} < \infty.$$

(If $x_{b^*} = 0$, consider $J_{k+1}^{(n)*}$ instead.)

We want to estimate $P_{k,n}^*$, the probability that the particle b^* reaches 0 *before* undergoing collision with any other particle. Suppose we modify our initial distribution by moving b^* from x_{b^*} to $(1 + \delta)^k n$, and w^* from x_{w^*} to $(1 + \delta)^{k+1} n$, while leaving all other particles unchanged. This is equivalent to shrinking the black interval $J_k^{(n)*} = [x_{b^*}, x_{w^*})$ to a black interval $J_k^{(n)} = [(1 + \delta)^k n, (1 + \delta)^{k+1})$, or to changing the colors of all unit segments in the set $J_k^* - J_k$ from black to white. Let $P_{k,n}^{(1)}$ denote the probability that with this new initial location, b^* reaches 0 before undergoing collision. It is clear from a sample path comparison of the process with the above two initial distributions, that

$$P_{k,n}^{(1)} \leqq P_{k,n}^*.$$

The situation in the *modified* initial distribution looks as in Fig. 1, in which



Heavy line denotes black and light line denotes white (Figs. 1 through 5).

FIG. 1.

FIG. 2.

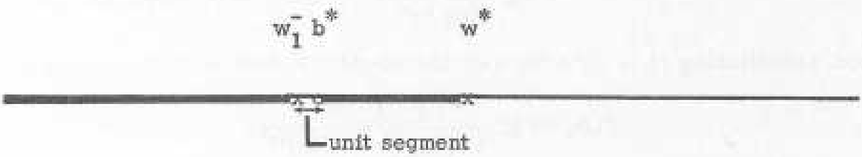


FIG. 3.

particles have been numbered to the left and right of b^* and w^*; o's denote the location of black particles, and x's of white.

Figure 2 is obtained from 1 by coloring black all unit segments to the left of w_1^-, and coloring white all those to the right of w^*. Let $P^{(2)}_{k,n}$ denote the probability that b^* reaches 0 before collision with the initial pattern in Fig. 2, and let us compare $P^{(1)}$ and $P^{(2)}$. There are two ways that b^* can fail to reach 0: (a) by virture of collision with a white particle on its left, (b) by virtue of a collision with a white particle on its right. Case (a) is equivalent to the black interval $J_k^{(n)}$ coming in contact with a black interval on its left. Whenever such an event occurs with an initial distribution as in Fig. 1, then a comparison of sample paths shows that it must also occur with the pattern in 2. Such a comparison also shows that if case (b) occurs in Fig. 1 it must also occur in Fig. 2. Hence

$$P^{(2)}_{k,n} \leqq P^{(1)}_{k,n} .$$

Finally, we modify the initial pattern from Fig. 2 to Fig. 3, by moving the left white neighbor w_1^- from its position at $x_{w_1^-}$, rightward to the point $((1+\delta)^k n - 1)$, and let $P^{(3)}_{k,n}$ be the corresponding probability that b^* reaches 0 before collision. Clearly $P^{(3)}_{k,n} \leqq P^{(2)}_{k,n}$, and hence, combining the above inequalities

$$P^{(3)}_{k,n} \leqq P^*_{k,n} . \tag{2}$$

Now $P^{(3)}_{k,n}$ can be described in terms of random walks.

Let $\{X_i; i = 1, 2, \cdots\}$ and $\{Y_i; i = 1, 2, \cdots\}$ be independent random variables with

$$P\{X_i = -1\} = 1 - P\{X_i = 0\} = p ,$$

and

$$P\{Y_i = -1\} = 1 - P(Y_i = 0\} = q ,$$

with $p > q$ being the same constants as specified before. Let $S_0 = X_0 = X_0(k) = (1+\delta)^{k+1}n$, and $T_0 = Y_0 = Y_0(k) = (1+\delta)^k n$,

$$S_j = S_j(k) = \textstyle\sum_{i=0}^j X_i\,; \qquad T_j = T_j(k) = \sum_{i=0}^j Y_i\,;$$
$$\hat{S}_j = S_j - X_0\,; \qquad \hat{T}_j = T_j - Y_0\,.$$

Let $\{X_i'\}$ be a process distributed as $\{X_i\}$, independent of $\{X_i\}$ and $\{Y_i\}$ and with $X_0' = Y_0 - 1$; and with the corresponding sums denoted by primes. Finally, let $N = N(k) =$ the 1st hitting time of 0 by $T_j(k)$, and define the event

$$\mathscr{E}^{(n)}(k) = \{S_j(k) > T_j(k) > S_j'(k), j = 1, \cdots, N\}\,.$$

(We suppress the dependence on n in the S's and T's.)

LEMMA. *There exists an $n_0 < \infty$, and a $\theta_0 > 0$ such that for all $n \geqq n_0$ and $k \geqq 1$, $P[\mathscr{E}^{(n)}(k)] > \theta_0$.*

PROOF.

$$\begin{aligned} P[\mathscr{E}^{(n)}(k)] &= \textstyle\sum_{j=1}^\infty P\{\mathscr{E}^{(n)}(k) \mid N = j\}P\{N = j\} \\ &\geqq \textstyle\sum_{j=1}^K P\{\mathscr{E}^{(n)}(k) \mid N = j\}P\{N = j\}\,, \end{aligned}$$

where we take $K = \langle (1+\varepsilon)T_0/q \rangle$, $\varepsilon > 0$ to be chosen later,

$$\begin{aligned} &\geqq P\{N \leqq K\}P\{S_j > T_j > S_j'; j = 1, \cdots, K\} \\ &\geqq P\{N \leqq K\}P\{S_j > T_0 \geqq T_j \geqq a_j > S_j'; j = 1, \cdots, K\} \end{aligned}$$

where $\{a_j\}$ is a sequence to be specified,

$$\begin{aligned} &= P\{N \leqq K\} \cdot P\{S_j > T_0, j = 1, \cdots, K\} \\ &\quad \times P\{T_0 \geqq T_j \geqq a_j, j = 1, \cdots, K\} \qquad (3) \\ &\quad \times P\{a_j \geqq S_j'; j = 1, \cdots, K\}\,. \end{aligned}$$

(It is understood that all the above factors depend on n and k.)

We now proceed to estimate these terms. First

$$\begin{aligned} P\{N < K\} &= P\left\{T_j(k) \text{ hits } 0 \text{ before } (1+\varepsilon)\frac{(1+\delta)^k n}{q}\right\} \\ &\geqq P\{T_{(1+\varepsilon)(1+\delta)^k n/q} < 0\} \qquad (4) \\ &= P\{\hat{T}_{(1+\varepsilon)(1+\delta)^k n/q} < -(1+\delta)^k n\} \\ &= P\left\{\frac{\hat{T}_{(1+\varepsilon)(1+\delta)^k n/q}}{(1+\varepsilon)(1+\delta)^k n/q} < -\frac{q}{(1+\varepsilon)}\right\} \to 1 \qquad \text{as } n \to \infty \end{aligned}$$

uniformly in k.

Next

$$\begin{aligned} P\{S_j > T_0, 1 \leqq j \leqq K\} &= P\{\hat{S}_j > -(1+\delta)^k \delta n, 1 \leqq j \leqq K\} \\ &= P\{\hat{S}_K > -(1+\delta)^k \delta n\}\,. \end{aligned}$$

Since $\delta > p/q$ we can choose ε so that $\delta/(1+\varepsilon) > 1$, and then the above

$$= P\left\{\frac{\hat{S}_K}{K} > -\frac{\delta q}{1+\varepsilon}\right\} = P\left\{\frac{\hat{S}_K}{K} > -p(1+\varepsilon')\right\}$$

for some $\varepsilon' > 0$. Since $K \to \infty$ as $n \to \infty$, and $\hat{S}_j$ is a sum of independent random variables with mean $-p$, we see that

$$P[S_j > T_0, 1 \leqq j \leqq K] \to 1 \qquad \text{as } n \to \infty . \tag{5}$$

Now we take $a_j = T_0 - \frac{1}{2}j(p+q)$. Then for all $K > 1$,

$$\begin{aligned} P\{T_0 \geqq T_j \geqq a_j; 1 \leqq j \leqq K\} &= P\left\{0 \geqq \hat{T}_j \geqq -\frac{j(p+q)}{2}, j = 1, \cdots, K\right\} \\ &\geqq P\left\{\hat{T}_j \geqq -\frac{j(p+q)}{2}, j = 1, 2, \cdots\right\} > 0 . \end{aligned}$$

Finally

$$P\{a_j \geqq S_j', j = 1, \cdots, K\} = P\left\{-j\frac{p+q}{2} \geqq \hat{S}_j' - 1, j = 1, 2, \cdots\right\} > 0 .$$

The lemma follows from (3), (4), (5) and the above inequalities.

Since clearly $P_{k,n}^{(2)} \geqq P[\mathscr{E}^{(n)}(k)]$, we have by (2) and the lemma that $P_{k,n}^{*} \geqq \theta_0 > 0$ for n sufficiently large. Thus the probability that b^* hits 0 before undergoing collision, or equivalently that $J_k^{(n)*}$ reaches 0, is $\geqq \theta_0 > 0$, where the lower bound θ_0 is independent of k.

Now pick any δ_1 such that $p/q < \delta_1 < \delta$. Let $\tau(b^*)$ denote the extinction time of b^*, or the time at which it reaches the origin, whichever comes first. Consider any initial black interval $J_k^{(n)} = (1+\delta)^k n[1, 1+\delta)$. At time τ this interval can have shrunk to a black interval no smaller than

$$[(1+\delta)^k n, (1+\delta)^{k+1} n - \tau) .$$

But for k sufficiently large (depending on τ)

$$[(1+\delta)^k n, (1+\delta)^{k+1} n - \tau) \subset (1+\delta)^k n[1, 1+\delta_1) .$$

Hence by conditioning on $\tau(b^*)$, we can easily argue that with probability one, there will be an infinite number of black intervals

$$(1+\delta)^k n[1, 1+\delta_1) , \quad k \quad \text{taken from a subsequence of} \quad \{1, 2, \cdots\} .$$

Now let b_1^* denote the left end-point of one of these, and argue exactly as above that the probability that b_1^* reaches the origin before extinction is $\geqq \theta_1 > 0$ for $n \geqq$ some n_1. Actually an inspection of the construction used in deriving this bound shows that somewhat more is true. If we denote by $\mathscr{F}_{\tau(b^*)}$ the history of the process (i.e., the σ-field generated by the process) up to and including $\tau(b^*)$, then $P\{b_1^* \text{ reaches 0 before extinction} \mid \mathscr{F}_{\tau(b^*)}\} \geqq \theta_1 > 0$, the inequality being *uniform* over all possible conditionings (states, realizations) of the process at and before $\tau(b^*)$.

We now continue in this fashion. Pick any δ_2 such that $p/q < \delta_2 < \delta_1$. Let $\tau(b_1^*) =$ the extinction or zero crossing time of b_1^*. Then at $\tau(b_1^*)$ there will be infinitely many black intervals $(1+\delta)^k n[1, 1+\delta_2)$. Let b_2^* denote one of these. Then $P\{b_2^* \text{ reaches } 0 \mid \mathscr{F}_{\tau(b_1^*)}\} \geqq \theta_2$. We thus choose sequences $\{\delta_i\}$,

$p/q < \delta_i$, δ_i decreasing, and then corresponding particles $\{b_i^*\}$; and show that

$$P\{b_{i+1}^* \text{ reaches } 0 \mid \mathscr{F}_{\tau(b_i^*)}\} \geqq \theta_{i+1} > 0 .$$

Moreover by taking $\delta_i \geqq \delta_0 > p/q$ we can guarantee $\theta_i >$ some $\theta > 0$ for all i. It thus follows that the probability that less than k out of the first n b_i^*'s reaches 0 goes to zero as $n \to \infty$ (for any k), and hence the conclusion of part (i).

PROOF OF (ii). Consider again the intervals

$$J_{k,\delta}^{(n)} = [(1+\delta)^k n, (1+\delta)^{k+1} n) , \qquad k = 0, 1, 2, \cdots ,$$

and again denote by $B_{k,\delta}^{(n)}$ the event that all unit segments in $J_{k,\delta}^{(n)}$ are black. Using 1(b) we see (as in (3)) that

$$P(B_{k,\delta}^{(n)}) \leqq \frac{K'}{k^{b\delta}[\log(1+\delta)n]^{b\delta}} ,$$

and

$$\textstyle\sum_k P(B_k) < \infty \qquad \text{if} \quad b\delta > 1 .$$

Thus for b sufficiently large, at most finitely events B_k occur. Thus to prove that only finitely many intervals cross 0, we need not concern ourselves with black intervals longer than the $J_k^{(n)}$'s.

Suppose a black interval is determined by a black particle b at its left end-point and a white particle w at its right. Let us say that the interval "reaches" the origin if the particle b does, and "crosses" 0 if b and w reach 0 (before extinction). Consider the intervals $J_{k,\delta}^{(n)}$ defined above and define the event $H_k(B)$ that at least one black interval having a left end-point in $J_{k,\delta}^{(n)}$ reaches 0. To prove (ii) it is sufficient (by Borel–Cantelli) to show that $\sum P[H_k(B)] < \infty$.

Now modify the initial state of the process by coloring all of $[0, (1+\delta)^k n]$ white, and all of $J_{k,n}^{(n)}$ black, thus producing a black interval which we denote by $\hat{J}_{k,n}^{(n)}$. (Note that due to possible overlap at the right end-point, $\hat{J}$ may be larger than J.) Let $\hat{H}_k(\beta)$ denote the event that $\hat{J}_{k,\delta}^{(n)}$ reaches 0 (under the modified initial distribution). Then a sample path comparison of the type discussed in the proof of (i) shows that

$$P[H_k(B)] \leqq P[\hat{H}_k(B)] .$$

Finally let $H_k'(B)$ denote the event that a black interval exactly equal to $J_{k,\delta}^{(n)}$ reaches 0, under the initial condition that $[0, (1+\delta)^k n]$ is white. Since only finitely many intervals can be longer than $[(1+\delta)^k n, (1+\delta)^{k+1} n]$ (thus putting a bound on the difference between $\hat{J}_{k,\delta}^{(n)}$ and $J_{k,\delta}^{(n)}$) it follows that $\sum P[\hat{H}_k(B)] < \infty$ for sufficiently small δ if

$$\sum P[H_k'(B)] < \infty$$

for sufficiently small δ.

We now turn to the proof of the fact that the last series converges. Let $I_0(n) = I_0 =$ the interval $[(1+\delta)n, (1+\delta')n)$, with $\delta' > \delta$ to be specified later. Define the events:

EVENT A. The interval I_0 has at least $K_1 \log\log n$ white unit segments any two of which are separated by a distance of at least $K_2 \log n$ (under the initial distribution).

EVENT B. At least one white sub-segment in I_0 (say $I_1(W)$) never becomes extinct and has grown to length at least $K_3 \log\log n$ within $c \log\log n$ steps.

EVENT C. Such a white interval $I_1(W)$ of length $K_3 \log\log n$ "consumes" $J_0(B)$ before the latter reaches the origin (i.e., at some time before the left (black) end point of $J_0(B)$ reaches 0, the left end-point of $I_1(W)$ is $\leqq$ that of $J_0(B)$).

Clearly

$$1 - P[H_0'(B)] \geqq P(ABC) = P(C \mid AB)P(B \mid A)P(A)\,. \tag{6}$$

Let us first estimate the number of white segments in I_0. Let $d = \delta' - \delta > 0$, and write

$$p_j = \frac{f_j \log\log j}{j} = P\{[j, j+1) \text{ is initially white}\}\,.$$

By hypothesis $f_j \geqq b$, where b will be chosen later to be sufficiently large, and hence $p_j \geqq \gamma n^{-1} \log\log n$ for $j \in I_0$, where $\gamma = b/(1+\delta')$. Then

$$P\{\text{at most } t \text{ whites in } I_0 \text{ initially}\}$$
$$\leqq \sum_{i=0}^{t} \binom{dn}{i} \left(\frac{\gamma \log\log n}{n}\right)^i \left(1 - \frac{\gamma \log\log n}{n}\right)^{dn-i}.$$

If $t \leqq K_1 \log\log n$, the above is

$$\leqq t \binom{dn}{t} \left(\frac{\gamma \log\log n}{n}\right)^t \left(1 - \frac{\gamma \log\log n}{n}\right)^{dn-t}$$
$$\leqq c\,\frac{(dn)^t}{t!} \left(\frac{\gamma \log\log n}{n}\right)^t \left(1 - \frac{\gamma \log\log n}{n}\right)^{-t} e^{-(\log\log n)d\gamma}$$

and using Stirling's formula

$$\leqq c'\,\frac{(ed\gamma)^t}{t^t} (\log\log n)^t \left(1 - \frac{\gamma \log\log n}{n}\right)^{-t} \left(\frac{1}{\log n}\right)^{d\gamma}.$$

Taking $t = K_1 \log\log n$, we get

$$P\{\text{at most } K_1 \log\log n \text{ whites}\} \leqq c'' \left(\frac{ed\gamma}{K_1}\right)^{K_1 \log\log n} \left(\frac{1}{\log n}\right)^{d\gamma} = c''(\log n)^{-K},$$

where $K = d\gamma - K_1 \log(ed\gamma/K_1)$.

Hence letting A_0 (A_0^+ resp.) denote the event that there are exactly (at least; resp.) $K_1 \log\log n$ white units in I_0 at time 0, we have

$$P(A_0^+) \geqq 1 - \text{constant} \cdot (\log n)^{-K}\,. \tag{7}$$

Furthermore $P(A) \geqq P(A \mid A_0^+)P(A_0^+) \geqq P(A \mid A_0)P(A_0^+)$. To estimate $P(A \mid A_0)$ note the following elementary occupancy problem: If N cells are arranged in a row, and each is independently occupied with probability p, then the conditional probability that a particular site is occupied, given that a total of K sites are occupied, equals K/N. The conditional probability that there exist two occupied sites separated by a distance $\leqq l$ (given that K are occupied) is thus bounded by constant $\cdot (K/N)^2(lN)$ (independent of p). A slight adaptation of this argument to our setting yields

$$\begin{aligned} &1 - P(A \mid A_0) \\ (8) \qquad &= 1 - P\{\text{no two whites in } I_0 \text{ are separated by less than } \\ &\qquad K_2 \log n \mid \text{there are exactly } K_1 \log\log n \text{ whites in } I_0\} \\ &\leqq \text{constant} \left(\frac{K_1 \log\log n}{dn}\right)^2 [(dn)(K_2 \log n)] \leqq \text{const.} \frac{(\log n)^2}{n}. \end{aligned}$$

Combining (7) and (8) we see that

$$\begin{aligned} (9) \qquad P(A) &\geqq \left[1 - c_1\left(\frac{1}{\log n}\right)^K\right]\left[1 - c_2\frac{(\log n)^2}{n}\right] \\ &\geqq 1 - c_3\left(\frac{1}{\log n}\right)^\beta, \end{aligned}$$

where β can be made arbitrarily large by taking b large, hence in turn γ and K large.

To bound $P(B \mid A)$ from below note that since $p > q$, it follows from elementary considerations that for suitable choice of c, there are $\lambda > 0$ and $n_0 < \infty$ such that for all $n \geqq n_0$

$$\begin{aligned} (10) \qquad &P\{\text{at time } c \log\log n \text{ a given unit white segment has not} \\ &\qquad \text{become extinct and has grown to length } \geqq K_3 \log\log n\} \geqq \lambda. \end{aligned}$$

(This can be seen by expressing the statement in terms of a simple random walk.) Furthermore within $c \log\log n$ steps a unit interval cannot grow to length $K_2 \log n$, and hence any white units which are initially separated by $K_2 \log n$ are still growing independently. Hence

$$(11) \qquad P(B \mid A) \geqq 1 - (1 - \lambda)^{K_1 \log\log n} \geqq 1 - \left(\frac{1}{\log n}\right)^2$$

for K_1 sufficiently large and $n \geqq n_0$.

Finally we estimate $P(C \mid AB)$. This is bounded from below by the probability that the black interval $J_0(B)$ is consumed by the white interval $I(W)$, as in Fig. 4 below, before the former reaches the origin.

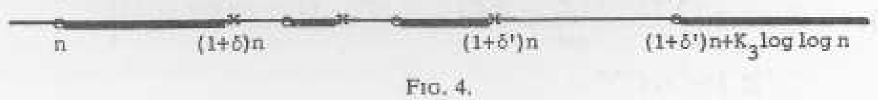


FIG. 4.

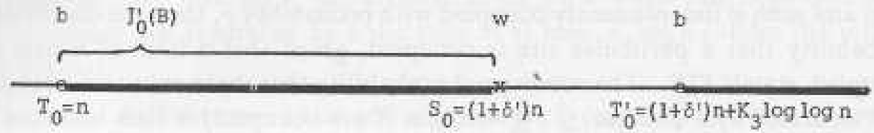


FIG. 5.

This probability is decreased if $J_0(B)$ is increased to $J_0'(B)$ as indicated in Fig. 5.

Let $\{T_j\}$, $\{S_j\}$, and $\{T_j'\}$ be independent random walks as defined before, but now take

$$T_0 = n\,, \qquad S_0 = (1+\delta')n\,, \qquad T_0' = (1+\delta')n + K_3 \log\log n\,.$$

Let N be the first passage time of $\{T_j\}$ to 0. Then

$$\begin{aligned} P\{C \mid AB\} &\geqq P\{S_N < T_N,\, T_j' > S_j,\, j = 1, \cdots, N\} \\ &\geqq P\{S_N < T_N,\, T_j' > S_j \text{ for all } j \geqq 1\} \\ &\geqq P\{T_j' > S_j \text{ for all } j \geqq 1\} - P\{S_N \geqq T_N\}\,. \end{aligned} \tag{12}$$

Now the first probability equals the probability that random walk $U_k = Z_0 + Z_1 + \cdots + Z_k$, with $\{Z_i\}$ independent and $= +1, 0, -1$ with probabilities

$$r^+ = p(1-q)\,, \qquad r^0 = 2pq\,, \qquad r^- = q(1-p)\,, \tag{13}$$

and with $Z_0 = K_3 \log\log n$, *never* hits the origin. Since $r^+ > r^-$, the probability of hitting 0 is, for sufficiently large n, bounded by

$$c\left(\frac{1}{\log n}\right)^{K_3 \log \alpha},$$

where $\alpha > 1$ and c is some constant. Hence (12)

$$\geqq \left[1 - c\left(\frac{1}{\log n}\right)^{K_3 \log \alpha}\right] - P\{S_N > T_N\}\,. \tag{14}$$

Also

$$\begin{aligned} P\{S_N \leqq T_N\} &= \textstyle\sum_{j=1}^{\infty} P\{S_N \leqq T_N \mid N = j\} P\{N = j\} \\ &= \textstyle\sum_{j=1}^{K} + \sum_{j=K+1}^{\infty} \geqq P\{S_K \leqq T_K\} P\{N \geqq K\}\,. \end{aligned} \tag{15}$$

Now take $K = n(1-\varepsilon)/q$, ε to be specified. Then

$$\begin{aligned} P\{N \geqq K\} &= P\{T_K \geqq 0\} = P\{\hat{T}_K \geqq -n\} \\ &= P\{\hat{T}_{n(1-\varepsilon)/q} \geqq -n\} \geqq 1 - c_1 e^{-\lambda_1 n}\,, \qquad \lambda_1 > 0\,, \end{aligned} \tag{16}$$

the last inequality following from a standard tail estimate for the central limit theorem (see, e.g., page 517 ff, Feller **2**). Also

$$P\{S_K < T_K\} = P\{Z_1 + \cdots + Z_K > \delta' n\}\,,$$

with Z_i distributed as in (13)

$$\begin{aligned} &= P\left\{\frac{\sum_{i=1}^{n(1-\varepsilon)/q} [Z_i - EZ_i]}{c n^{1/2}} > n^{1/2}\,\frac{\delta' - (1-\varepsilon)(p/q - 1)}{c}\right\} \\ &\geqq 1 - c_2 e^{-\lambda_2 n}\,, \end{aligned} \tag{17}$$

provided we choose δ' and ε are sufficiently small so that

$$\frac{\delta'}{1-\varepsilon} < \frac{p}{q} - 1 . \tag{18}$$

Hence by (15), (16), (17)

$$P\{S_N \leqq T_N\} \geqq 1 - c'e^{-\lambda n} , \qquad \lambda > 0 , \tag{19}$$

or $P\{S_N > T_N\} < c'e^{-\lambda n}$.

Inserting this in (14), we thus conclude that

$$P\{C \mid AB\} \geqq 1 - \text{const.} \left(\frac{1}{\log n}\right)^{K_3 \log \alpha} . \tag{20}$$

We take K_3 sufficiently large so that $K_3 \log \alpha \geqq 2$. Going back to (6), and using (9), (11), and (20), we see that for $n \geqq$ some n_0,

$$1 - P[H_0'(B)] \geqq \left[1 - \frac{\text{const.}}{(\log n)^{\delta}}\right]\left[1 - \left(\frac{1}{\log n}\right)^2\right]\left[1 - \frac{\text{const.}}{(\log n)^{K_3 \log \alpha}}\right]$$

$$\geqq 1 - \frac{\text{const.}}{(\log n)^2} .$$

Hence

$$P[H_k'(B)] \leqq \frac{\text{const.}}{[\log (1+\delta)^k n]^2}$$

and

$$\textstyle\sum_{k=0}^{\infty} P[H_k'(B)] < \infty .$$

Hence by the Borel–Cantelli lemma only finitely many $J_k(B)$'s reach 0. This proves (ii) and the theorem.

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