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Note

Remark on a Theorem of Lindström

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In a recent paper Lindström [1] proves a theorem on finite sets and he also proves a transfinite extension. In this note we only concern ourselves with the transfinite case of Lindström's theorem. He in fact proves the following theorem: Let $|\mathscr{S}| = \kappa$ be an infinite set, and let A_α, $1 \leqslant \alpha < \omega_m$, the initial ordinal of cardinality m, $m > \kappa$, be subsets of $\mathscr{S}$. Then for every $p < m$ there are p disjoint sets of the indices I_γ, $1 \leqslant \gamma < \omega_p$, so that the p sets

$$\bigcup_{j \in I_\gamma} A_j$$

are all equal.

We are going to prove the following slightly stronger result.

DEFINITION. $cf(m)$ denotes the smallest cardinal so that m is the sum of $cf(m)$ cardinals smaller than m.

THEOREM. *Let $|\mathscr{S}| = \kappa > \aleph_0$, and let A_α, $1 \leqslant \alpha < \omega_m$, where $m > \kappa$, and even $cf(m) > \kappa$, be m subsets of $\mathscr{S}$. Then there are m disjoint sets of indices I_γ, so that the m sets*

$$\bigcup_{j \in I_\gamma} A_j$$

are all equal.

If $cf(m) \leqslant \kappa$, the theorem is not true.

Let $\{x_\alpha\}$, $1 \leqslant \alpha < \omega_\kappa$, be the elements of S. An element is said to be bad if it is contained in fewer than m sets A_α. Throw away all the bad elements and the sets A_α containing them. But if $m > \kappa$ we have thrown away fewer than m sets and we are left with a set $\mathscr{S}_1 \subset \mathscr{S}$ (perhaps $|\mathscr{S}_1| < |\mathscr{S}|$)

and sets $A_i \in \mathscr{S}_1$, $1 \leqslant i < \omega_m$, so that every element of $\mathscr{S}_1$ is contained in m sets A_{α_i}. Note that the A_{α_i} all occur among the sets A_α since $A_{\alpha_i} \cap (\mathscr{S} - \mathscr{S}_1) = \varnothing$.

Now there clearly are m disjoint sets of indices I_γ, $1 \leqslant \gamma < \omega_m$, so that

$$\bigcup_{\alpha_i \in I_\gamma} A_{\alpha_i} = \mathscr{S}_1 .$$

In fact, we can construct the sets I_γ so that

$$| I_\gamma | \leqslant \kappa$$

and every α_i occurs in an I_γ. This can be done by a simple transfinite induction. Suppose we have already constructed $p < m$ sets I_γ satisfying $\bigcup_{\alpha_i \in I_\gamma} A_{\alpha_i} = \mathscr{S}_1$, $| I_\gamma | \leqslant \kappa$, and well order the indices $\{\alpha_i\}$, $1 \leqslant \alpha_i < \omega_m$. Let α_j be the first index which does not occur in $\bigcup I_\gamma$ where γ runs through the p sets which we have already constructed. We construct a new set $I_{\gamma'}$ which is disjoint from $\bigcup I_\gamma$ and so that $\bigcup_{\alpha_i \in I_{\gamma'}} A_{\alpha_i} = \mathscr{S}_1$ and $\alpha_j \in I_{\gamma'}$.

First of all, we put α_j in $I_{j'}$, and for each element x_j of $\mathscr{S}_1$, we choose a set containing it and which is such that it has not yet been used. Since every element of $\mathscr{S}_1$ is contained in m sets and we have used so far fewer than m sets, our construction can clearly be carried out and we obtain the required decomposition of the index set, and this completes the proof of our theorem.

Clearly, if $cf(m) \leqslant \kappa$, our theorem cannot hold. If $m \leqslant \kappa$, our sets can be disjoint if $cf(m) \leqslant \kappa < m$. Put $m = \bigcup_\beta g_\beta$, $1 \leqslant \beta < \omega_g$, $g \leqslant \kappa$. Let $x_\beta \in g$, $1 \leqslant \beta \leqslant g$, and consider any g_β sets containing x_β but not containing any x_δ for $\delta < \beta$. It is clear that our $\bigcup g_\beta = m$ sets do not satisfy our theorem.

References

1. B. Lindström, A theorem on families of sets, *J. Combinatorial Theory* (*A*) **13** (1972), 274–277.

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