# ON ORTHOGONAL POLYNOMIALS WITH REGULARLY DISTRIBUTED ZEROS

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[Received 14 August 1973]

## 1. Introduction

Let $d\alpha(x)$ be a non-negative measure on $(-\infty, \infty)$ for which all moments

$$\mu_m(d\alpha) = \int_{-\infty}^{\infty} x^m \, d\alpha(x) \quad (m = 0, 1, \ldots)$$

exist and are all finite. We consider the orthonormal polynomials

$$p_n(d\alpha, x) = \gamma_n(d\alpha) \prod_{k=1}^{n} [x - x_{kn}(d\alpha)] \tag{1.1}$$

which satisfy $\gamma_n(d\alpha) > 0$ and $\int p_n(d\alpha) p_m(d\alpha)\, d\alpha(x) = \delta_{mn}$, where $\delta_{mn}$ is the Kronecker symbol. The zeros $x_{kn}(d\alpha)$ of $p_n(d\alpha, x)$ are real and simple. We assume that they are ordered increasingly. If no misunderstanding can arise, we write $x_{kn}$ for $x_{kn}(d\alpha)$ (resp. $x_{kn}(w)$, see below). Let us denote by $N_n(d\alpha, t)$ the number of integers $k$ for which

$$x_{1n}(d\alpha) - x_{nn}(d\alpha) \geqslant t[x_{1n}(d\alpha) - x_{nn}(d\alpha)]$$

holds. The distribution function of the zeros is defined, when it exists, as

$$\beta(t) = \lim_{n\to\infty} n^{-1} N_n(d\alpha, t) \quad (0 \leqslant t \leqslant 1). \tag{1.2}$$

We are here concerned with the case when the distribution function is given by

$$\beta_0(t) = \frac{1}{2} - \frac{1}{\pi} \operatorname{arc\,sin}(2t-1). \tag{1.3}$$

In this case the points $\theta_{kn} = \operatorname{arc\,sin} x_{kn}$ are equidistributed in Weyl's sense.

A non-negative measure $d\alpha$ for which the array $x_{kn}(d\alpha)$ has the distribution function $\beta_0(t)$ will be called an *arc-sine measure*. If $d\alpha(x) = w(x)\,dx$ is absolutely continuous, we apply, replacing $d\alpha$ by $w$, the notations $p_n(w, x)$, $\gamma_n(w)$, $x_{kn}(w)$ and call a non-negative $w(x)$ an *arc-sine weight* if $d\alpha(x) = w(x)\,dx$ is an arc-sine measure. A fairly complete treatise of arc-sine weights with compact support is given in [**9**] by Ullman.

The *restricted support* of a weight $w(x)$ is defined as the set $\{x : w(x) > 0\}$. The *support* of $w(x)$ can be characterized as the set of points $\xi$ for which every interval containing $\xi$ contains a subset with positive measure of the restricted support of $w$. It was proved by Erdős and Turán ([**3**]) that a $w(x)$ having support $[-1,1]$ is arc-sine provided that its restricted support has Lebesgue measure equal to 2. This, as well as another criterion for arc-sine weights, established by Geronimus ([**7**]), is treated also in [**9**].

Arc-sine weights with non-compact support were introduced by Erdős in [**2**].

The case when the support of the measure $d\alpha$ is contained in $[-1,1]$ and the two points $-1, 1$ belong to this support is of particular interest. We have then $x_{1n}(d\alpha) \to 1$, $x_{nn}(d\alpha) \to -1$ and (1.2) can be rewritten as

$$\lim_{n\to\infty} n^{-1} \sum_{k:x_{kn}(d\alpha)\geqslant T} 1 = \frac{1}{\pi} \operatorname{arc}\cos T \quad (-1 \leqslant T \leqslant 1). \tag{1.4}$$

For the measures $d\alpha$, resp. weights $w$, whose support is contained in $[-1,1]$, we apply the term *arc-sine on* $[-1,1]$ if the array $\{x_{kn}(d\alpha)\}$, resp. $\{x_{kn}(w)\}$, satisfies (1.4).

Our results are as follows.

THEOREM 1.1. (a) *The condition*

$$\varlimsup_{n\to\infty} {}^{n-1}\sqrt{(\gamma_{n-1}(d\alpha))}[x_{1n}(d\alpha)-x_{nn}(d\alpha)] \leqslant 4 \tag{1.5}$$

*implies that $d\alpha$ is arc-sine.*

(b) *It follows from* (1.5) *that*

$$\lim_{n\to\infty} {}^{n-1}\sqrt{(\gamma_{n-1}(d\alpha))}[x_{1n}(d\alpha)-x_{nn}(d\alpha)] = 4. \tag{1.6}$$

See also Theorem 4.2 for a more general result.

We show that the arc-sine weights with infinite support studied by the first of us in [**2**] satisfy (1.6), but the weights $w_\alpha(x) = \exp\{-|x|^\alpha\}$, $\alpha > 0$, are not arc-sine. It is further proved by a counter-example that even the stronger sufficient condition (1.6) is not necessary in general. The case is different if $w(x)$ has compact support.

THEOREM 1.2. *A weight $w$, the support of which is contained in $[-1,1]$, is arc-sine on $[-1,1]$ if and only if*

$$\varlimsup_{n\to\infty} {}^{n}\sqrt{(\gamma_n(w))} \leqslant 2. \tag{1.7}$$

We note that by Ullman's Lemma 1.2 in [**9**], the support of $w$ is precisely $[-1,1]$. We do not make use of this observation. Also, Theorem 1.2 was conjectured by Ullman in [**9**], part 7. He proved the weaker statement that if the restricted support of $w$ is a determining set (see Definition 1.1) then condition (1.7) is sufficient ([**9**], Theorem 1.6(b)). The sufficiency part of Theorem 1.2 can be generalized to measures $d\alpha$ which are not necessarily absolutely continuous (see Theorem 3.1 below).

DEFINITION 1.1 (Ullman, [**9**], Definition 1.4). We say that $A \subseteq [-1, 1]$ is a *determining set* if all weights $w(x)$, the restricted support of which contain $A$, are arc-sine on $[-1, 1]$.

Let us denote by $C(A)$ the capacity (that is, inner logarithmic capacity) of the set $A$ and by $|A|$ its outer (linear) Lebesgue measure. Note that the capacity of $[-1, 1]$ is $\frac{1}{2}$.

DEFINITION 1.2. We say that $A \subseteq [-1, 1]$ has *minimal capacity* $\frac{1}{2}$ if for every $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that for every $B$ having Lebesgue measure less than $\varepsilon$ we have $C(A \setminus B) > \frac{1}{2} - \varepsilon$.

THEOREM 1.3a. *A measurable subset $A$ of $[-1, 1]$ is a determining set if and only if it has minimal capacity $\frac{1}{2}$.*

Theorem 1.3a was stated as a conjecture by Erdős in several lectures held in the last thirty years; see [**2**].

THEOREM 1.3b. *A measurable subset $A$ of $[-1, 1]$ is a determining set if and only if it is a 'good set' (in the sense of Erdős, [**2**]).*

## 2. Sufficiency of condition (1.5)

We denote by $T_n(X) = \cos(n \arccos x)$ the $n$th Chebychev polynomial of the first kind. The zeros of $T_n(x)$ are $t_{kn} = \cos[(2k-1)/2n]\pi$.

LEMMA 2.1. *We have for every $d\alpha$,*

$$\varliminf_{n\to\infty} \sqrt[n-1]{(\gamma_{n-1}(d\alpha))}[x_{1n}(d\alpha) - x_{nn}(d\alpha)] \geqslant 4. \tag{2.1}$$

*Proof.* Let

$$x_{kn} = \tfrac{1}{2}(x_{1n} + x_{nn}) + \tfrac{1}{2}\tau_{kn}(x_{1n} - x_{nn}), \tag{2.2}$$

then $|\tau_{kn}| \leqslant 1$ $(k = 1, 2, \ldots, n)$.

By applying the Lagrange interpolation formula with nodes $x_{kn}$, we have

$$T_{n-1}[2(x_{1n} - x_{nn})^{-1}(z - \tfrac{1}{2}(x_{1n} + x_{nn}))] = \sum_{k=1}^{n} l_{kn}(z) T_{n-1}(\tau_{kn}). \tag{2.3}$$

By [**4**], formula III (6.3),

$$l_{kn}(z) = \frac{\gamma_{n-1}(d\alpha)}{\gamma_n(d\alpha)} \lambda_{kn} \frac{p_{n-1}(d\alpha, x_{kn})}{z - x_{kn}} p_n(d\alpha, z). \tag{2.4}$$

The $\lambda_{kn}$ are the Christoffel numbers with respect to $d\alpha$. Comparing highest coefficients in (2.3) and applying (2.4), we obtain

$$2^{2n-3}(x_{1n}-x_{nn})^{-n+1} = \gamma_{n-1}(d\alpha)\sum_{k=1}^{n}\lambda_{kn}p_{n-1}(d\alpha, x_{kn})T_{n-1}(\tau_{kn}). \tag{2.5}$$

Since $|\tau_{kn}| \leqslant 1$ implies $|T_{n-1}(\tau_{kn})| \leqslant 1$, we have by the quadrature formula

$$\begin{aligned}\left[\frac{2^{2n-3}}{(x_{1n}-x_{nn})^{n-1}\gamma_{n-1}(d\alpha)}\right]^2 &\leqslant \left[\sum_{k=1}^{n}\lambda_{kn}|p_{n-1}(d\alpha, x_{kn})|\right]^2 \\ &\leqslant \sum_{k=1}^{n}\lambda_{kn}\sum_{k=1}^{n}\lambda_{kn}p_{n-1}{}^2(d\alpha, x_{kn}) \\ &= \int_{-\infty}^{\infty}d\alpha(x)\int_{-\infty}^{\infty}p_{n-1}{}^2(d\alpha, x)\,d\alpha(x) = \mu_0(d\alpha) < \infty.\end{aligned} \tag{2.6}$$

(2.1) is a consequence of (2.6).

Let $z = \frac{1}{2}(x_{1n}+x_{nn})+\frac{1}{2}(x_{1n}-x_{nn})\zeta$. By (2.3) and (2.4),

$$|T_{n-1}(\zeta)| \leqslant \frac{|p_n(d\alpha, z)|}{\gamma_n(d\alpha)}\gamma_{n-1}(d\alpha)\sum_{k=1}^{n}\lambda_{kn}|p_{n-1}(d\alpha, x_{kn})|\max_k\frac{1}{|z-x_{kn}|}.$$

Let us observe that $z-x_{kn} = \frac{1}{2}(x_{1n}-x_{nn})(\zeta-\tau_{kn})$, the last factor does not exceed $2(x_{1n}-x_{nn})^{-1}[\Delta(\zeta)]^{-1}$, where $\Delta(\zeta)$ denotes the euclidean distance of $\zeta$ from the interval $[-1, 1]$. From the second half of (2.6), we obtain

$$|T_{n-1}(\zeta)| \leqslant \frac{|p_n(d\alpha, z)|}{\gamma_n(d\alpha)}\gamma_{n-1}(d\alpha)\frac{2}{x_{1n}-x_{nn}}\frac{[\mu_0(d\alpha)]^{\frac{1}{2}}}{\Delta(\zeta)}. \tag{2.7}$$

In (2.7) we take logarithms on both sides and divide by $n$. After rearranging terms, we get

$$\begin{aligned}\frac{1}{n}\log\frac{\gamma_n(d\alpha)}{|p_n(d\alpha, z)|} &= \frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{|z-x_{kn}|} \\ &= \log\frac{2}{x_{1n}-x_{nn}}+\frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{|\zeta-\tau_{kn}|} \\ &\leqslant \frac{1}{n}\log\frac{2}{x_{1n}-x_{nn}}+\frac{1}{n}\log\gamma_{n-1}(d\alpha)+\frac{1}{n}\log\frac{2^{n-2}}{|T_{n-1}(\zeta)|} \\ &\qquad -\frac{n-2}{n}\log 2+\frac{1}{n}\log\frac{[\mu_0(d\alpha)]^{\frac{1}{2}}}{\Delta(\zeta)},\end{aligned}$$

that is,

$$(2.8)\qquad \frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{|\zeta-\tau_{kn}|}\leqslant\left(1-\frac{1}{n}\right)\log\{\tfrac{1}{4}(x_{1n}-x_{nn})\sqrt[n-1]{(\gamma_{n-1}(d\alpha))}\}$$
$$+\frac{1}{n}\log\frac{2^{n-2}}{|T_{n-1}(\zeta)|}+\frac{1}{n}\log\left(2\frac{[\mu_0(d\alpha)]^{\frac{1}{2}}}{\Delta(\zeta)}\right).$$

LEMMA 2.2. *We have, for every $d\alpha$ and every $\zeta\notin[-1,1]$,*

$$(2.9)\quad \overline{\lim_{n\to\infty}}\, n^{-1}\sum_{k=1}^{n}\log\frac{1}{|\tau_{kn}-\zeta|}\leqslant\frac{1}{\pi}\int_{-1}^{1}\log\frac{1}{|x-\zeta|}\frac{dx}{\sqrt{(1-x^2)}}$$
$$+\log\left\{\overline{\lim_{n\to\infty}}[\tfrac{1}{4}\sqrt[n-1]{(\gamma_{n-1}(d\alpha))}(x_{1n}-x_{nn})]\right\}.$$

*Proof.*

$$\frac{\pi}{n-1}\log\frac{2^{n-2}}{|T_{n-1}(\zeta)|}=\frac{\pi}{n-1}\sum_{k=1}^{n-1}\log\frac{1}{|\zeta-t_{k,n-1}|}$$

is a Riemann sum of the integral

$$\int_0^{\pi}\log\frac{1}{|\zeta-\cos\theta|}d\theta=\int_{-1}^{1}\log\frac{1}{|\zeta-x|}\frac{dx}{\sqrt{(1-x^2)}}.$$

Applying this fact, we obtain (2.9) from (2.8).

*Proof of Theorem* 1.1. (a) Let $\mathscr{P}(x)=c\prod(x-\zeta_j)$ be an arbitrary polynomial whose zeros are situated outside $[-1,1]$. We insert $\zeta=\zeta_j$ in (2.9) and add up:

$$(2.10)\qquad \overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{|\mathscr{P}(\tau_{kn})|}\leqslant\frac{1}{\pi}\int_{-1}^{1}\log\frac{1}{|\mathscr{P}(x)|}\frac{dx}{\sqrt{(1-x^2)}}.$$

Now let $f(x)$ be a bounded upper semicontinuous function in $[-1,1]$. Then there exists a sequence of polynomials $\{\mathscr{P}_\nu\}$ which satisfy, for $x\in[-1,1]$,

$$(2.11)\qquad \mathscr{P}_{\nu+1}(x)>\mathscr{P}_\nu(x)>\ldots>\mathscr{P}_1(x)>c>0$$

and

$$(2.12)\qquad \lim_{\nu\to\infty}\log\frac{1}{\mathscr{P}_\nu(x)}=f(x).$$

By (2.10), we have

$$\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}f(\tau_{kn})\leqslant\overline{\lim_{n\to\infty}}\frac{1}{n}\sum_{k=1}^{n}\log\frac{1}{\mathscr{P}_\nu(\tau_{kn})}$$
$$\leqslant\frac{1}{\pi}\int_{-1}^{1}\log\frac{1}{\mathscr{P}_\nu(x)}\frac{dx}{\sqrt{(1-x^2)}}\quad(\nu=1,2,\ldots).$$

Let $\nu \to \infty$, then it follows by dominated convergence from (2.11) and (2.12) that

$$\overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=1}^{n} f(\tau_{kn}) \leqslant \frac{1}{\pi} \int_{-1}^{1} f(x) \frac{dx}{\sqrt{(1-x^2)}}. \tag{2.13}$$

Let $T \in [-1,1]$. Inserting in (2.17) for $f$ the characteristic function of the interval $[T,1]$ (resp. $[-1,T]$) we find that the sums

$$\Sigma_n^{(1)} = \frac{1}{n} \sum_{k:\tau_{kn}\geqslant T} 1 \quad \text{and} \quad \Sigma_n^{(2)} = \frac{1}{n} \sum_{k:\tau_{kn}\leqslant T} 1$$

satisfy

$$\overline{\lim_{n\to\infty}} \Sigma_n^{(1)} \leqslant \frac{1}{\pi} \int_{T}^{1} \frac{dx}{\sqrt{(1-x^2)}} \quad \text{and} \quad \overline{\lim_{n\to\infty}} \Sigma_n^{(2)} \leqslant \frac{1}{\pi} \int_{-1}^{T} \frac{dx}{\sqrt{(1-x^2)}}. \tag{2.14}$$

Clearly $\Sigma_n^{(1)} + \Sigma_n^{(2)} \geqslant 1$, thus

$$\begin{aligned} \underline{\lim_{n\to\infty}} \Sigma_n^{(1)} &\geqslant 1 - \overline{\lim_{n\to\infty}} \Sigma_n^{(2)} \\ &\geqslant 1 - \frac{1}{\pi} \int_{-1}^{T} \frac{dx}{\sqrt{(1-x^2)}} = \frac{1}{\pi} \int_{T}^{1} \frac{dx}{\sqrt{(1-x^2)}} = \frac{1}{\pi} \arccos T. \end{aligned} \tag{2.15}$$

By (2.14) and (2.15),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k:\tau_{kn}\geqslant T} 1 = \frac{1}{\pi} \arccos T; \tag{2.16}$$

hence $d\alpha$ is arc-sine on $[-1,1]$.

Assertion (b) follows from Lemma 2.1.

## 3. Conditions for arc-sine weights on $[-1, 1]$

By $\mathscr{T}$ we denote closed subintervals of $[-1,1]$ and by $\|f\|_{\mathscr{T}}$ the supremum norm of $f(x)$ on $\mathscr{T}$. Let $\mathfrak{P}_n$ be the set of all polynomials with degree not exceeding $n$, $\mathfrak{P}_n^* \subseteq \mathfrak{P}_n$ the set of monic polynomials of degree $n$, that is, $\mathscr{P}_n \in \mathfrak{P}_n^*$ if and only if $\mathscr{P}_n(x) - x^n \in \mathfrak{P}_{n-1}$. We are going to investigate the monic orthogonal polynomials

$$\omega_n(d\alpha, x) = [\gamma_n(d\alpha)]^{-1} p_n(d\alpha, x). \tag{3.1}$$

*In this as well as in the next section we consider only distributions $d\alpha$ (resp. weights $w(x)$) the support of which is contained in $[-1,1]$.*

The following two known inequalities will be applied.

CHEBYCHEV–BERNSTEIN INEQUALITY (Bernstein, [**1**]). *We have, for every $\mathscr{P}_n \in \mathfrak{P}_n$ and every $z \notin [-1,1]$,*

$$|\mathscr{P}_n(z)| \leqslant |T_n(z)| \, \|\mathscr{P}_n\|_{[-1,1]}. \tag{3.2}$$

Remez inequality (Remez, [**8**]; Freud, [**4**], Lemma III.7.3). *We have, for every* $\mathscr{P}_n \in \mathfrak{P}_n$,

$$\|\mathscr{P}_n\|_{[-1,1]} \leqslant T_n\left(\frac{4}{|M|}-1\right), \tag{3.3}$$

*where* $|M|$ *is the Lebesgue measure of the set*

$$M = \{x\colon |\mathscr{P}_n(x)| \leqslant 1\} \cap [-1,1]. \tag{3.4}$$

Lemma 3.1. *If the array* $\{\tau_{kn} \in [-1,1],\ k = 1,2,\ldots,n;\ n = 1,2,\ldots\}$ *has arc-sine distribution, that is, satisfies* (2.16), *then*

$$\omega_n(z) = (z-\tau_{1n})(z-\tau_{2n})\ldots(z-\tau_{nn})$$

*satisfies*

$$\lim_{n\to\infty} {}^{n}\!\sqrt{(\|\omega_n\|_{\mathscr{T}})} = \tfrac{1}{2} \tag{3.5}$$

*for every* $\mathscr{T} \subseteq [-1,1]$.

*Proof.* By (2.16), the equation

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} f(\tau_{kn}) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(x)\,dx}{\sqrt{(1-x^2)}}$$

is valid if $f$ is the characteristic function of an interval. Consequently it holds for every $f$ continuous in $[-1,1]$. By putting $f(t) = \log|z-t|$, which is continuous for every $z \notin [-1,1]$, we get

$$\begin{aligned}\lim_{n\to\infty} {}^{n}\!\sqrt{(|\omega_n(z)|)} &= \frac{1}{\pi}\int_{-1}^{1} \log|z-x| \frac{dx}{\sqrt{(1-x^2)}} \\ &= \lim_{n\to\infty} {}^{n}\!\sqrt{(2^{-n+1}|T_n(z)|)} = \tfrac{1}{2}|z+\sqrt{(z^2-1)}| \\ &= \tfrac{1}{2}\varphi(z), \quad \text{by definition.}\end{aligned} \tag{3.6}$$

The second part we obtained from the fact that the roots of $T_n(z)$ are arc-sine-distributed. The curve $C_\delta\colon \varphi(z) = 1+\delta$ surrounds $[-1,1]$ for every $\delta > 0$; from the maximum principle as applied to $\omega_n(z)$ inside $C_\delta$ and by letting $\delta$ tend to zero, we obtain

$$\overline{\lim_{n\to\infty}}\, {}^{n}\!\sqrt{(\|\omega_n\|_{[-1,1]})} \leqslant \tfrac{1}{2}. \tag{3.7}$$

Now let $\mathscr{T} = [a,b] \subseteq [-1,1]$. Applying (3.2) to

$$\mathscr{P}_n(z) = \omega_n(\tfrac{1}{2}(a+b) + \tfrac{1}{2}(b-a)z)$$

and $z = i\varepsilon$, we get

$$\begin{aligned}\tfrac{1}{2}\varphi(\tfrac{1}{2}(a+b)+i\varepsilon) &= \lim_{n\to\infty} {}^{n}\!\sqrt{(|\mathscr{P}_n(i\varepsilon)|)} \leqslant \lim_{n\to\infty} {}^{n}\!\sqrt{(|T_n(i\varepsilon)|)}\, \underline{\lim}_{n\to\infty}\, {}^{n}\!\sqrt{(\|\mathscr{P}_n\|_{[-1,1]})} \\ &= \varphi(i\varepsilon)\, \underline{\lim}_{n\to\infty}\, {}^{n}\!\sqrt{(\|\omega_n\|_{\mathscr{T}})}.\end{aligned}$$

Thus, since $\varphi$ is continuous and $\varphi(\zeta) = 1$ for $\zeta \in [-1, 1]$,

$$\varliminf_{n\to\infty} {}^n\sqrt{(\|\omega_n\|_{\mathscr{T}})} \geqslant \tfrac{1}{2} \lim_{\varepsilon\to 0} \frac{\varphi(\frac{1}{2}(a+b)+i\varepsilon)}{\varphi(i\varepsilon)} = \tfrac{1}{2}. \tag{3.8}$$

Now (3.4) follows from (3.7), from the relation $\|\omega_n\|_{\mathscr{T}} \leqslant \|\omega_n\|_{[-1,1]}$ and from (3.8).

LEMMA 3.2. *For every $p_n \in \mathfrak{P}_n$, every real interval $\mathscr{T}$, and every $0 < \varepsilon < 1$, there exists a measurable subset $\mathscr{T}_\varepsilon$ of $\mathscr{T}$ of measure not less than $\psi(\varepsilon)|\mathscr{T}|$, where $\psi(\varepsilon) = \frac{1}{4}\varepsilon^2 - \frac{1}{16}\varepsilon^4$, such that, for every $x \in \mathscr{T}_\varepsilon$, we have*

$$|p_n(x)| > (1-\varepsilon)^n \|p_n\|_{\mathscr{T}}. \tag{3.9}$$

*Proof.* By a linear transformation, we can take $\mathscr{T} = [-1, 1]$, $|\mathscr{T}| = 2$. The *Remez inequality*, as applied to $\mathscr{P}_n(x) = (1-\varepsilon)^{-n} p_n(x)/\|p_n\|_{\mathscr{T}}$, gives

$$(1-\varepsilon)^{-n} \leqslant T_n(x_M) \leqslant (x_M + \sqrt{(x_M{}^2 - 1)})^n, \tag{3.10}$$

where $x_M = (4/|M|) - 1$ and $M$ is defined by (3.4).

A direct calculation shows that

$$\xi + \sqrt{(\xi^2 - 1)} \leqslant (1-\varepsilon)^{-1} \quad (1 \leqslant \xi \leqslant 1 + \tfrac{1}{2}\varepsilon^2). \tag{3.11}$$

By (3.10) and (3.11), we have $(4/|M|) - 1 = x_M > 1 + \frac{1}{2}\varepsilon^2$; hence

$$2 - |M| > \tfrac{1}{2}\varepsilon^2(1 + \tfrac{1}{4}\varepsilon^2)^{-1} > \tfrac{1}{2}(\varepsilon^2 - \tfrac{1}{4}\varepsilon^4) = \psi(\varepsilon)|\mathscr{T}|,$$

and on the set $[-1, 1] \setminus M$, of measure $2 - |M| > \psi(\varepsilon)|\mathscr{T}|$, we have $|\mathscr{P}_n(x)| > 1$, that is, $|p_n(x)| > (1-\varepsilon)^n \|p_n\|_{\mathscr{T}}$.

*Proof of Theorem* 1.2. The condition $\operatorname{supp} w \subseteq [-1, 1]$ implies $x_{1n}(w) - x_{nn}(w) < 2$, so (1.8) implies (1.5). By (1.8) and (2.1), we have $x_{1n}(w) - x_{nn}(w) \to 2$, that is, $x_{1n}(w) \to 1$ and $x_{nn}(w) \to -1$. This, together with Theorem 1.1, shows that $w$ is arc-sine on $[-1, 1]$.

We turn to the proof that if $w$ is arc-sine on $[-1, 1]$ then (1.7) holds.

We choose a sufficiently small $\Delta$ for which the set

$$\mathfrak{M}_\Delta(w) = \{x \in [-1, 1] : w(x) \geqslant \Delta\}$$

has positive measure. Then, for every $0 < \delta < 1$, there exists an interval $\mathscr{T}_\delta \subseteq [-1, 1]$ for which $|\mathscr{T}_\delta \cap \mathfrak{M}_\Delta(w)| > (1-\delta)|\mathscr{T}_\delta|$. We choose any $\varepsilon$ such that $0 < \varepsilon < 1$ and choose $\mathscr{T}_\delta$ with $\delta < \frac{1}{2}\psi(\varepsilon)$. We assume that $w$ is arc-sine on $[-1, 1]$. Then by Lemma 3.1, we have $\lim_{n\to\infty} {}^n\sqrt{(\|\omega_n(w, x)\|_{\mathscr{T}_\delta})} = \frac{1}{2}$, that is, for sufficiently large $n$,

$$\|\omega_n(w, x)\|_{\mathscr{T}_\delta} \geqslant (1-\varepsilon)^n 2^{-n}.$$

By Lemma 3.2, $\mathscr{T}_\delta$ has a subset $\mathscr{J}_\delta$ of measure greater than $\psi(\varepsilon)|\mathscr{T}_\delta|$, where

$$|\omega_n(w, x)| \geqslant (1-\varepsilon)^{2n} 2^{-n}. \tag{3.12}$$

By construction, $\mathscr{J}_\delta \cap \mathfrak{M}_\Delta(w)$ has a common subset $\mathfrak{M}_\varepsilon$ of measure $|\mathfrak{M}_\varepsilon| > \frac{1}{2}\psi(\varepsilon)$, so (3.12) is valid for $x \in \mathfrak{M}_\varepsilon$. For the points

$$x \in \mathfrak{M}_\varepsilon \subseteq \mathfrak{M}_\Delta(w)$$

we have also $\omega(x) \geqslant \Delta$. From these and (3.1) we infer that, for sufficiently large $n$,

$$\begin{aligned}\frac{1}{\gamma_n^2(w)} = \int_{-1}^{1} \omega_n^2(w,x)w(x)\,dx &\geqslant \int_{\mathfrak{M}_\varepsilon} \omega_n^2(w,x)w(x)\,dx \\ &\geqslant |\mathfrak{M}_\varepsilon|\,\Delta(1-\varepsilon)^{4n}2^{-2n} \\ &\geqslant \tfrac{1}{2}\psi(\varepsilon)\Delta(1-\varepsilon)^{4n}2^{-2n},\end{aligned}$$

that is,

$$\overline{\lim_{n\to\infty}}\, \sqrt[n]{(\gamma_n(w))} \leqslant 2(1-\varepsilon)^{-2}.$$

Letting $\varepsilon$ tend to zero, we see that (1.7) holds.

THEOREM 3.1. *Let $w$ be arc-sine on $[-1,1]$; further let* $\operatorname{supp} d\alpha \subseteq [-1,1]$ *and let $\alpha'(x) \geqslant Kw(x)$ hold for a constant $K > 0$ and almost every $x \in [-1,1]$; then also $d\alpha$ is arc-sine on $[-1,1]$.*

*Proof.* Since $p_n(w)$ and $p_n(Kw)$ have the same zeros, we can take $K = 1$. We have

$$\begin{aligned}\text{(3.13)}\qquad \frac{1}{\gamma_n^2(w)} &= \inf_{Q \in \mathfrak{P}_n^*} \int_{-1}^{1} Q^2(x)w(x)\,dx \\ &\leqslant \int_{-1}^{1} \{[\gamma_n(d\alpha)]^{-1} p_n(d\alpha,x)\}^2 w(x)\,dx \\ &\leqslant \frac{1}{\gamma_n^2(d\alpha)} \int_{-1}^{1} p_n^2(d\alpha,x)\,d\alpha(x) = \frac{1}{\gamma_n^2(d\alpha)}.\end{aligned}$$

Since $w$ is arc-sine on $[-1,1]$,

$$\text{(3.14)}\qquad \overline{\lim_{n\to\infty}}\, \sqrt[n]{(\gamma_n(dx))} \leqslant 2.$$

Since $\operatorname{supp} d\alpha \subseteq [-1,1]$, we have $-1 < x_{nn}(d\alpha) < x_{1n}(d\alpha) < 1$, so that by Lemma 2.1 and (3.14) $x_{1n}(d\alpha) \to 1$, $x_{nn}(d\alpha) \to -1$. Thus the conditions of Theorem 1.1 are satisfied and consequently $d\alpha$ is arc-sine on $[-1,1]$.

## 4. Investigation of certain weights with infinite support

We denote by $c_1, c_2, \ldots$ positive numbers independent of $n$ but possibly dependent on the choice of the weight.

In [5], Freud introduced the weights

$$w_Q(x) = \exp\{-2Q(|x|)\} \quad (-\infty < x < \infty),$$

where $Q(x)$ $(0 \leqslant x < \infty)$ is a positive increasing differentiable function and $x^\rho Q'(x)$ $(x \geqslant 0)$ is increasing for some $\rho < 1$. By our condition,

$$Q(x) = Q(0) + \int_0^x Q'(t)\,dt \leqslant Q(0) + x^\rho Q'(x) \int_0^x t^{-\rho}\,dt \tag{4.1}$$

$$= Q(0) + (1-\rho)^{-1} x Q'(x),$$

so the moments $\mu_m(w_Q)$ are finite because

$$Q(x) \geqslant Q(1) + (1-\rho)^{-1} Q'(1) x^{1-\rho}.$$

We denote by $q_s$ $(s \geqslant 0)$ the solution of the equation $q_s Q'(q_s) = s$.

It is proved in [5] that

$$c_1 q_n \leqslant x_{1n}(w_Q) \leqslant c_2 q_n. \tag{4.2}$$

Since $w_Q$ is even, we have

$$x_{nn}(w_Q) = -x_{1n}(w_Q). \tag{4.3}$$

THEOREM 4.1. *If $w_Q$ is arc-sine then* (1.5) *and* (1.6) *are satisfied.*

Note that Theorem 4.1 and Theorem 1.5 together show that (1.5) as well as (1.6) are necessary and sufficient conditions for $w_Q$ to be arc-sine.

*Proof.* By assumption $([x_{1n}(w_Q)]^{-n}[\gamma_n(w_Q)]^{-1} p_n(w_Q, x_{1n}x)) = (\omega_n(w_Q, x))$ is a sequence of monic polynomials which is arc-sine on $[-1,1]$. Let $\mathscr{T}(\eta) = [-\eta, \eta]$. By Lemma 3.1, we have, for every $0 < \eta < 1$ and every $\varepsilon > 0$,

$$\| \omega_n(w_Q) \|_{\mathscr{T}(\eta)} \geqslant 2^{-n}(1-\varepsilon)^n \quad (n \geqslant c_3(\varepsilon)). \tag{4.4}$$

By Lemma 3.2, $\mathscr{T}(\eta)$ has a measurable subset $\mathscr{T}_\varepsilon(\eta)$ of measure at least $2\eta\psi(\varepsilon)$, so

$$|\omega_n(w_Q, x)| \geqslant 2^{-n}(1-\varepsilon)^{2n} \quad (x \in \mathscr{T}_\varepsilon(\eta),\ n \geqslant c_3(\varepsilon)). \tag{4.5}$$

If $t \in \mathscr{T}_\varepsilon(\eta) \subseteq \mathscr{T}(\eta)$, we have by (4.2) and (4.3), provided that $\eta c_2 < 1$,

$$-\log w_Q(t x_{1n}) \leqslant 2Q(\eta x_{1n}) \leqslant 2Q(0) + (1-\rho)^{-1} \eta x_{1n} Q'(\eta x_{1n}) \tag{4.6}$$

$$\leqslant 2Q(0) + (1-\rho)^{-1} c_2 \eta q_n Q'(\eta c_2 q_n)$$

$$\leqslant 2Q(0) + (1-\rho)^{-1} c_2 \eta (c_2 \eta)^{-\rho} Q'(q_n) = 2Q(0) + c_4 \eta^{1-\rho} n.$$

By the transformation $x = x_{1n} t$,

$$1 = \int_{-\infty}^{\infty} p_n^2(w_Q, x) w_Q(x)\,dx = [x_{1n}(w_Q)]^{n+1} \gamma_n(w_Q) \tag{4.7}$$

$$\times \int_{\mathscr{T}_\varepsilon(\eta)} \omega_n^2(w_Q, t) w_Q(x_{1n} t)\,dt$$

$$\geqslant \mathscr{T}_\varepsilon(\eta) 2^{-2n} (1-\varepsilon)^{4n} \exp\{-2Q(0) - c_4 \eta^{1-\rho} n\}$$

$$\geqslant c_5 \eta \psi(\varepsilon) 2^{-2n} (1-\varepsilon)^{4n} \exp\{-c_4 \eta^{1-\rho} n\},$$

the second half by (4.5) and (4.6). Since (4.7) must hold for arbitrary small $\eta$ and $\varepsilon$, we infer that

$$\overline{\lim_{n\to\infty}} \{[x_{1,n-1}(w_Q)]^{n-1} \sqrt[n-1]{(\gamma_{n-1}(w_Q))}\} \leqslant 2. \tag{4.8}$$

The zeros of $p_n(w_Q)$ and $p_{n-1}(w_Q)$ separate each other, so

$$x_{2n}(w_Q) < x_{1,n-1}(w_Q) < x_{1n}(w_Q).$$

Since $w_Q$ is arc-sine by assumption, we have $x_{2n}(w_Q)/x_{1n}(w_Q) \to 1$; consequently $x_{1,n-1}(w_Q)/x_{1n}(w_Q) \to 1$. Combining this with (4.8) and (4.3), we see that (1.5) is valid. By Theorem 1.1, this implies that (1.6) also is satisfied.

Remark. Erdős investigated† in [**2**] the weights $w_R(x) = \exp\{-2R(x)\}$ where the (not necessarily differentiable) function $R(x)$ satisfies, for every $\varepsilon > 0$,

$$R(y) > 2R(x) \quad (|y| > (1+\varepsilon)|x| > c_6(\varepsilon)). \tag{4.9}$$

It is proved in [**2**] that $w_R$ is arc-sine and the proof implies that (1.6) is valid in this case.

Theorem 4.2. *If, for an increasing subsequence $(n_j)$ of the natural numbers, we have*

$$\overline{\lim_{j\to\infty}} \{\sqrt[n_j-1]{(\gamma_{n_j-1}(d\alpha))}(x_{1n}(d\alpha) - x_{nn}(d\alpha)) \leqslant 4 \tag{4.10}$$

*then, putting $x_{kn} = \frac{1}{2}(x_{1n}+x_{nn}) + \frac{1}{2}(x_{1n}-x_{nn})\tau_{kn}$, we have*

$$\lim_{j\to\infty} n_j^{-1} \sum_{k:\,\tau_{kn}\geqslant T} 1 = \frac{1}{\pi}\arccos T. \tag{4.11}$$

*Proof.* If $n_j = j$, this is just Theorem 1.1. The proof of Theorem 4.2 follows by replacing $n$ by $n_j$ in the proof of Theorem 1.1. Details are left to the reader.

Theorem 4.3. *If $Q^*(x)$ satisfies, besides the conditions indicated for $Q(x)$, the inequality*

$$Q^*(2x) \leqslant c_7 Q^*(x) \tag{4.12}$$

*then*

$$\underline{\lim}_{n\to\infty} x_{1n}(w_{Q^*})\, \sqrt[n-1]{(\gamma_{n-1}(w_{Q^*}))} > 2. \tag{4.13}$$

† We have made an obvious change of notation.

*Proof.* Let $(n_j)$ be an increasing subsequence of the natural numbers for which

$$(4.14)\quad \lim_{j\to\infty} x_{1n_j}(w_{Q^*})\,{}^{n_j-1}\!\sqrt{(\gamma_{n_j-1}(w_{Q^*})} = \lim_{n\to\infty} x_{1n}(w_{Q^*})\,{}^{n-1}\!\sqrt{(\gamma_{n-1}(w_{Q^*}))}.$$

If (4.11) is not satisfied for the sequence $(n_j)$ then (4.13) is a consequence of Theorem 4.2. Thus we can assume in what follows that (4.11) holds. We consider the monic polynomials of degree $n_j-1$

$$(4.15)\qquad \omega^*_{n_j-1}(x) = 2^{-n_j+m+2}(x_{1,n_j})^{n_j-m-1}x^m T_{n_j-m-1}(x/x_{1,n_j}).$$

Here $x_{kn_j} = x_{kn_j}(w_{Q^*})$. Then by the minimum property (3.13),

(4.16)

$$\gamma_{n_j-1}{}^{-2}(w_{Q^*}) \leqslant \int_{-\infty}^{\infty} [w^*_{n_j-1}(x)]^2 w_{Q^*}(x)\,dx$$

$$= 2^{-2n_j+2m+4}(x_{1,n_j})^{2n_j-2m-2}\int_{-\infty}^{\infty} x^{2m}T_{n_j-m-1}{}^2(x/x_{1,n_j})w_{Q^*}(x)\,dx.$$

We apply the Gauss–Jacobi quadrature formula to the integral (4.16) and take $|T_n(x)| \leqslant 1$ for $|x| \leqslant 1$ into consideration; then

$$(4.17)\quad \int_{-\infty}^{\infty} x^{2m}T^2_{n_j-m-1}(x/x_{1n_j})w_{Q^*}(x)\,dx$$

$$= \sum_{k=1}^{n_j} \lambda_{kn_j}(w_{Q^*})x^{2m}_{kn_j}T_{n_j-m-1}{}^2(x_{kn_j}/x_{1n_j})$$

$$\leqslant \sum_{k=1}^{n_j} \lambda_{kn_j}(w_{Q^*})x^{2m}_{k,n_j}$$

$$\leqslant (x_{1n_j}/4)^{2m}\sum_{k=1}^{n_j} \lambda_{kn_j}(w_{Q^*}) + x^{2m}_{1n_j}\sum_{|x_{kn_j}|>x_{1n_j}/4} \lambda_{kn}(w_{Q^*}).$$

By the quadrature formula,

$$(4.18)\qquad \sum_{k=1}^{n_j} \lambda_{k,n_j}(w_{Q^*}) = \mu_0(w_{Q^*}).$$

It follows from (4.11) that, for sufficiently great $n_j$, there exist roots $x_{ln_j}$ of $p_{n_j}(w_{Q^*})$ situated in $[x_{1n_j}/5, x_{1n_j}/4]$. Thus by the Markov–Stieltjes inequality ([**4**], § 1.5) and by symmetry,

$$(4.19)\quad \sum_{x_{k,n_j}<-x_{1,n_j}/5} \lambda_{k,n_j}(w_{Q^*}) = \sum_{x_{k,n_j}>x_{1,n_j}/5} \lambda_{k,n_j}(w_{Q^*}) \leqslant \int_{x_{1,n_j}/5}^{\infty} w_{Q^*}(t)\,dt.$$

Since $x^\rho Q'(x)$ is increasing, we have

$$Q^*(x) \geqslant \int_{\frac12 x}^{x} Q^*(t)\,dt \geqslant Q^*(\tfrac12 x)(\tfrac12 x)^\rho \int_{\frac12 x}^{x} t^{-\rho}\,dt \geqslant c_8 x Q^{*\prime}(x).$$

Denoting by $q_s^*$ the solution of the equation $q_s^* Q'(q_s^*) = s$, we obtain, by (4.2) and (4.12),

$$\left\{\begin{array}{l} Q^*(\tfrac{1}{5}x_{1n}) \geqslant Q^*(\tfrac{1}{5}c_1 q_n^*) \geqslant \tfrac{1}{5}c_1 c_8 q_n^*, \\ Q^*(\tfrac{1}{10}c_1 q_n^*) \geqslant c_9 q_n^* Q^*(q_n^*) = c_9 n. \end{array}\right. \tag{4.20}$$

By (4.19) and (4.20),

$$\sum_{|x_{k,n_j}|>x_{1n_j}/4} \lambda_{kn_j} \leqslant 2\int_{x_{1,n_j}/5} e^{-2Q^*(x)}\,dx$$

$$\leqslant 2\exp\{-Q^*(x_{1,n_j}/5)\}\int_0^\infty e^{-Q^*(x)}\,dx \leqslant c_{10}e^{-c_9 n}.$$

By formulae (4.17)–(4.20),

$$\int_{-\infty}^{\infty} x^{2m} T_{n_j-m-1}{}^2(x/x_{1,n_j}) w_{Q^*}(x)\,dx \leqslant x_{1,n_j}{}^{2m}[\mu_0(w_{Q^*})4^{-2m} + c_{10}e^{-c_9 n}];$$

hence, by (4.16),

$$\gamma_{n_j-1}{}^{-2}(w_{Q^*})(x_{1,n_j})^{-2n_j+2}2^{2n_j-2} \leqslant 4[\mu_0(w_Q^*)2^{-m} + c_{10}2^m e^{-c_9 n_j}]. \tag{4.21}$$

Up to now we have not disposed of the integer $m$. Let us put $m = [c_9 n_j/2\log 2]$, that is, $2^m \sim \exp\{\tfrac{1}{2}c_9 n_j\}$. Inserting this (4.21), we see that the limit (4.14) is greater than $2e^{\frac{1}{4}c_9} > 2$.

Corollary. *Let* $w_Q(x) = \exp\{-2Q(|x|)\}$, *where* $Q(x)$ *is differentiable,* $x^\rho Q'(x)$ $(x \geqslant 0)$ *is increasing for some* $\rho < 1$, *and* $0 < Q'(2x) \leqslant c_7 Q'(x)$ *for* $x > 0$; *then* $w_Q$ *is not arc-sine.*

This corollary is a consequence of Theorem 4.1 and Theorem 4.3 since $x_{nn}(w_Q) = -x_{1n}(w_Q)$. We observe that our corollary implies that $w_\alpha(x) = \exp(-|x|^\alpha)$ is not arc-sine for any $\alpha > 0$. This was stated without proof by Erdős in [**2**].

As a last item of our paper, we show that *the sufficient condition* (4.10) *is not necessary for* (4.11).

Lemma 4.1. *If the weight* $W(x)$ *is even and decreasing for* $x > 0$, *we have, for every* $\xi > 0$ *and* $\eta > 0$,

$$\tfrac{1}{2}[c_{11}\eta^{2n-1}W(\eta)]^{1/(2n-2)} \leqslant x_{1n}(W) \leqslant \xi + c_{12}(2/\xi)^{2n-1}\int_\xi^\infty x^{2n-1}W(x)\,dx. \tag{4.22}$$

Lemma 4.1 is proved in [**5**].

Let $n_0 = 0$, $n_1 = 1$, …, $n_{k+1} = e^{n_k}$ $(k = 1, 2, \ldots)$ and

$$W(x) = e^{-n_k} \quad (n_{k-1} < |x| \leqslant n_k;\ k = 1, 2, \ldots). \tag{4.23}$$

Then

$$\int_{n_k}^{\infty} x^{2n_k} W(x)\,dx \leqslant 2e^{-n_{k+1/2}} \quad (k \geqslant c_{13}). \tag{4.24}$$

Inserting $\xi = \eta = n_k$ in (4.22), we get

$$c_{14} n_k \leqslant x_{1n_k}(W) \leqslant n_k + c_{15}. \tag{4.25}$$

We put $\nu = n_k$, $\nu_1 = n_{k-1} = \log\nu$, $\mu = [\nu/\log\nu]+1$, and $x_\nu = x_{1\nu}(W)$. The polynomial $x^\mu T_{\nu-\mu}(x/x_\nu)$ has leading coefficient $2^{\nu-\mu-1}x_\nu^{-\nu+\mu}$; thus, by the extremum property of $\gamma_\nu(W)$,

$$\begin{aligned}
\left[\frac{2^{\nu-\mu-1}}{\gamma_\nu(W)x_\nu^{\nu-\mu}}\right]^2 &\leqslant \int_{-\infty}^{\infty} x^{2\mu}[T_{\nu-\mu}(x/x_\nu)]^2 W(x)\,dx \\
&\leqslant 2\int_0^{\nu_1} x^{2\mu}\,dx + 2e^{-\nu}\int_{\nu_1}^{x_\nu} x^{2\mu}\,dx \\
&\qquad + 2^{2\nu}x_\nu^{-2\nu+2\mu}\int_{x_\nu}^{\infty} x^{2\nu}W(x)\,dx \\
&\leqslant 2\nu_1^{2\mu+1} + 2e^{-\nu}x_\nu^{2\mu+1} + 2^{2\nu}x_\nu^{2\mu+1}\exp\{-e^{-\nu}/2\} \\
&\leqslant c_{16}x_\nu^{2\mu+1}e^{-\nu}\exp\left\{c_{17}\frac{\nu\log\log\nu}{\log\nu}\right\}.
\end{aligned} \tag{4.26}$$

In consequence of (4.26) and $x_\nu = x_{1\nu} = -x_{\nu\nu}$, the left-hand side of (4.10) is greater than $4\sqrt{e} > 4$, that is, (4.10) is not valid for the choice of $d\alpha = W\,dx$. In spite of that, we show that $W$ is arc-sine.

Let us suppose the contrary. Then there exists $\delta > 0$ such that the maximum modulus of the monic polynomial of degree $\nu$ in $t$, $\gamma_\nu^{-1}(W)x_\nu^{-\nu}p_\nu(W, x_\nu t)$ ($t \in [-1, 1]$), exceeds $2^{-\nu}(1+\delta)^{2\nu}$ and consequently, by Lemma 3.2,

$$|p_\nu(W; x)| \geqslant \gamma_\nu(W)x_\nu^{\nu}2^{-\nu}(1+\delta)^{\nu} \quad (x \in M_\nu), \tag{4.27}$$

where $M_\nu \subseteq [-x_\nu, x_\nu]$ and $|M_\nu| > 2x_\nu\psi(\delta)$. Since $x_\nu < \nu + O(1)$, (4.27) is valid for a subset $M_\nu^*$ of $[-\nu, \nu]$ satisfying $|M_\nu^*| > x_\nu\psi(\delta)$ if $\nu$ is sufficiently great. We infer that

$$\begin{aligned}
1 = \int_{-\infty}^{\infty} p_\nu^2(W, x)W(x)\,dx &\geqslant \int_{M_\nu^*} p_\nu^2(W, x)W(x)\,dx \\
&\geqslant x_\nu\psi(\delta)\gamma_\nu^2(W)x_\nu^{2\nu}2^{-2\nu}(1+\delta)^{\nu}e^{-\nu};
\end{aligned} \tag{4.28}$$

but (4.28) contradicts (4.26), which means that our assumption that $W$ is not arc-sine was false. Thus $W$ furnishes the example indicated.

## 5. On determining sets

The lower capacity $\mathscr{L}(A)$ of a set $A \subseteq [-1,1]$ is defined by

$$\mathscr{L}(A) = \inf_{\substack{B \subseteq A \\ |A \setminus B| = 0}} C(B). \tag{5.1}$$

LEMMA 5.1 (Ullman). *A measurable subset $A$ of $[-1,1]$ is a determining set if and only if*

$$\mathscr{L}(A) = \tfrac{1}{2}. \tag{5.2}$$

*Proof.* (5.2) is necessary by [**9**], Theorem 1.2, and [**9**], Lemma 1.2. In order to prove that (5.2) is sufficient, it is enough to show that the following additional hypothesis, assumed in [**9**], Theorem 1.2, is satisfied: for every interval $\mathscr{T} \subseteq [-1,1]$ we have $|A \cap \mathscr{T}| > 0$. In fact, supposing the contrary, we would have $\frac{1}{2} = \mathscr{L}(A) \leqslant C([-1,1] \setminus \mathscr{T}) < \frac{1}{2}$ (the last part: for example, [**9**], Lemma 5.4). Thus $|A \cap \mathscr{T}| > 0$.

In order to prove Theorem 1.3, we prove a more general result concerning stability of capacities.†

Let $\nu$ be a $\sigma$-additive Borel measure on the plane and $A$ a $\nu$-measurable point set of the plane; we denote the outer measure of $B$ by $\nu(B)$. We define the lower $\nu$-capacity $C_\nu(A)$ of $A$ as follows: for $\varepsilon > 0$, let $\mathscr{K}(\varepsilon)$ denote the set of compact subsets $K$ of $A$ satisfying $\nu(A \setminus K) < \varepsilon$. Let

$$C_\varepsilon(\nu, A) = \inf_{K \in \mathscr{K}(\varepsilon)} C(K); \tag{5.3}$$

clearly $C_\varepsilon(\nu, A)$ is an increasing function of $\varepsilon$. We define

$$C(\nu, A) = \lim_{\varepsilon \to 0} C_\varepsilon(\nu, A). \tag{5.4}$$

LEMMA 4.2. *For every $\nu$-measurable plane set $A$, there exists a subset $A^\nu \subseteq A$ for which $\nu(A \setminus A^\nu) = 0$ and*

$$C(A^\nu) = C(\nu, A). \tag{5.5}$$

*Proof.* Let $\varepsilon_n = 2^{-n}$ $(n = 1, 2, \ldots)$. By our definitions, there exist compact sets $K \subseteq A$ $(n = 1, 2, \ldots)$ such that

$$\nu(A \setminus K_n) \leqslant \varepsilon_n \tag{5.6}$$

and

$$C_{\varepsilon_n}(\nu, A) \leqslant C(K_n) \leqslant C_{\varepsilon_n}(\nu, A) + \varepsilon_n. \tag{5.7}$$

We apply the same notations as in Tsuji's book [**11**] and let $\mu_n$ be the equilibrium distribution on $K_n$ and $\mathscr{U}(\mu_n, z)$ the conductor potential of

† A detailed proof of the following Lemma 5.2 was published by Freud in [**6**]. Here we repeat the proof briefly.

$K_n$. By [11], § II.2, there exists a sequence $(n_j)$ such that $\mu_{n_j}$ converges to a Borel measure $\mu$. We define

$$A^\nu = \varliminf_{j\to\infty} K_n = \bigcup_{m=1}^{\infty} \bigcap_{j=m}^{\infty} K_{n_j} \subseteq A.$$

It follows that

$$\nu(A \setminus A^\nu) \leqslant \sum_{j=m}^{\infty} \nu(A \setminus K_{n_j}) \leqslant \sum_{j=m}^{\infty} \varepsilon_{n_j} \leqslant 2^{-m+1},$$

that is $\nu(A \setminus A^\nu) = 0$, as required.

We say that a property is satisfied *almost everywhere* (in short, *a.e.*) if the exceptional set is a Borel set of zero capacity. By the definition of $\mu_{n_j}$, we have, for every $z$,

$$\mathscr{U}(\mu_{n_j}, z) \leqslant \log \frac{1}{C(K_{n_j})} \tag{5.8}$$

and

$$\mathscr{U}(\mu_{n_j}, z) = \log \frac{1}{C(K_{n_j})} \quad \text{a.e. } z \in K_{n_j}. \tag{5.9}$$

By the lower-envelope principle (de la Vallée-Poussin, [10], II.69, or [9], Lemma 5.3) we infer from (5.10) (5.7), (4.8), and (5.9) that

$$\mathscr{U}(\mu, z) = \varliminf_{j\to\infty} \mathscr{U}(\mu_{n_j}, z) \leqslant \lim_{\varepsilon\to 0} \log \frac{1}{C_\varepsilon(\nu, A)} = \log \frac{1}{C(\nu, A)} \quad \text{a.e. } z \tag{5.10}$$

and that the sign of equality holds in (5.10) a.e. $z \in A^\nu$. In consequence of these properties, $\mu$ is the equilibrium distribution of a set covering $A^\nu$ and $C(A^\nu) \leqslant C(\nu, A)$.

Since $A^\nu \subseteq A$ and $\nu(A \setminus A^\nu)$, we have $C_\varepsilon(\nu, A) \leqslant C(A^\nu)$ for every $\varepsilon > 0$; when $\varepsilon \to 0$, we get $C(A^\nu) = C(\nu, A)$.

*Proof of Theorem* 1.3. Let $\lambda$ denote the linear Lebesgue measure on $[-1, 1]$. By Lemma 5.2 we have, for every measurable $A \subseteq [-1, 1]$, $\mathscr{L}(A) \leqslant C(A^\lambda) = C(\lambda, A)$. By [9] (see Lemma 3.3), there exists a subset $A_0 \subseteq A$ satisfying $C(A_0) = \mathscr{L}(A)$ and $|A \setminus A_0| = 0$. By (4.3)

$$C_\varepsilon(\lambda, A) \leqslant C(A_0) = \mathscr{L}(A)$$

for every $\varepsilon > 0$. This implies $C(\lambda, A) \leqslant \mathscr{L}(A)$, so $\mathscr{L}(A) = C(\lambda, A)$. For a 'good set' $A$, we have, by Definition 1.2, $C(\lambda, A) = \frac{1}{2}$, that is, $\mathscr{L}(A) = \frac{1}{2}$. Now Theorem 1.3 follows from Lemma 5.1.

## REFERENCES


1. S. N. **Bernstein**, *Leçons sur les propriétés extrémales et la meilleure approximation des fonctions analytiques d'une variable réelle* (Gauthier-Villars, Paris, 1926).
2. P. **Erdős**, 'On the distribution of the roots of orthogonal polynomials', *Proc. Conf. on Constructive Theory of Functions* (Budapest, 1969), ed. G. Alexits and S. B. Steckhin (Akadémiai Kiadó, Budapest, 1972).
3. —— and P. **Turán**, 'On interpolation, III', *Ann. of Math.* 41 (1940) 510–55.
4. G. **Freud**, *Orthogonale polynome* (Birkhäuser, Basel, 1969); English trans. by L. Földes (Pergamon, New York, 1971).
5. —— 'On the greatest zero of an orthogonal polynomial, II', *Acta Sci. Math. (Szeged)*, to appear.
6. —— 'On a class of sets introduced by P. Erdős', *Proc. Internat. Colloq. on infinite and finite sets* (Keszthely, 1973).
7. L. Ja. **Geronimus**, *Orthogonal polynomials* (Consultants' Bureau, New York, 1961).
8. E. J. **Remez**, 'Sur une propriété des polynomes de Tchebycheff', *Commun. Inst. Sci. Kharkow* 13 (1936) 93–95.
9. J. L. **Ullman**, 'On the regular behaviour of orthogonal polynomials', *Proc. London Math. Soc.* (3) 24 (1972) 119–48.
10. Ch. J. **de la Vallée-Poussin**, *Le potentiel logarithmique* (Gauthier-Villars, Paris, 1949).
11. M. **Tsuji**, *Potential theory* (Maruzen, Tokyo, 1959).



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