Canad. Math. Bull. Vol. 17 (4), 1974.

ON ABUNDANT-LIKE NUMBERS

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Problem 188, [3], stated: Apart from finitely many primes p show that if n_p is the smallest abundant number for which p is the smallest prime divisor of n_p, then n_p is not squarefree.

Let $2=p_1<p_2<\cdots$ be the sequence of consecutive primes. Denote by $n_k^{(c)}$ the smallest integer for which p_k is the smallest prime divisor of $n_k^{(c)}$ and $\sigma(n_k^{(c)})\geq cn_k^{(c)}$ where $\sigma(n)$ denotes the sum of divisors of n. Van Lint's proof, [3], gives without any essential change that there are only a finite number of squarefree integers which are $n_k^{(c)}$'s for some $c\geq 2$. In fact perhaps 6 is the only such integer. This could no doubt be decided without too much difficulty with a little computation.

Note that $n_2^{(2)}=945=3^3\cdot 5\cdot 7$. I will prove that $n_k^{(2)}$ is cubefree for all $k>k_0$, the exceptional cases could easily be enumerated. The cases $1<c<2$ causes unexpected difficulties which I have not been able to clear up completely. I will use the methods developed in the paper of Ramunujan on highly composite numbers [1]. A well known result on primes states that for every s, [2],

$$\sum_{p<x}\frac{1}{p}=\log\log x+B+0\left(\frac{1}{(\log x)^s}\right). \tag{1}$$

(1) implies

$$\sum_{x<p<x^{1+a}}\frac{1}{p}=\log(1+a)+0\left(\frac{1}{(\log x)^s}\right). \tag{2}$$

It would be interesting to decide whether

$$\sum_{x<p<x^{1+a}}\frac{1}{p}-\log(1+a) \tag{3}$$

changes sign infinitely often. I do not know if this question has been investigated.

THEOREM 1. *$n_k^{(2)}$ is cubefree for all $k>k_0$.*

Clearly (see [1])

$$k_k^{(2)}=\prod_{i=0}^{l}p_{k+i}^{\alpha_i},\qquad \alpha_0\geq\alpha_1\geq\cdots\geq\alpha_l. \tag{4}$$

It is easy to see that

$$\exp\left(\sum_{i=1}^{l}\frac{1}{p_{k+i}-1}\right)>\frac{\sigma(n_k^{(c)})}{n_k^{(c)}}\geq\exp\left(\sum_{i=1}^{l}\frac{1}{p_{k+i}}-\sum_{i=1}^{l}\frac{1}{p_{k+i}^2}\right).$$

This, together with the definition of $n_k^{(c)}$, and a simple computation imply

$$\sum_{i=1}^{l} \frac{1}{p_{k+i}} = \log c + 0\left(\frac{1}{k}\right)$$

and hence by (2) we have

$$\lim_{k\to\infty} \frac{p_{k+l}}{p_k^c} = 1. \tag{5}$$

Let $c=2$. We show that if $\varepsilon>0$ is small enough then for every u such that $p_{k+u}<(1+\varepsilon)p_k$. We have

$$\alpha_{k+u} \geq 2. \tag{6}$$

If (6) would be false put

$$N = n_k^{(2)} p_{k+u} p_{k+u+1} p_{k+u+2} p_{k+l}^{-1} p_{k+l-1}^{-1} < n_k^{(2)} \tag{7}$$

by (5) and $p_{k+u+2}<2p_k$. Further for $k>k_0$, $p_{k+u+2}<(1+2\varepsilon)p_k$ by the prime number theorem. Thus for sufficiently small ε we have by a simple computation

$$\frac{\sigma(N)}{N} > \frac{\sigma(n_k^{(2)})}{n_k^{(2)}}. \tag{8}$$

(7) and (8) contradict the definition of $n_k^{(2)}$ and thus (6) is proved.

Now we prove Theorem 1. Let p_{k+u} be the greatest prime not exceeding $(1+\varepsilon)p_k$. By the prime number theorem

$$p_{k+u} > \left(1+\frac{\varepsilon}{2}\right) p_k.$$

Assume $\alpha_k \geq 3$. Put $N_1 = n_k^{(2)} p_{k+l+1} p_k^{-1} p_{k+u}^{-1}$. By (5), $N_1 < n_k^{(2)}$ and by a simple computation $\sigma(N_1)/N_1 > \sigma(n_k^{(2)})/n_k^{(2)}$, which again contradicts the definition of $n_k^{(c)}$. This proves Theorem 1.

THEOREM 2. *$n_k^{(2)} = \prod_{i=0}^{u} p_{k+i}^2 \prod_{i=u+1}^{l} p_{k+i}$ where*

$$\lim_{k=\infty} \frac{p_{k+l}}{p_k^2} = 1, \qquad \lim_{k=\infty} \frac{p_{k+u}}{p_k} = 2^{1/2}. \tag{9}$$

The first equation of (9) is (5), the proof of the second is similar to the proof of Theorem 1 and we leave it to the reader.

Henceforth we assume $1<c<2$. It seems likely that for every c there are infinitely many values of k for which $n_k^{(c)}$ is squarefree and also there are infinitely many values of k for which $n_k^{(c)}$ is not squarefree. I can not prove this. Denote by A the set of those values c for which $n_k^{(c)}$ is infinitely often not squarefree and B denotes the set of those c's for which $n_k^{(c)}$ is infinitely often squarefree.

THEOREM 3. *A, B and $A \cap B$ are everywhere dense in* $(1, 2)$.

We only give the proof for the set A, for the other two sets the proof is similar. Let $1 \leq u_1 < v_1 \leq 2$. It suffices to show that there is a c in A with $u_1 < c < v_1$. Let k_1 be sufficiently large and let l_1 be the smallest integer for which

$$\prod_{i=0}^{l}\left(1+\frac{1}{p_{k_1+i}}\right) = \sigma\left(\prod_{i=0}^{l_1} p_{k_1+i}\right)\Big/\prod_{i=0}^{l_1} p_{k_1+i} > u_1 \tag{10}$$

Put $x_1 = \prod_{i=0}^{l_1} p_{k_1+i}$. We show that for every α satisfying

$$u_1 < \frac{\sigma(x_1)}{x_1} < \alpha < \frac{\sigma(p_{k_1}x_1)}{p_{k_1}x_1} < v_1 \tag{11}$$

we have

$$n_{k_1}^{(\alpha)} = p_{k_1}x_1. \tag{12}$$

To prove (12) write

$$n_{k_1}^{(\alpha)} = \prod_{i=1}^{j} p_{k_1+i}^{\alpha_i}, \qquad \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_j.$$

We show $\alpha_0=2$, $\alpha_1=1$, $j=l_1$ which implies (12). Assume first $\alpha_1 \geq 2$. For sufficiently large k_1 we have from (5)

$$T = n_{k_1}^{(\alpha)} p_{k_1+j+1} p_{k_1}^{-1} p_{k_1+1}^{-1} < n_{k_1}^{(\alpha)} \quad \text{and} \quad \frac{\sigma(T)}{T} > \frac{\sigma(n_{k_1}^{(\alpha)})}{n_{k_1}^{(\alpha)}}$$

which contradicts the definition of $n_k^{(\alpha)}$. Thus $\alpha_1=1, j \leq l_1$ follows from (5) and (11) and $\alpha_0 < 3$ follows like $\alpha_1=1$. Thus by (10) $j=l$ and (12) is proved. Thus for the interval (11) $n_k^{(\alpha)}$ is not squarefree. Now put

$$u_2 = \frac{\sigma(x_1)}{x_1}, \qquad v_2 = \frac{\sigma(p_{k_1}x_1)}{p_{k_1}x_1}.$$

Let p_{k_2} be sufficiently large and repeat the same argument for (u_2, v_2) which we just need for (u_1, v_1). We then obtain $x_2 = \prod_{i=0}^{l_2} p_{k_2+i}$ so that for every α in $u_2 < \sigma(x_2)/x_2 < \alpha < \sigma(p_{k_2}x_2)/p_{k_2}x_2 < v_2$ $n_{k_2}^{(\alpha)} = p_{k_2}x_2$ and is thus not squarefree. This construction can be repeated indefinitely and let c be the unique common point of the intervals (u_i, v_i), $i=1, 2, \ldots$. Clearly $n_{k_1}^{(c)} = p_{k_2}x_i$ is not squarefree for infinitely many integers k_i or c is in A which completes the proof of Theorem 3.

I can prove that B has measure 1 and that for a certain α every $1 < c < 1+\alpha$ is in B. I can not prove the same for A. I do not give these proofs since it seems very likely that every c, $1 < c < 2$ is in $A \cap B$.

Let $r > 2$ be an integer. It is not difficult to prove by the method used in the proof of Theorem 1 that $p_k^r \mid n_k^{(r)}$ for all $k > k_0(r)$, but for $k > k_0(r)$, $p_k^{r+1} \mid n_k^{(r)}$ i.e. $n_k^{(r)}$ is divisible by an rth power but not an $(r+1)$st power.