MATHEMATICAL STRUCTURES — COMPUTATIONAL
MATHEMATICS — MATHEMATICAL MODELLING
Papers dedicated to Professor L. Iliev's 60th Anniversary
Sofia, 1974, p. 225-234

CHEBYSHEV RATIONAL APPROXIMATION TO ENTIRE FUNCTIONS IN $[0, \infty]$

P. Erdős, A. R. Reddy

Dedicated to Professor L. Iliev's 60th Anniversary

Summary. Let $f(z)$ be an entire function with non-negative coefficients. Put

$$\min \max_{0 \leq x < \infty} |1/f(z) - 1/g_n(z)| = A_n(z),$$

where the minimum is taken over all polynomials of degree not exceeding n. The authors obtain various inequalities for $A_n(f)$, e. g. they prove that if $f(z)$ is of infinite order then for every $\varepsilon > 0$ $A_n(f) > e^{-\varepsilon n}$ holds for infinitely many values of n, but if $f(z)$ is of finite order then for every $\varepsilon > 0$, $A_n(f) < c^n$ holds for infinitely many n.

Introduction: Quite recently Chebyshev rational approximation to certain entire functions on the whole positive axis has attracted the attention of many mathematicians. In this respect the papers ([3—7, 9]) are worth mentioning. All these papers have been devoted only to entire functions of finite order. On the other hand, methods developed and used in these papers are valid only to entire functions of finite order. In this paper we develop a method by which we can get results for functions of zero, finite as well as for infinite orders. We also obtain lower bounds for $\lambda_{0,n}$, the Chebyshev constants for $1/f$ on $[0, \infty)$. Besides this, we obtain much more precise information in the case of functions of zero order. In fact we give an example which shows clearly how much closely one can approximate entire functions of small growth.

Notation. For any non-negative integer n, π_n denotes the collection of real polynomials of degree at most n. Then let

$$\lambda_{0,n} = \inf_{P_n \in \pi_n} \| 1/f - 1/P_n \|_{[0,\infty)}$$

denote the Chebyshev constants for $1/f(x)$ on $[0, \infty)$

Theorems:

Theorem 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function with* $a_0 > 0$ *and* $a_n \geq 0$ $(n \geq 1)$. *Then for any* $\varepsilon > 0$, *there are infinitely many values of n such that*

(1) $$\lambda_{0,n} \leqq \exp(-n/(\log n)^{1+\varepsilon}).$$

It will be clear from the proof that (1) holds for every $e^{-m/g(m)}$, where $g(m)$ is increasing regularly and $\sum 1/mg(m) < \infty$.

Proof. Since $f(z)$ is entire, $|a_n|^{1/n} \to 0$. Put $|a_n|^{-1/n} = U_n$, then $U_n \to \infty$. Now it is easy to see from the convergence of

$$\prod_{k=4}^{\infty} (1 + 1/(k \log k(\log\log k)^2))$$

that there are arbitrarily large values of n for which, for every $l > 0$

(2) $$U_{n+l} > U_n \prod_{t=1}^{l} (1 + 1/((n+t) \log(n+t)(\log\log(n+t))^2)).$$

From (2) we get with $l = n$

(3) $$U_{2n} > U_n(1 + 1/(2 \log n(\log\log n)^2)).$$

Let $S_n(x)$ denote the n-th partial sum of $f(x)$. Now we prove under the uniform norm

(4) $$|1/f(x) - 1/S_{2n}(x)| < \exp(-2n/(\log 2n)^{1+\varepsilon}), \quad \forall x > 0$$

and all large n. From the definition of $\lambda_{0,n}$ (1) follows from (4).

To prove (4), observe that on the one hand we have for all $x \geqq 0$

(5) $$0 \leqq 1/S_{2n}(x) - 1/f(x) \leqq 1/S_{2n}(x) \leqq 1/a_n x^n.$$

Now for any $\varepsilon > 0$, let $x \geqq U_n(1 + 1/(\log n)^{1+\varepsilon/2})$, then

$$a_n x^n \geqq (1 + 1/(\log n)^{1+\varepsilon/2})^n \geqq \exp(n/2(\log n)^{1+\varepsilon/2})^* > \exp(2n/(\log 2n)^{1+\varepsilon}).$$

In other words (4) holds for $x \geqq U_n(1 + 1/(\log n)^{1+\varepsilon/2})$.

Now let $x < U_n(1 + 1/(\log n)^{1+\varepsilon/2})$. Then for $n \geqq n_1$,

(6) $$0 \leqq 1/S_{2n}(x) - 1/f(x) = (f(x) - S_{2n}(x))/f(x)S_{2n}(x) \leqq a_0^{-2} \sum_{k=2n+1}^{\infty} a_k x^k.$$

By (2) and (3) we have for $k > 2n$

(7) $$a_k < U_n^{-k}(1 + 1/2 \log n(\log\log n)^2)^{-k}.$$

Thus, from (6) and (7) for $x < U_n(1 + 1/(\log n)^{1+\varepsilon/2})$, we obtain

$$a_0^{-2} \sum_{k=2n+1}^{\infty} a_k x^k < a_0^{-2} \sum_{k=2n+1}^{\infty} U_n^{-k}(1 + 1/\log n(\log\log n)^2)^{-k} U_n^k(1 + 1/(\log n)^{1+\varepsilon/2})^k$$

$$= a_0^{-2} \sum_{k=2n+1}^{\infty} ((1 + 1/(\log n)^{1+\varepsilon/2})(1 + 1/2(\log n)(\log\log n)^2)^{-1})^k$$

* ε may not be the same at each occurrence.

$$<a_0^{-2}\sum_{k=2n+1}^{\infty}(1-1/4(\log n)(\log\log n)^2)^k<\exp(-2n/(\log 2n)^{1+\varepsilon})$$

as stated.

Theorem 2. *Let $f(z)$ be an entire function of infinite order with non-negative coefficients. Then for any $\varepsilon>0$ there are infinitely many values of m such that*

$$\lambda_{0,m}\geqq e^{-\varepsilon m}. \tag{8}$$

Proof. Let us assume on the contrary the following:

$$|1/f(x)-1/P_n(x)|<e^{-\varepsilon n} \tag{9}$$

is valid for all large n and all $0\leqq x<\infty$. Since $f(z)$ is of infinite order, for every r, there are arbitrarily large values of t_r for which

$$(f(t_r))^r<f(t_r(1+1/r)). \tag{10}$$

It is possible to choose for any t_r and $\varepsilon>0$ sufficiently large n, such that

$$f(t_r)=e^{\varepsilon n/8}. \tag{11}$$

From (9) and (11), we get for $0<x\leqq t_r$

$$\max|P_n(x)|<e^{\varepsilon n/4}. \tag{12}$$

Now it follows from (12), for sufficiently large r, in the interval $0<x\leqq t_r(1+1/r)$ along with (9) of [10, p. 68],

$$\max|P_n(x)|<e^{\varepsilon n/2}. \tag{13}$$

Take $x=t_r(1+1/r)$, then

$$f(t_r(1+1/r))>(f(t_r))^r=e^{r\varepsilon n/8}>e^{2\varepsilon n}. \tag{14}$$

That is

$$0<1/P_n(x)-e^{-2\varepsilon n}<1/P_n(x)-1/f(t_r(1+1/r)). \tag{15}$$

From (15) it is easy to verify that

$$|1/P_n(x)-e^{-2\varepsilon n}|>e^{-\varepsilon n}\quad\text{for}\quad 0<x\leqq t_r(1+1/r),$$

which contradicts our earlier assumption (9). Hence the theorem is proved.

Theorem 3. *Let $f(z)=\sum_{n=0}^{\infty}a_nz^n$ be an entire function of finite order ϱ with $a_0>0$ and $a_n\geqq 0$ $(n\geqq 1)$. Then for any $\varepsilon>0$,*

$$\varliminf_{n\to\infty}(\lambda_{0,n})^{(\varrho+\varepsilon)/n}\leqq 0.08. \tag{16}$$

Proof. Since $f(z)$ is an entire function of finite order ϱ, we get for any $\varepsilon>0$

$$\lim_{n\to\infty}n^{1/(\varrho+\varepsilon)}|a_n|^{1/n}=0\qquad([2,\text{ p. }9]) \tag{17}$$

put $U_n=a_n^{-1/n}$, then $U_n n^{-1/(\varrho+\varepsilon)}\to\infty$. Then there exist infinitely many n for which

$$(18)\qquad U_{n+l}(n+l)^{-1/(\varrho+\varepsilon)}\geqq U_n n^{-1/(\varrho+\varepsilon)}\qquad (l=0,\ 1,\ 2,\dots).$$

Let $U_n\geqq(1.6)^{1/(\varrho+\varepsilon)}$.
Then as in the case of the proof of Theorem 1, we get

$$(19)\qquad |1/f(x)-1/S_{2n}(x)|\leqq 1/S_{2n}(x)\leqq 1/a_n x^n\leqq(1.6)^{-n/(\varrho+\varepsilon)}.$$

On the other hand let $x<U_n(1.6)^{1/(\varrho+\varepsilon)}$.
For any $k>n$, we get from (18),

$$k^{1/(\varrho+\varepsilon)}|a_k|^{1/k}<U_n^{-1}n^{1/(\varrho+\varepsilon)}.$$

That is

$$(20)\qquad |a_k|<U_n^{-k}(n/k)^{k/(\varrho+\varepsilon)}.$$

Then as earlier

$$|1/f(x)-1/S_{2n}(x)|\leqq a_0^{-2}\sum_{k=n+1}^{\infty}a_k x^k$$

$$(21)\qquad <a_0^{-2}\sum_{k=2n+1}^{\infty}U_n^{-k}(n/k)^{k/(\varrho+\varepsilon)}(1.6)^{k/(\varrho+\varepsilon)}U_n^k$$

$$=a_0^{-2}\sum_{k=2n+1}^{\infty}(1.6n/k)^{k/(\varrho+\varepsilon)}=a_0^{-2}(1.6n/(2n+1))^{(2n+1)/(\varrho+\varepsilon)}$$

$$\times(2n+1)^{1/(\varrho+\varepsilon)}/((2n+1)^{1/(\varrho+\varepsilon)}-(1.6n)^{1/(\varrho+\varepsilon)}).$$

Hence from (19) and (21) for all $0\leqq x<\infty$ along with the definition of $\lambda_{0,n}$

$$\varliminf_{n\to\infty}(\lambda_{0,n})^{(\varrho+\varepsilon)/n}\leqq\max(1/\sqrt{1.6},\ 0.8)=0.8.$$

Remarks. We can replace $\varliminf$ by $\varlimsup$ for certain functions of regular growth.

Examples:

$$f(x)=1+\sum_{n=2}^{\infty}x^n(n\log n)^{-n/\varrho};$$

$$f(x)=\sum_{n=0}^{\infty}x^n(\sigma e\varrho/n)^{n/\varrho},\quad 0<\sigma<\infty,\quad 0<\varrho<\infty,$$

Theorem 4. *Let* $f(z)=\sum_{n=0}^{\infty}a_n z^n$ *be any entire function with* $a_0>0$ *and* $a_n\geq 0$ $(n\geq 1)$. *Let* $M(r)=\max_{|z|=r}|f(z)|$ *and*

$$1 \leq \overline{\lim_{r\to\infty}} (\log\log M(r))/(\log\log r) = \lambda < 2.$$

Then for any $\varepsilon > 0$,

(22) $$\underline{\lim_{n\to\infty}} (\lambda_{0,n})^{n^{-1/(\lambda-1+\varepsilon)}} = 0.$$

Proof. We get from [8, Theorems 1 and 3],

$$\overline{\lim_{n\to\infty}} (\log n)/\log\left(\frac{1}{n}\log|1/a_n|\right) = \lambda - 1.$$

From this we get as earlier for any $\varepsilon > 0$

(23) $$\lim_{n\to\infty} |a_n|^{1/n} \exp(n^{1/(\lambda-1+\varepsilon)}) = 0.$$

(24) Put $U_n = a_n^{-1/n}$ then $U_n \exp(-n^{1/(\lambda-1+\varepsilon)}) \to \infty$.

Then there exist infinitely many n for which

(25) $$U_{n+l} \exp(-(n+l)^{1/(\lambda-1+\varepsilon)}) \geq U_n \exp(-n^{1/(\lambda-1+\varepsilon)}).$$

Now let $x \geq (2\theta)^{n^{1/(\lambda-1+\varepsilon)}}$, where $1 < \theta < e/2$.
Then

$$S_{2n}(x) \geq a_n x^n \geq a_n U_n^n (2\theta)^{n \cdot n^{1/(\lambda-1+\varepsilon)}} = (2\theta)^{n^{1+1(\lambda-1+\varepsilon)}}.$$

Hence as usual

(26) $$|1/f(x) - 1/S_{2n}(x)| \leq (2\theta)^{-n \cdot n^{1/(\lambda-1+\varepsilon)}}.$$

On the other hand let $x < U_n (2\theta)^{n^{1/(\lambda-1+\varepsilon)}}$. Then as earlier it is easy to see that for any $k > n$,

(27) $$|a_k| \leq U_n^{-k} \exp(k(n^{1/(\lambda-1+\varepsilon)} - k^{1/(\lambda-1+\varepsilon)})).$$

Therefore,

$$\sum_{k=2n+1}^{\infty} a_k x^k < \sum_{k=2n+1}^{\infty} (2\theta)^{kn^{1/(\lambda-1+\varepsilon)k}} \exp(k(n^{1/(\lambda-1+\varepsilon)} - k^{1/(\lambda-1+\varepsilon)}))$$

$$\leq \sum_{k=2n+1}^{\infty} ((2\theta e)^{n^{1/(\lambda-1+\varepsilon)}})^k \exp(-k^{1/(\lambda-1+\varepsilon)})$$

(28) $$\leq ((2\theta e)^{n^{1/(\lambda-1+\varepsilon)}} \exp(-(2n)^{1/(\lambda-1+\varepsilon)}))^{2n+1}$$

$$\times((1 + (2\theta)^{n^{1/(\lambda-1+\varepsilon)}} \exp(n^{1/(\lambda-1+\varepsilon)}))/\exp((2n)^{1/(\lambda-1+\varepsilon)} + \dots)$$

$$\leq ((2\theta)^{n^{1/(\lambda-1+\varepsilon)}} \exp(-n^{1/(\lambda-1+\varepsilon)}(2^{1/(\lambda-1+\varepsilon)} - 1)))^{2n+1}$$

$$\times(\exp((2n)^{1/(\lambda-1+\varepsilon)}))/(\exp((2n)^{1/(\lambda-1+\varepsilon)} - (2\theta)^{n^{1/(\lambda-1+\varepsilon)}}).$$

Therefore we get from (26) and (28) along with the definition of $\lambda_{0,n}$

$$\underline{\lim_{n\to\infty}} (\lambda_{0,n})^{-1/(\lambda-1+\varepsilon)} = 0.$$

Theorem 5. *Let* $f(x)=\sum_{j=0}^{\infty} q^{j^k} x^j$, *where* $0<q<1$ *and* $2\leqq k<\infty$. *Then*

$$(29)\qquad q\leqq \varliminf_{n\to\infty} (\lambda_{0,n})^{1/n^k}\leqq \varlimsup_{n\to\infty}(\lambda_{0,n})^{1/n^k}\leqq q^{(1-2^{1-k})}.$$

Proof. Let us write for convenience $a_n=q^{n^k}=1/d_1d_2\ldots d_n$. Where d_n is positive and strictly increasing to ∞ with n, S_n denotes the n-th partial sum of $f(x)$. Then

$$0\leqq 1/S_{2n-1}(x)-1/f(x)=(f(x)-S_{2n-1}(x))/f(x)S_{2n-1}(x)$$

$$\leqq S_{2n-1}^{-2}(x)\sum_{k=2n}^{\infty} a_k\, x^k \leqq a_{2n}x^{2n}a_n^{-2}x^{-2n}\sum_{j=0}^{\infty} a_{2n+j}x^j/a_{2n},$$

because $S_{2n-1}^2(x)\geqq a_n^2x^{2n}$.
Therefore

$$(30)\qquad 0\leqq 1/S_{2n-1}(x)-1/f(x)\leqq a_{2n}a_n^{-2}\sum_{j=0}^{\infty} d_{2n+1}^{-j}x^j.$$

Now we get from (30) for all $0<x\leqq d_{2n}$,

$$0\leqq 1/S_{2n-1}(x)-1/f(x)\leqq a_{2n}a_n^{-2}\sum_{j=0}^{\infty}(d_{2n}/d_{2n+1})^j=a_{2n}a_n^{-2}d_{2n+1}/(d_{2n+1}-d_{2n}).$$

That is,

$$(31)\qquad 0\leqq 1/S_{2n-1}(x)-1/f(x)\leqq \frac{d_1d_2\ldots d_n}{d_{n+1}d_{n+2}\ldots d_{2n}}\cdot\frac{d_{2n+1}}{d_{2n+1}-d_{2n}}.$$

On the other hand let $x\geqq d_{2n}$, then

$$0\leqq 1/S_{2n-1}(x)-1/f(x)\leqq 1/S_{2n-1}(x)\leqq 1/d_nx^n$$

$$(32)\qquad \leqq 1/a_nd_{2n}^n=d_1d_2\ldots d_nd_{2n}^{-n}.$$

By comparing (31) and (32), it is easy to see that for all large n, (31) is larger than (32). Let

$$(33)\qquad \Delta_n\equiv \sup_{0\leqq x<\infty} |1/S_n(x)-1/f(x)|,\quad \forall n\geqq n_0.$$

Now substitute $a_n=q^{n^k}$ $(k\geqq 2)$ in (31), then we obtain

$$(34)\qquad 0\leqq 1/S_{2n-1}(x)-1/f(x)\leqq q^{(2n)^k(1-2^{1-k})}/(1-q^{2(n+1)^k-n^k-(n+2)^k}).$$

Then from (33) and (34), we obtain

$$(35)\qquad \varlimsup_{n\to\infty}(\Delta_{2n-1})^{(2n)^{-k}}\leqq q^{1-2^{1-k}}.$$

(35) is true for every large integer n of the form $2n_p-1$ $(p=1, 1, 3,\ldots)$. Let $2n_p-1\leqq n<2n_p+1$, then

$$\Delta_n^{(n+1)^{-k}} \leqq \Delta_{2n_p-1}^{(n+1)^{-k}} = \Delta_{2n_p-1}^{(2n_p)^k (2n_p)^{-k}(n+1)^{-k}} \leqq (\Delta_{2n_p-1}^{(2np)^{-k}})^{(2n_p/(2n_p+2))^k}$$

therefore

$$\overline{\lim_{n\to\infty}} (\lambda_{0,n})^{n^{-k}} = \overline{\lim_{n\to\infty}} \Delta_n^{n^{-k}}$$

$$\leqq \lim_{p\to\infty} (\Delta_{2n_p-1}^{(2n_p)^{-k}})^{(2n_p/(2n_p+2))^k} \leqq q^{1-2^{1-k}}.$$

We can also show easily that $\underline{\lim}_{n\to\infty} (\lambda_{0,n})^{n^{-k}} \geqq q$.

Let $f(z) = \sum_{j=0}^{\infty} q^{j^k} z^j$ $(k \geqq 2)$, let $M(r) = \max_{|z|=r} |f(z)|$, then it is known [8, Theorems 1, 3] that,

$$\overline{\lim_{x\to\infty}} (\log\log M(x))/\log\log x = k/(k-1). \tag{36}$$

From (36) we get for any $\varepsilon > 0$, there is an r_0, such that for all $r \geqq r_0(\varepsilon)$,

$$M(r) \leqq \exp((\log r)^{k(1+\varepsilon)/(k-1)}). \tag{37}$$

For all $0 \leqq x \leqq r$, we have

$$0 \leqq f(x) \leqq f(r) = M(r) \leqq \exp((\log r)^{k(1+\varepsilon)/(k-1)}), \quad k \geqq 2. \tag{38}$$

From the definition of $\lambda_{0,n}$ we know that

$$\lambda_{0,n} \equiv \inf_{P_n \in \pi_r} \| 1/f(x) - 1/P_n(x) \|_{[0,\infty)}. \tag{39}$$

Now we pick only those P_n which give best approximation in the sense of (39), and we denote them by P_n^*. Then

$$1/f - 1/P_n^* \leqq \lambda_{0,n}. \tag{40}$$

We choose in (38), $r = \exp(n^{(k-1)/k(1+\varepsilon)})$ then $\exp((\log r)^{k(1+\varepsilon)/(k-1)}) = e^n$, then $f(x) \leqq e^n < 1/\lambda_{0,n}$, which is valid for all large n, because of

$$\overline{\lim_{n\to\infty}} (\lambda_{0,n})^{k^{-n}} \leqq q^{1-2^{1-k}}.$$

Now (40) gives with a simple calculation

$$-f^2(x)/(1/\lambda_{0,n} + f(x)) \leqq P_n^*(x) - f(x) \leqq f^2(x)/(1/\lambda_{0,n} - f(x)), \ 0 \leqq x \leqq r. \tag{41}$$

From (41) we get

$$| P_n^*(x) - f(x) | \leqq f^2(x)/(1/\lambda_{0,n} - f(x)) \leqq e^{2n}/(1/\lambda_{0,n} - e^n),$$
$$0 \leqq x \leqq r, \tag{42}$$

because $f^2(x)/(1/\lambda_{0,n} - f(x))$ is an increasing function of x.

Now let

(43) $$E_n = \inf_{r_n \in \pi_n} \{ \max_{0 \leq x \leq r} | r_n(x) - f(x) | \}, \quad \forall n \geq 0.$$

From (42) and (43) we get

(44) $$E_n \leq e^{2n}/(1/\lambda_{0,n} - e^n), \quad \forall n \geq n_0.$$

To get the lower bound for E_n, we transform the interval $[0, r = \exp(n^{k(1+\varepsilon)/(k-1)})]$ into the interval $[-1, 1]$ by means of the linear transformation

$$x = \frac{t+1}{2} \exp(n^{(k-1)/k(1+\varepsilon)}), \quad -1 \leq t \leq 1.$$

The function $g(t) = f\left(\frac{t+1}{2} \exp(n^{(k-1)/k(1+\varepsilon)})\right)$ is also an entire function of t. From the statement of the theorem the coefficients of $f(x)$ are clearly non-negative now by using a result of S. N. Bernstein ([1], (16), p. 10) we get

$$E_n \geq g^{(n+1)}(-1)/2^n(n+1) = f^{(n+1)}(0) \exp(n^{(k-1)/k(1+\varepsilon)}(n+1)) 2^{-2n-1}/(n+1)!$$

that is,

(45) $$E_n \geq a_{n+1} 2^{-2n-1} \exp(n^{(k-1)/k(1+\varepsilon)}(n+1)).$$

Hence by (44) and (45), we get

(46) $$a_{n+1} 2^{-2n-1} \exp(n^{(k-1)/k(1+\varepsilon)}(n+1)) \leq e^{2n}/(1/\lambda_{0,n} - e^n).$$

A simple calculation based on (46) gives us by observing the fact that $k \geq 2$ and $a_n = q^{n^k}$,

$$\varliminf_{n \to \infty} (\lambda_{0,n})^{n^{-k}} \geq q.$$

Theorem 6. *Let $f(x)$ be a real valued continuous function (not $\equiv 0$) on any finite interval $[0, b]$ and assume that there exist a sequence of real polynomials $\{P_n(x)\}_0^\infty$ with $P_n \in \pi_n$ for each $n \geq 0$, and a real number $R > 1$ such that*

(47) $$\varlimsup_{n \to \infty} \{\| 1/f(x) - 1/P_n(x) \|\}^{n^{-\alpha}} \leq 1/R < 1$$

for any $0 < \alpha < 1$. Then $f(x)$ is infinitely differentiable on $[0, b]$.

Proof. Let $M(b) = \| f \|_{[0, b]}$; $0 \leq b < \infty$. For any R, with $R > R_1 > 1$, it follows from (41), that there exists a positive integer $n_1(R_1)$ such that

(48) $$\| 1/f(x) - 1/P_n(x) \| \leq R_1^{-n^\alpha} \quad \text{for all} \quad n > n_1(R_1).$$

Now for any fixed $b > 0$ and $\alpha > 0$, we can find a least positive integer $n_2 = n_2(b)$ such that

(49) $$R_1^{n^\alpha} > R_1^{n^\alpha} - M(b) \geq R_1^{n^\alpha} 2^{-1} \quad \text{for all} \quad n > n_2(b).$$

We get from (48) and (49) with a simple calculation that

$$(50)\qquad \|P_n - f\| \leqq 2M^2(b)R_1^{-n^\alpha} \quad \text{for all} \quad n \geqq \max(n_1, n_2) = n_3.$$

Denote by

$$(51)\qquad E_n(f;\ 0,\ b) = \inf_{Q_n \in \pi_n} \|f - Q_n\|_{[0,\ b]}.$$

From (50) and (51) we get

$$(52)\qquad E_n(f;\ 0,\ b) \leqq 2M^2(b)R_1^{-n^\alpha}, \quad n \geqq n_3.$$

From (52) we get

$$(53)\qquad \overline{\lim_{n\to\infty}}\ E_n^{-n^\alpha} < 1.$$

From (53) we get for any positive integer

$$(54)\qquad \overline{\lim_{n\to\infty}}\ n^r E_n(f;\ 0,\ b) = 0.$$

Then it is known ([10, p. 350]) that f is infinitely differentiable on $[0, b]$.

Remarks:

In conclusion it may be pointed out that it is possible to obtain much more information than in [3] using the method of Theorem 5 for certain entire functions. For instance, let $f(z) = 1 + \sum_{n=1}^{\infty} (\log n/n)^{n/\varrho} z^n$, where $0 < \varrho < \infty$, this is an entire function of order ϱ and type infinity, satisfying the assumptions of Theorem 5 of [3], but the conclusion is

$$\overline{\lim_{n\to\infty}}\ (\lambda_{0,n})^{1/n} < 1.$$

It is easy to show for this function by adopting the method of Theorem 5 that

$$\overline{\lim_{n\to\infty}}\ (\lambda_{0,n})^{1/n} \leqq 2^{-1/\varrho}.$$

Further, let

$$g(z) = 1 + \sum_{n=1}^{\infty} z^n / 1^1 2^2 3^3 \ldots n^n,$$

this is an entire function of order zero with

$$\overline{\lim_{n\to\infty}}\ (\log\log M(r))/\log\log r = \lambda = 2.$$

For this function, we can show easily by using the method of Theorem 5 that

$$\overline{\lim_{n\to\infty}}\ (\lambda_{0,n})^{1/n^2} = 0,$$

improving the conclusion of our Theorem 4 for this function.

Similarly, there exist many entire functions satisfying certain growth conditions for which we can get better conclusions than some of the theorems presented here.

REFERENCES

1. S. N. Bernstein. Leçons sur les proprietes extremales et la meilleure approximation des functions analytiques d'une variable reelle. Paris, 1926.
2. R. P. Boas. Entire functions. New York, 1954.
3. W. J. Cody, G. Meinardus, R. S. Varga. Chebyshev rational approximations to e^{-x} in $[0, +\infty]$ and applications to heat conduction problems. *J. Approximation Theory*, **2**, 1969, 50—65.
4. G. Meinardus, R. S. Varga. Chebyshev rational approximation to certain entire functions in $[0, +\infty)$. *J. Approximation Theory*, **3**, 1970, 300—309.
5. G. Meinardus et al. Converse Theorems and extension in Chebyshev rational approximation to certain entire functions in $[0, +\infty)$. *Bull. Amer. Math. Soc.*, **77**, 1971, 460—461; *Trans. Amer. Math. Soc.*, **170**, 1972, 171—185.
6. D. J. Newman. Rational approximation to e^{-x}. *J. Approximation Theory*, **10**, 1974, 301—303.
7. A. R. Reddy. A contribution to rational Chebyshev approximation to certain entire functions in $[0, \infty]$. *J. Approximation Theory*, **11**, 1974, 85—96.
8. A. R. Reddy. On entire Dirchlet Series of zero order. *Tôhoku Math. J.*, **16**, 1966, 144—155.
9. A. Schonhage. Zur Rationalen Approximirbarkeit von e^{-x} über $[0, \infty]$. *J. Approximation Theory*, **7**, 1973, 395—398.
10. A. F. Timan. Theory of approximation of functions of a real variable. New York, 1963.

Mathematical Institute of the
Hungarian Academy of Sciences
Budapest Hungary

Department of Mathematics
Michigan State University
Michigan USA

Received on 20. III. 1973