THE ASYMMETRIC PROPELLER

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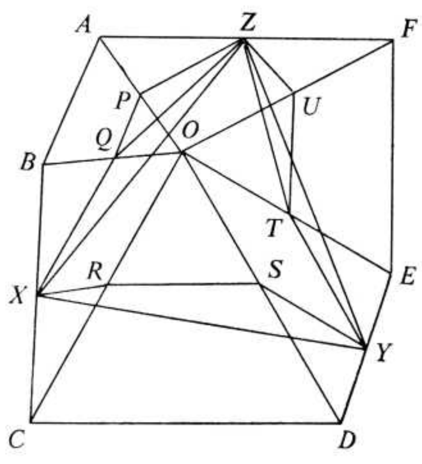
In this note, we prove an extension of a known elementary geometric result in two ways, i.e., synthetically and by complex numbers. Then we show that the result characterizes closed curves of 6-fold symmetry.

THEOREM. *If OAB, OCD, OEF are equilateral triangles, each labeled in the same clockwise or counterclockwise direction (and not necessarily congruent), then X, Y, Z, the midpoints of BC, DE and FA, are vertices of an equilateral triangle.*



Synthetic Proof. Let P, Q, R, S, T, U denote the midpoints of OA, OB, OC, OD, OE, and OF, respectively. Then,

$$PQ = PO = ZU \quad \text{and} \quad \measuredangle\,(PQ, ZU) = 60°;$$

$$PZ = OU = UT \quad \text{and} \quad \measuredangle\,(PZ, UT) = 60°.$$

Thus, $\measuredangle\, QPZ = \measuredangle\, ZUT$ and triangles QPZ and ZUT are congruent with a 60° mutual inclination between corresponding sides. Then, $QZ = ZT$ with $\measuredangle\, QZT = 60°$. Since $QX = OR = OS = TY$ with $\measuredangle\,(QX, TY) = 60°$, triangles ZQX and ZTY are congruent. Finally, $ZX = ZY$ with $\measuredangle\, XZY = 60°$, giving the desired result.

NOTE. Triangles ZQT, YUR and XSP are equilateral. This can be shown directly, as with triangle ZQT above, or by allowing one of the triangles OAB, OCD, OEF to degenerate to a point-triangle at O and applying the main theorem.

This proof applies for any rotation of one or more of the triangles OAB, OCD and OEF about O. Thus the triangles in the initial configuration may be separate, contiguous or overlapping in any manner.

Proof by Complex Numbers. In the figure, $z_1(OA)$, $z_2(OC)$, $z_3(OE)$ denote arbitrary complex numbers and $\lambda = e^{i\pi/3}$. We now have to show that $\lambda z_1 + z_2$, $\lambda z_2 + z_3$, $\lambda z_3 + z_1$ are the vertices of an equilateral triangle, i.e.,

$$(\lambda z_3 + z_1) - (\lambda z_2 + z_3) = \lambda^2\{(\lambda z_2 + z_3) - (\lambda z_1 + z_2)\}$$

or

$$\lambda(z_3 - z_2) + z_1 - z_3 = \lambda^3(z_2 - z_1) + \lambda^2(z_3 - z_2).$$

Since $\lambda^3 = -1$ and $\lambda - \lambda^2 = 1$, the result follows.

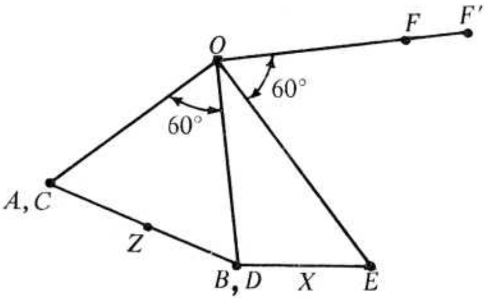
Proofs using complex numbers may also be found in the Amer. Math. Monthly, Aug.–Sept. 1968, Problem B–1 of the William Lowell Putnam Mathematical Competition and in H. Eves, *A Survey of Geometry*, II, Allyn and Bacon, Boston, 1965, p. 184. In these two solutions, however, the superfluous conditions $|z_1| = |z_2| = |z_3|$ as well as non-overlapping were assumed.

We now show that the result given by our theorem characterizes curves of 6-fold symmetry.

THEOREM. *AB, CD, EF are arbitrary chords (in the same sense) of a given closed curve, starlike with respect to O, and which subtend 60° angles from point O. If X, Y, Z, the respective midpoints of DE, FA, BC, are vertices of an equilateral triangle, then the curve must be one of 6-fold symmetry (with respect to O).*

Proof. We first show that there exists a chord PQ such that POQ is equilateral. Let OR be a shortest radius from O to the curve and then let OR' denote the radius making 60° with OR. It follows by continuity that as we rotate the radius OR about O up to 60°, $OR' - OR$ must have a zero value. An equilateral triangle POQ still exists even if we dropped the starlike assumption for the curve. In this case, we would apply P. Lévy's chord theorem (see H. Hadwiger, H. Debrunner, V. Klee, *Combinatorial Geometry in the Plane*, Holt, Rinehart and Winston, N.Y., 1964, p. 23).

Let points A and C be fixed points coinciding with P and let points B and D be fixed coinciding with Q. OE is an arbitrary radius and $OF' = OE$.



It follows from our first theorem that the midpoint of AF must coincide with that of AF'. Thus, $OF = OF'$ and the curve is one of 6-fold symmetry.