SOME EXTREMAL PROPERTIES CONCERNING TRANSITIVITY IN GRAPHS

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In this note we consider only non-trivial labelled oriented graphs, i.e. digraphs D having at least one arc, no loops, and for each pair of points a and b of D at most one of the arcs ab and ba is in D. D is *transitive* if arc ac is in D whenever arcs ab and bc are in D. We investigate the number of arcs of the largest transitive subgraph contained in a (round robin) tournament, i.e. a complete oriented graph. Denote by $F(n)$ the greatest integer so that every tournament on n points contains a transitive subgraph of $F(n)$ arcs. We will prove

$$\frac{1}{4}\binom{n}{2} < F(n) < \frac{1}{4}\binom{n}{2} + (\alpha + 2\sqrt{2})\, n^{\frac{3}{2}}$$

where c any constant greater than $2^{-\frac{5}{4}}\sqrt{\log 2}$.

A set of arcs in a tournament T is called consistent if the set does not contain an oriented cycle or in other words if it is possible to relabel the points in such a way that if the arc $u_i u_j$ is in T then $i > j$. Clearly every transitive subgraph is consistent but the converse is not true. Denote by $f(n)$ the greatest integer so that every tournament on n points contains a set of $f(n)$ consistent arcs. ERDŐS and MOON proved [1]

$$\frac{1}{2}\binom{n}{2} + c_1 n < f(n) < \frac{1}{2}\binom{n}{2} + \left(\frac{1}{2} + o(1)\right)(n^3 \log n)^{\frac{1}{2}}$$

where c_1 is a suitable positive constant.

The lower bound has been improved to $\frac{1}{2}\binom{n}{2} + c_2 n^{\frac{3}{2}}$ by Joel SPENCER in a recent article [2].

We will call the graph D *dibipartite* if the vertices of D can be split into two sets A and B so that every arc of D is from a point of A to a point of B.

Our first theorem is not concerned with transitivity, however it is essential for the proof of a later result. In this theorem c_3 and c_4 are suitably chosen positive constants.

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THEOREM 1. *For all tournaments T_n on n points, with $o\left(2^{\binom{n}{2}}\right)$ exceptions, the largest dibipartite subgraph of T_n contains less than $\frac{1}{4}\binom{n}{2}+\alpha n^{\frac{3}{2}}$ arcs where α is any constant greater than $2^{-\frac{5}{4}}\sqrt{\log 2}$.*

PROOF. Let $t(n)$ be the number of tournaments T_n containing a dibipartite subgraph with more than $\frac{1}{4}\binom{n}{2}+\alpha n^{\frac{3}{2}}$ arcs. Then since there are at most $\binom{n}{r}\binom{r(n-r)}{t}2^{\binom{r}{2}+\binom{n-r}{2}}$ tournaments T_n containing a dibipartite subgraph with t arcs originating from a set of r points of T_n and terminating in the remaining set of $n-r$ points we have for n sufficiently large

$$t(n)\leq \sum_{0\leq r\leq n}\ \sum_{t\geq \frac{1}{4}\binom{n}{2}+\alpha n^{3/2}}\binom{n}{r}\binom{r(n-r)}{t}2^{\binom{r}{2}+\binom{n-r}{2}}\leq$$

$$\leq \max_{(2-\sqrt{2})n\leq 4r\leq (2+\sqrt{2})n} n\, 2^{n+\binom{r}{2}+\binom{n-r}{2}}\sum_{t\geq \frac{n^2}{8}+\beta n^{\frac{3}{2}}}\binom{r(n-r)}{t}$$

where β is any constant satisfying $2^{-\frac{5}{4}}\sqrt{\log 2}<\beta<\alpha$. We set $m=r(n-r)$, let γ be any constant satisfying $2^{\frac{1}{4}}\sqrt{\log 2}<\gamma<2\sqrt{2}\,\beta$ and from Stirling's formula obtain

$$\sum_{t\geq \frac{n^2}{8}+\beta n^{\frac{3}{2}}}\binom{r(n-r)}{t}\leq \sum_{t\geq \frac{m}{2}+\gamma m^{\frac{3}{4}}}\binom{m}{t}\leq \frac{c_3\sqrt{m}\,2^m}{\left(1-4\gamma^2 m^{-\frac{1}{2}}\right)^{\frac{m+1}{2}}}\left(\frac{1-2\gamma m^{-\frac{1}{4}}}{1+2\gamma m^{-\frac{1}{4}}}\right)^{\gamma m^{\frac{3}{4}}}\leq$$

$$\leq \frac{c_3\sqrt{m}\,2^m}{e^{-2\gamma^2\sqrt{m}-9\gamma^4}}\,e^{-4\gamma^2\sqrt{m}}.$$

The exponential expressions base e follow from the inequalities $\log\frac{1-x}{1+x}<-2x$, $x^2<1$ and $1-x>e^{-x-x^2}$, $0<x<\frac{1}{2}$. As a consequence we have, since $r(n-r)\geq\frac{n^2}{8}$

$$t(n)\leq c_4 n^2\, 2^{\binom{n}{2}+n-2\gamma^2\sqrt{r(n-r)}\log_2 e}=o\left(2^{\binom{n}{2}}\right).$$

THEOREM 2. *For all tournaments T_n on n points, with $o\left(2^{\binom{n}{2}}\right)$ exceptions, if T_n contains a transitive subgraph S with $f(n)$ arcs then S contains a dibipartite subgraph with more than $f(n)-2\sqrt{2}\,n^{\frac{3}{2}}$ arcs.*

PROOF. We may assume that no point of S has more than $\sqrt{2n}$ arcs to it from points of S and more than $\sqrt{2n}$ arcs from it to points of S. To see this let s be a point of S with arcs $r_i s$, $1 \leqq i \leqq p$ and st_j, $1 \leqq j \leqq q$ in S. By transitivity of S each arc $r_i t_j$ is also in S so there are at most

$$n \binom{n-1}{p} \binom{n-1-p}{q} 2^{\binom{n}{2} - p - q - pq} \leqq n\, 2^{\binom{n}{2} + 2n - p - q - pq}$$

such tournaments. Consequently

$$\sum_{\sqrt{2n} \leqq p,q \leqq n} n\, 2^{\binom{n}{2} 2n - p - q - pq} \leqq n^3\, 2^{\binom{n}{2} - \sqrt{2n}} = o\left(2^{\binom{n}{2}}\right).$$

Suppose now T_n contains a transitive subgraph S having $f(n)$ arcs, the points of which may be partitioned into subsets U, V and W where U is the set of those points of S having arcs to more than $\sqrt{2n}$ points of S, V is the set of those points of S having arcs from more than $\sqrt{2n}$ points of S, and W is the set of those points of S having at most $\sqrt{2n}$ arcs to points of S and at most $\sqrt{2n}$ arcs from points of S. Now since there are at most $\sqrt{2n}\,|U|$ arcs in S to points of U, at most $\sqrt{2n}\,|V|$ arcs in S from points of V, at most $\sqrt{2n}\,|W|$ arcs in S to points of W and at most $\sqrt{2n}\,|W|$ arcs in S from points of W there are more than $f(n) - 2\sqrt{2}\,n^{\frac{3}{2}}$ arcs in S from points of U to points of V thus forming the required dibipartite subgraph of S.

These two results combine to give us

THEOREM 3. *The largest transitive subgraph of a non-trivial oriented graph D contains more than a fourth of the arcs of D. For all tournaments T_n on n points, with $o\left(2^{\binom{n}{2}}\right)$ exceptions, the largest transitive subgraph of T_n contains fewer than $\frac{1}{4}\binom{n}{2} + (\alpha + 2\sqrt{2})n^{\frac{3}{2}}$ arcs where α is any constant greater than $2^{-\frac{5}{4}}\sqrt{\log 2}$.*

PROOF. It is easily shown by induction on the number of points that more than half the edges of a non-trivial undirected graph are contained in a bipartite subgraph. Hence more than a fourth of the arcs of D are contained in a dibipartite subgraph and this gives the first assertion of the theorem.

To prove the second part let T_n be a tournament with more than $\frac{1}{4}\binom{n}{2} + (\alpha + 2\sqrt{2})n^{\frac{3}{2}}$ arcs in a transitive subgraph. Then by Theorem 2 T_n contains a dibipartite subgraph with more than $\frac{1}{4}\binom{n}{2} + \alpha\, n^{\frac{3}{2}}$ arcs but by Theorem 1 there are at most $o(2^{\binom{n}{2}})$ such T_n.

Our final theorem provides an interesting result which should be compared to Theorem 2.

THEOREM 4. *The largest dibipartite subgraph of a non-trivial transitive graph T contains more than half the arcs of T and this bound is best.*

PROOF. Let O_T be the set of those points of T whose outdegree is equal to or larger than their indegree and I_T be the set of remaining points. We will show by induction on the number n of points of T that more than half its arcs are from a point in O_T to a point in I_T. This is trivial if $n = 2$.

We will use the fact that removal of a point and its incident arcs from a transitive graph results in a transitive graph. We will also frequently use the following property concerning indegree (id) and outdegree (od) which we shall call Property t. If arc ab is in a transitive graph then $\mathrm{od}(a) \geq 1 + \mathrm{od}(b)$ and $1 + \mathrm{id}(a) \leq \mathrm{id}(b)$.

Assume $n > 2$ and the assertion holds for all non-trivial transitive graphs with fewer than n points. We consider two cases:

(i). There is a point p of T for which $\mathrm{id}(p) = 1 + \mathrm{od}(p) = 1 + a$. Let U be T with p and its incident arcs removed. Then, since T is transitive, U is non-trivial and transitive. We wish to show a) $O_U \supseteq O_T$ and b) $I_U \supseteq I_T - \{p\}$ for then equality will hold in a) and b) and the theorem will follow in this case, from the inductive hypothesis and the fact that p is in I_T and, by Property t, each arc to p is from a point in O_T.

To prove a) let r be a point of O_T. Now either r and p are not adjacent and hence r is in O_U or, by Property t, arc rp is in T in which case, again by Property t, $\mathrm{id}(r) \leq a$ and $\mathrm{od}(r) \geq a + 1$ and hence r is in O_U. The inclusion of b) is proved in a similar manner.

(ii). There is no point p of T for which $\mathrm{id}(p) = 1 + \mathrm{od}(p)$. In this case we choose a point q in O_T which is not adjacent to any points of O_T and designate by V the graph remaining when q and its incident arcs are deleted from T. Such a q must exist since T has no cycles. Our hypotheses guarantee that V is not trivial. It suffices now to show a) $O_V \supseteq O_T - \{q\}$ and b) $I_V \supseteq I_T$.

To prove a) let $r \neq q$ be a point of O_T. Then either r and p are not adjacent and hence r is in O_V or rq is in T in which case, by Property t, $\mathrm{od}(r) \geq 1 + \mathrm{od}(q) \geq \mathrm{id}(q) \geq 1 + \mathrm{id}(r)$ and hence r is in O_V.

To prove b) let r be a point of I_T. Then $\mathrm{id}(r) \geq 2 + \mathrm{od}(r)$ and r must be in I_V.

To show the bound is best consider the transitive tournament T_n on n points. There are at most $\binom{n}{2} - \binom{a}{2} - \binom{n-a}{2}$ arcs from a subset of a points of T_n to the remaining $n - a$ points, and so the largest dibipartite subgraph of T_n contains at most $\frac{1}{2}\left(1 + \frac{1}{n-1}\right)$ of the arcs of T_n.

REFERENCES

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