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OSCULATION VERTICES
IN ARRANGEMENTS OF CURVES

1. INTRODUCTION

In 1897 de Rocquigny [18] raised the problem of determining the maximal number of points of tangency possible in a system of n mutually non-crossing circles in the plane. The problem seems to have been forgotten until an analogous question concerning 'Apollonian arrangements' of curves was raised and the solution indicated in [11]. The aim of the present note is to indicate the easy solutions to those questions together with some results on osculation vertices in more general arrangements of circles or curves.

We begin by defining a number of concepts; our terminology is patterned after that of [11], modified to suit our present needs. A finite family $\mathscr{C} = \{C_1, ..., C_n\}$ of $n = n(\mathscr{C})$ simple closed curves is called an *arrangement of curves* provided the intersection $C_i \cap C_j$ of any two curves in $\mathscr{C}$ is either empty, or a single point, or a pair of points at which the curves cross each other. Each point which belongs to two or more curves is called a *vertex* of the arrangement. If a vertex is the only intersection point of some two curves it is called an *osculation vertex*, and the two curves are said to osculate each other at that vertex. (It should be noted that through an osculation vertex there may pass additional curves of the arrangement, either osculating other curves at the vertex, or crossing them.) An arrangement of curves $\mathscr{C}$ is called *Apollonian* provided all pairs of its curves have either empty or one-point intersections; hence all its vertices are osculation vertices. We shall denote by $f_0(\mathscr{C})$ and $\omega(\mathscr{C})$ the number of all vertices and the number of osculation vertices of the arrangement $\mathscr{C}$. Clearly $\omega(\mathscr{C}) = 0$ if $n(\mathscr{C}) = 1$, and $\omega(\mathscr{C}) \leqslant 1$ if $n(\mathscr{C}) = 2$; also selfevident is the relation $\omega(\mathscr{C}) + f_0(\mathscr{C}) \leqslant n(\mathscr{C})$ $(n(\mathscr{C}) - 1)$. However, for Apollonian arrangements we have much better results:

THEOREM 1. *If $\mathscr{C}$ is an Apollonian arrangement of $n = n(\mathscr{C}) \geqslant 3$ curves then*

$$\omega(\mathscr{C}) \leqslant 3n - 6;$$

moreover, equality is possible for each $n \geqslant 3$ even for Apollonian arrangements of circles.

The relationship between $\omega(\mathscr{C})$ and $n(\mathscr{C})$ is much more complicated if non-Apollonian arrangements of curves are considered. Using the notation $\omega(n) = \max\{\omega(\mathscr{C}) \mid n(\mathscr{C}) = n\}$ we have:

THEOREM 2. *There exist constants* $c^* > 0$ *and* c^{**} *such that for all* n

$$c^* n^{4/3} \leqslant \omega(n) \leqslant c^{**} n^{5/3}.$$

Very little is known concerning the rate of growth of $\omega(\mathscr{C})$ for arrangements $\mathscr{C}$ of n circles. Denoting by $\omega^*(n)$ the maximum of $\omega(\mathscr{C})$ for all such arrangements, the best result we could establish is:

THEOREM 3. *There exist constants* $c > 0$ *and* n_0 *such that*

$$\omega^*(n) \geqslant n^{1 + c/\log\log n}$$

for all integers $n > n_0$.

Sections 2, 3, and 4 are devoted to the proofs of the above theorems; various remarks, bibliographical references, and open questions are discussed in Section 5.

2. THE PROOF OF THEOREM 1

We shall first prove that $\omega(\mathscr{C}) \leqslant 3n(\mathscr{C}) - 6$, using induction on the number $k(\mathscr{C})$ of curves C in $\mathscr{C}$ which have the property that both regions of C contain points of other curves of $\mathscr{C}$. (The two *regions* of a simple closed curve C are the two connected components of the (open) complement of C in the plane.)

In case $k(\mathscr{C}) = 0$ we start by observing that we may assume, without loss of generality, that each vertex of $\mathscr{C}$ is *simple*, i.e. belongs to precisely two curves of $\mathscr{C}$. Indeed, if three or more curves osculate each other at one vertex, that vertex may be made simple by suitably 'retracting' some of the curves. An illustration is given in Figure 1, in which the 'empty' region of each curve is indicated by shading. Next, assuming all vertices of $\mathscr{C}$ simple, we associate
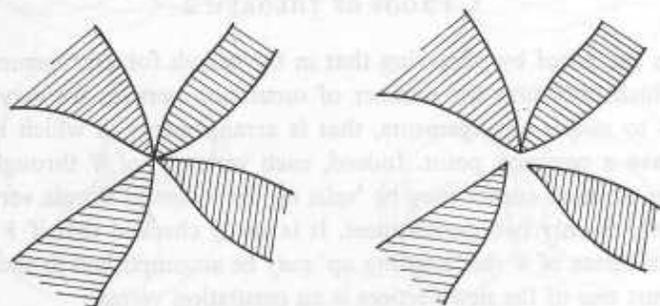

Fig. 1.

with $\mathscr{C}$ a planar graph $\mathscr{G}$ as follows: To each curve $C_i \in \mathscr{C}$ we make correspond a node N_i of $\mathscr{G}$ located in the 'empty' region D_i of C_i. Nodes N_i and N_j of $\mathscr{G}$ are connected by an edge E_{ij} if and only if C_i and C_j osculate each other at a vertex V_{ij} of $\mathscr{C}$; in that case E_{ij} consists of an open arc in D_i with endpoints N_i and V_{ij} and an arc in D_j with endpoints V_{ij} and N_j, together with the points N_i, V_{ij}, and N_j. By the Jordan curve theorem these arcs can be chosen so that $\mathscr{G}$ is a planar graph. Therefore the number of edges in $\mathscr{G}$, and thus also the number of vertices in $\mathscr{C}$, is at most $3n - 6$, and our assertion is established for $k(\mathscr{C}) = 0$.

If $k(\mathscr{C}) \geqslant 1$ we take any curve $C \in \mathscr{C}$ for which each region meets other curves of $\mathscr{C}$ and use it to form two 'smaller' Apollonian arrangements: The arrangement $\mathscr{C}_1$ consists of C and of those curves of $\mathscr{C}$ contained in the closure of one of the regions of C, while $\mathscr{C}_2$ consists of C and of the other curves of $\mathscr{C}$ (that is, those contained in the closure of the remaining region of C). Clearly $k(\mathscr{C}_1) < k(\mathscr{C})$, $k(\mathscr{C}_2) < k(\mathscr{C})$, and $n(\mathscr{C}_1) + n(\mathscr{C}_2) = n(\mathscr{C}) + 1$. Therefore, by the inductive assumption,

$$\omega(\mathscr{C}) \leqslant \omega(\mathscr{C}_1) + \omega(\mathscr{C}_2) \leqslant 3n(\mathscr{C}_1) - 6 + 3n(\mathscr{C}_2) - \\ - 6 \leqslant 3n - 9 < 3n - 6$$

provided $n(\mathscr{C}_1) \geqslant 3$ and $n(\mathscr{C}_2) \geqslant 3$. But if $n(\mathscr{C}_2) = 2$ then

$$\omega(\mathscr{C}) \leqslant \omega(\mathscr{C}_1) + 1 \leqslant 3n(\mathscr{C}_1) - 6 + 1 < 3n(\mathscr{C}) - 6;$$

similarly if $n(\mathscr{C}_1) = 2$.

This completes the proof of the inequality of Theorem 1.

In order to establish the equality assertion for Apollonian arrangements of circles, we only have to observe that for $n = 3$ three mutually touching circles have $3 = 3n - 6$ osculation vertices, and that additional circles, each osculating three previous ones, may be added in arbitrary numbers.

This completes the proof of Theorem 1.

3. PROOF OF THEOREM 2

We begin the proof by observing that in the search for arrangements of n curves which maximize the number of osculation vertices we may restrict ourselves to *simple* arrangements, that is arrangements in which no three curves have a common point. Indeed, each vertex V of $\mathscr{C}$ through which pass three or more curves may be 'split up' into several simple vertices, at each of which only two curves meet. It is easily checked that if V was an osculation vertex of $\mathscr{C}$ the 'splitting up' may be accomplished in such a way that at least one of the new vertices is an osculation vertex.

In order to establish the upper bound on $\omega(\mathscr{C})$, we associate with each

simple arrangement $\mathscr{C} = \{C_1, \ldots, C_n\}$ of curves an abstract graph $\mathscr{K}$ with nodes $N_1, \ldots, N_n$. Nodes N_i and N_j determine an edge of $\mathscr{K}$ if and only if $C_i \cap C_j$ is an osculation vertex of $\mathscr{C}$. The key to our proof is the following observation:

LEMMA 1. *The graph $\mathscr{K}$ contains no subgraph isomorphic to the bipartite graph $\mathscr{K}(3,9)$.*

Proof of Lemma 1. Three curves of a simple arrangement determine one of the thirteen non-isomorphic subarrangements shown by the heavy curves in Figure 2. In each of the thirteen cases, the lightly drawn curves indicate the various non-isomorphic ways in which another curve can osculate the first three. The number of such ways is indicated near each subarrangement;
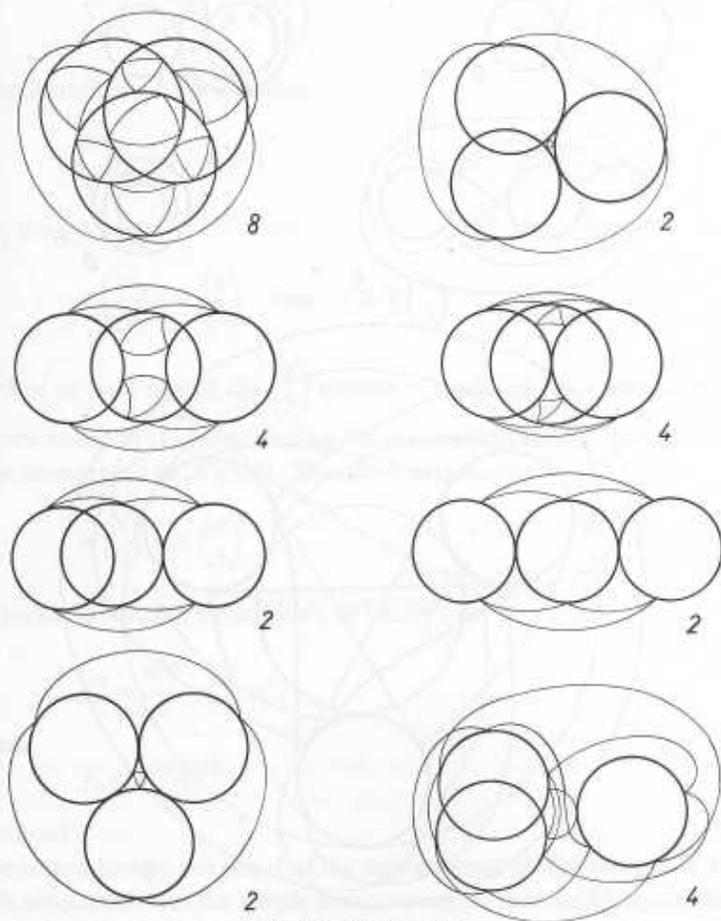

Fig. 2.   (First part.)

in two cases it equals 8, in all the others it is smaller. Simple reasoning using the Jordan curve theorem shows that in each case it is impossible to add another curve osculating the three starting ones without violating the assumption that all the curves form a simple arrangement. Therefore it is not possible to find, in any simple arrangement, nine curves each of which osculates each of three other curves of the arrangement. This completes the proof of Lemma 1.

As another ingredient of the proof we need:

LEMMA 2. *There exists an absolute constant c^{**} with the following property: If $\mathcal{K}$ is a graph with n nodes and if $\mathcal{K}$ contains no subgraph isomorphic to the*
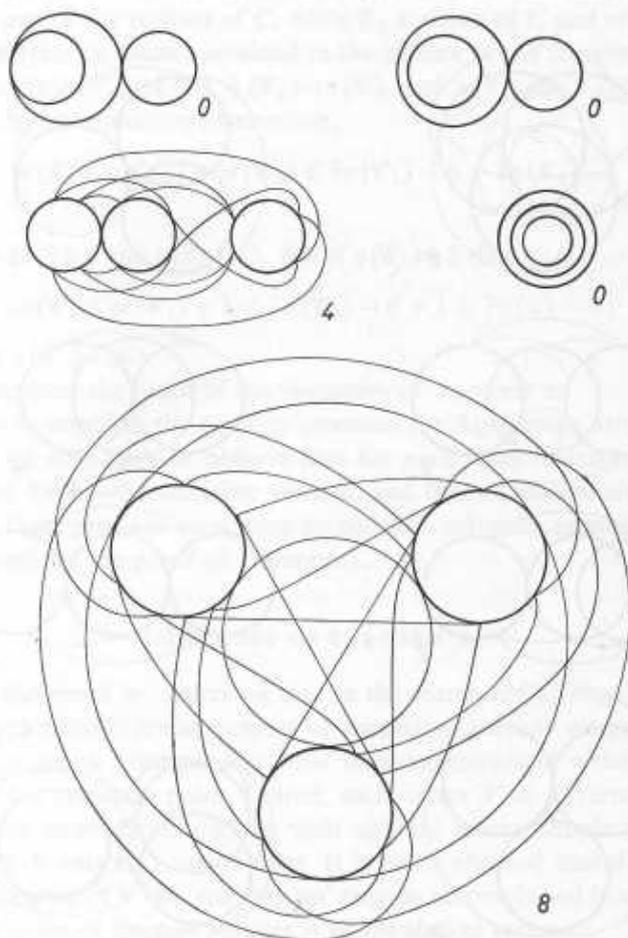

Fig. 2.   (Second part.)

bipartite graph $\mathscr{H}(3,9)$ *then the number* f_1 *of edges of* $\mathscr{H}$ *satisfies* $f_1 \leqslant c^{**}n^{5/3}$.

Proof of Lemma 2. We shall say that a triplet $\{N', N'', N'''\}$ of nodes of $\mathscr{H}$ is *associated* with a node N of $\mathscr{H}$ provided $\mathscr{H}$ contains the edges NN', NN'', and NN'''. We may compute the total number t of associations of triplets with nodes by observing that if a node N_i has valence $v(N_i)$ then N_i is associated with $\binom{v(N_i)}{3}$ triplets; therefore

$$t = \sum_{i=1}^{n} \binom{v(N_i)}{3}.$$

Denoting by $\bar{v}$ the average valence of the nodes of $\mathscr{H}$ (so that $\bar{v} = 2f_1/n$) we have

$$\sum_{i=1}^{n} \binom{v(N_i)}{3} \geqslant n\binom{\bar{v}}{3}$$

by the convexity of the function

$$\binom{x}{3} \quad \text{for} \quad x \geqslant 1.$$

Now, if we have

$$n\binom{\bar{v}}{3} \geqslant 9\binom{n}{3} \quad \text{then} \quad t \geqslant 9\binom{n}{3}$$

and thus at least one of the $\binom{n}{3}$ triplets of nodes of $\mathscr{H}$ is associated with 9 or more nodes of $\mathscr{H}$-contradicting the assumption that $\mathscr{H}$ contains no subgraph isomorphic to $\mathscr{H}(3,9)$. Therefore necessarily

$$n\binom{\bar{v}}{3} < 9\binom{n}{3}$$

and so, for a suitable constant c', $n\bar{v}^3 < c'n^3$, or

$$f_1^3 = \frac{n^3\bar{v}^3}{8} \leqslant \frac{c'}{8}n^5,$$

so that

$$f_1 \leqslant c^{**}n^{5/3}$$

as claimed.

We return now to the proof of the upper bound of Theorem 2. If $\mathscr{H}$ is the graph associated with the simple arrangement $\mathscr{C}$, then $\omega(\mathscr{C})$ equals the number of edges in $\mathscr{H}$. But by Lemma 1 $\mathscr{H}$ contains no subgraph isomorphic to

$\mathcal{K}(3,9)$. Therefore, using Lemma 2, we have

$$\omega(\mathscr{C}) = f_1 \leqslant c^{**} n^{5/3},$$

and so

$$\omega(n) \leqslant c^{**} n^{5/3},$$

as claimed.

In order to establish the lower bound we recall the following lemma due to Jarník (see [14, Satz 2]):

LEMMA 3. *For each $L > 0$ there exists a curve $C(L)$ with the following properties:*

(i) *$C(L)$ is a twice continuously differentiable, strictly convex curve of length L;*

(ii) *$C(L)$ is symmetric with respect to each of the coordinate axes;*

(iii) *$C(L)$ passes through*

$$\varphi(L) = \frac{3}{\sqrt[3]{2\pi}} L^{2/3} + O(L^{1/3})$$

lattice points (that is points both coordinates of which are integers).

We note that each $C(L)$ is centered at the origin $(0,0)$, and that $(x,y) \in C(L)$ implies that the translate of $C(L)$ centered at $(2x, 2y)$ osculates $C(L)$ at (x,y). For each integer L we now construct the arrangement $\mathscr{C}(n)$ formed by the family of $n = L^2$ translates $C_{x,y}$ of $C(L)$ with $1 \leqslant x, y \leqslant L$, where $C_{x,y}$ is centered at the lattice point (x, y). Since for each set of 4 lattice points symmetrically situated on $C_{x,y}$ at least one is an osculation vertex of $\mathscr{C}(n)$, the total number of osculating pairs in $\mathscr{C}(n)$ will be at least

$$\tfrac{1}{2} n \cdot \tfrac{1}{4} \varphi(L) = c_1 n L^{2/3} + n O(L^{1/3}) = c_1 n^{4/3} + O(n^{7/6}).$$

By suitable perturbations the arrangement $\mathscr{C}(n)$ may be changed into a *simple* arrangement $\mathscr{C}^*(n)$ such that each osculating pair in $\mathscr{C}(n)$ becomes an osculating pair in $\mathscr{C}^*(n)$, and thus $\omega(\mathscr{C}^*(n)) = c_1 n^{4/3} + O(n^{7/6})$. Therefore, for a suitable positive c^* we have

$$\omega(n) \geqslant c^* n^{4/3}$$

for all n, and the proof of Theorem 2 is completed.

(3) Various extensions and refinements of Jarník's [14] result (Lemma 3 above) have recently been obtained by Swinnerton-Dyer [20]. His results may be applied, instead of Lemma 3, in order to obtain results related to the lower bound of Theorem 2. As an example we mention:

For every $\varepsilon > 0$ and for every integer $k \geqslant 2$, there exists a strictly convex, k times continuously differentiable curve C, and a number $c = c(k, \varepsilon) > 0$

with the following property: For every n there exists an arrangement $\mathscr{C}(n)$ of n curves, each homothetic with C, such that $\omega(\mathscr{C}(n)) \geq cn^{1-\varepsilon+1/2k}$.

(4) The number of ways in which an integer may be expressed as a sum of two squares has been investigated already by Fermat, Gauss, and Jacobi (see [13] for references), and belongs to the well-known parts of elementary number theory. Nevertheless, we were unable to locate an explicit statement of Lemma 4 in the literature, and have therefore included a sketch of its proof.

(5) Among our results, Theorem 3 is clearly the least satisfactory since it gives no upper bound for $\omega^*(n)$. We conjecture that $n^{1+c/\log\log n}$ is the order of magnitude of $\omega^*(n)$, but it would be of interest to show that $\omega^*(n) = o(n^{1+\varepsilon})$ for each $\varepsilon > 0$, or at least $\omega^*(n) < cn^{3/2}$.

(6) It is easily seen that judicious choices of the number s used in the proof of Theorem 3 yield simple arrangements $\mathscr{C}(n)$. In other words, Theorem 3 remains valid even if in the definition of $\omega^*(n)$ we consider only arrangements for which no point belongs to three circles.

4. PROOF OF THEOREM 3

The proof of Theorem 3 is analogous to the proof of the lower bound in Theorem 2. Instead of Jarnik's [14] construction we use the following lemma, in which $\psi(m)$ denotes the number of distinct solutions in positive integers x, y of the Diophantine equation $x^2 + y^2 = m$; in other words, $4\psi(m)$ is the number of lattice points on the circle of radius $\sqrt{m}$ centered at the origin.

LEMMA 4. *For each $\varepsilon > 0$ and $m > m_0 = m_0(\varepsilon)$ there exists a $k < m$ such that*

$$\psi(k) > m^{(1-\varepsilon)\log 2/\log\log m}.$$

Proof of Lemma 4. Let $p_1 = 5 < p_2 = 13 < p_3 < \cdots$ be the sequence of consecutive primes of the form $p_i \equiv 1 \pmod 4$. The prime number theorem for arithmetic progressions (see, for example, Davenport [4], Prachar [17]) implies that

(*) $p_r = 2(1 + o(1)) r \log r.$

For a given m let r be chosen so that

$$\prod_{i=1}^{r} p_i < m \leq \prod_{i=1}^{r+1} p_i.$$

Then, on the one hand it is well known (see, for example, Hardy-Wright [13, pp. 242 and 238]) that $\psi(\prod_{i=1}^{r} p_i) = 2^r$. On the other hand, by Stirling's

formula and relation (*) we have

$$r = (1 + o(1)) \frac{\log m}{\log \log m}.$$

Putting $k = \prod_{i=1}^{r} p_i$ this completes the proof of Lemma 4.

The proof of Theorem 3 is now easy. We consider first the arrangement $\mathscr{C}^*(n^*)$ formed by $n^* = k^2$ circles, each of radius $\sqrt{k}$, centered at lattice points (x, y) with $1 \leqslant x, y \leqslant k$. Each of them passes through $4\psi(k)$ lattice points, and all those lattice points are contained in the square $1 - [\sqrt{k}] \leqslant x, y \leqslant k + [\sqrt{k}]$. There are $4k^2 \psi(k)$ incidences between lattice points and the circles of $\mathscr{C}^*(n^*)$. We observe that if a circle with radius r centered at a point A passes through a point B then the circle with radius $r + s$ centered at A osculates the circle with radius s centered at B, the osculation point belonging to the line determined by A and B. Now we construct an arrangement $\mathscr{C}(n)$ consisting of $n = k^2 + (k + 2[\sqrt{k}])^2$ circles. $\mathscr{C}(n)$ contains k^2 circles, each of radius $s + \sqrt{k}$, centered at lattice points (x, y) with $1 \leqslant x, y \leqslant k$, and $(k + 2[\sqrt{k}])^2$ circles, each of radius s, centered at lattice points (x, y) with $1 - [\sqrt{k}] \leqslant x, y \leqslant k + [\sqrt{k}]$, where s is a suitable number (for example, $s = \frac{1}{4}$). By the above we have

$$\omega(\mathscr{C}(n)) \geqslant 4k^2 \psi(k) \geqslant 4k^{2 + (1 - \varepsilon) \log 2 / \log \log k} \geqslant n^{1 + c^*/\log \log n}$$

for a suitable $c^* > 0$ and all sufficiently large n of the form $n = k^2 + (k + 2[\sqrt{k}])^2$; from this follows the estimate $\omega^*(n) \geqslant n^{1 + c/\log \log n}$ valid for a suitable positive $c < c^*$ and all sufficiently large n.

This completes the proof of Theorem 3.

5. REMARKS

(1) In view of Lemma 1 it may be inquired what other graphs may (or may not) be subgraphs of graphs $\mathscr{K}$ associated with simple arrangements of curves, or with various special kinds of arrangements. For example, $\mathscr{K}(4, 6)$ may be associated (see Figure 3), but it seems that neither $\mathscr{K}(4, 7)$ nor $\mathscr{K}(5, 5)$ are possible. Unfortunately, the upper bound of Theorem 2 could not be improved using our method of proof even if those observations were established. On the other hand, it is easily seen that $\mathscr{K}(3, 8)$ may be present even in graphs associated with arrangements of circles; therefore our methods do not yield any better upper bounds for arrangements of circles than for general arrangements of curves.

(2) Our Lemma 2 is only a very special case of a family of results that were initiated by a problem of Zarankiewicz [22]. Bounds of the form $c(j, k)n^{2 - 1/j}$ for the number of edges in graphs with n nodes containing no subgraphs
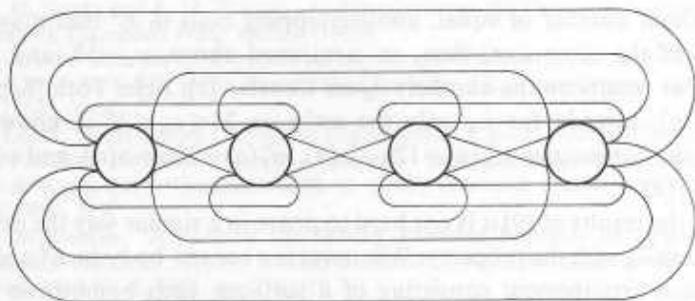
Fig. 3.

isomorphic to the bipartite graph $\mathscr{K}(j, k)$ were first mentioned in Kővári-Sós-Turán [15], where explicit formulae are given for the case $j = k$. We reproduced the proof of the special case of Lemma 2 (after which the general argument can easily be patterned) in order to make the present paper more self-contained, and also since we feel that it deserves to be more widely known. The general question (compare Turán [21] how many edges may a graph with n nodes have without possessing subgraphs of specific types has been the topic of many investigations; recent papers which provide access to the abundant literature are Erdős [5], [6], Simonovits [19], and Guy [12].

(7) De Rocquigny [18] also asked about the maximal possible number $\omega_3^s(n)$ of points of osculation in a family of n mutually non-crossing spheres in Euclidean 3-space ('Apollonian arrangements' of spheres in E^3). While the precise answer is still not known, some estimates may be found easily. We recall, first, the well known fact that a ball in E^3 may be osculated by at most $\kappa_3 = 12$ mutually nonoverlapping balls of equal or larger size. (This fact, asserted by Newton in 1694, was first established by R. Hoppe in 1874 (see [1]); for a survey of known results on this interesting problem, and for historical references, see Coxeter [2]). The simple appearance of the result is deceptive; compare, for example, the fallacious arguments of Fauquembergue [7]. For other results on the 'Hadwiger numbers', 'Newton numbers', and similar notions, and for references to additional literature, (see Fejes Tóth [8], in particular pages 200 and 214). Next, given any Apollonian arrangement of spheres in E^3 none of which bounds another, it follows that any smallest sphere osculates at most 12 others. By arguments similar to those used in the proof of Theorem 1 it follows easily that (for $n \geqslant 3$) $\omega_3^s(n) \leqslant 12n - 24$. However, this estimate is probably far from best possible; we conjecture that $\omega_3^s(n) = 6n - cn^{2/3} + o(n^{1/3})$ for a suitable constant $c > 0$.

For analogously defined Apollonian arrangements of n spheres in E^d one may similarly prove the rather crude estimate $\omega_d(n) \leqslant \kappa_d n$. Here κ_d denotes

the maximal number of equal, nonoverlapping balls in E^d that may touch another of the same size; thus, as mentioned above, $\kappa_3 = 12$, and clearly $\kappa_2 = 6$. For results on the numbers κ_d see Coxeter [2], Fejes Tóth [8, p. 214], Leech [16]; already for κ_4 only the estimate $24 \leqslant \kappa_4 \leqslant 26$ is known. We venture the conjectures $\omega_4^s(n) = 12n + o(n)$, $\omega_5^s(n) = 20n + o(n)$, and $\omega_6^s(n) = 36n + o(n)$.

Using the results of [9] it is not hard to prove in a similar way the existence of numbers λ_d with the property: Whenever is a convex body in E^d and $\mathscr{C}$ an Apollonian arrangement consisting of n surfaces, each homothetic to the boundary of K, then the number of osculation vertices of $\mathscr{C}$ is at most $\lambda_d n$.

It may seem that the arrangements just considered are too special, and that it would be more meaningful to investigate Apollonian arrangements of arbitrary surfaces (homeomorphic to the $(d-1)$-sphere) in E^d. However, already in E^3 it is possible to find, for each n, n nonoverlapping, smooth, rotund convex bodies, the boundaries of which form an Apollonian arrangement with $\binom{n}{2}$ osculation vertices (see Danzer-Grünbaum-Klee [3, p. 151] for polyhedral examples, which may be easily modified into the type of families required here). On the other hand, it seems that a less trivial estimate should exist for the number of osculation vertices in Apollonian arrangements of boundaries of centrally symmetric convex bodies in E^d (compare [10]).

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