JOURNAL OF COMBINATORIAL THEORY (A) 14, 341-363 (1973)

Euclidean Ramsey Theorems. I

P. ERDÖS

Mathematics Institute, Hungarian Academy of Science, Budapest 9, Hungary

R. L. GRAHAM

Bell Telephone Laboratories, Murray Hill, New Jersey 07971

P. MONTGOMERY

Systems Development Corporation, Huntsville Alabama

B. L. ROTHSCHILD[1]

University of California, Los Angeles, California 90024

J. SPENCER[1]

Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

AND

E. G. STRAUS[2]

University of California, Los Angeles, California 90024

Received November 5, 1972

DEDICATED TO THE MEMORY OF THEODORE S. MOTZKIN

The general Ramsey problem can be described as follows: Let A and B be two sets, and R a subset of $A \times B$. For $a \in A$ denote by $R(a)$ the set $\{b \in B \mid (a, b) \in R\}$. R is called r-Ramsey if for any r-part partition of B there is some $a \in A$ with $R(a)$ in one part. We investigate questions of whether or not certain R are r-Ramsey where B is a Euclidean space and R is defined geometrically.

Work on this paper was supported in part by: [1]NSF Grant GP-33580X and [2]NSF Grant GP-28696.

1. Introduction

We are concerned in this paper with problems of the type illustrated as follows: Is it true that for any partition of the Euclidean plane into two classes (we say that the plane is *two-colored*), there exists a set of three points all in the same class forming the vertices of an equilateral triangle of side length 1? (We call such a set *monochromatic.*)

In this example the answer is "no," as can be seen by dividing the points (x, y) into two classes according to the parity of $[2y/\sqrt{3}]$. On the other hand, if we two-color the points of Euclidean 4-space, we have only to look at the five points of an equilateral simplex of side length 1 to see that there must be a monochromatic equilateral triangle of side length 1.

These examples, then, suggest the following general question: Let K be a finite set of points in Euclidean m-space for some m. Then is there an integer n, depending only on K and the integer r, such that for any r-coloring of Euclidean n-space there is a monochromatic configuration K' congruent to K?

In the case of an equilateral triangle with $r = 2$, we saw that the answer is "yes," and that the minimal possible value for n satisfies $2 < n \leqslant 4$. We shall see later that the exact number is $n = 3$.

These questions can be considered special cases of the general Ramsey problem, described as follows: Let A and B be two sets, and R a subset of $A \times B$. For $a \in A$ denote by $R(a)$ the set $\{b \in B \mid (a, b) \in R\}$. R is said to have the *Ramsey property for r colors* if for every partitioning of B into r classes (r-coloring of B), there is an $a \in A$ such that $R(a)$ is contained in only one class (monochromatic). The general Ramsey problem is to characterize those R for which the Ramsey property holds. For instance, suppose A is the set of l-subsets of an n-set S, and B is the set of k-subsets of S. Let $R = \{(a, b) \mid b \subseteq a\}$.

Theorem 1 (Ramsey [7]). *If n is large enough (depending only on l, k, r), R satisfies the Ramsey property for r colors.*

The type of questions we are concerned with here, as indicated above, are questions in which R is determined by geometric considerations. For instance, in the example above, B is the set of points of Euclidean n-space, E^n, and A is the set of triples of these points forming equilateral triangles of side 1. R is just the inclusion relation. We saw that for $r = 2$ and $n = 4$ the Ramsey property holds, while for $r = 2$ and $n = 2$ it does not.

The theorem of van der Waerden [9] on arithmetic progressions was the first important case in which R was determined geometrically. In this case we can take B to be the positive integer points on the real line, A the subsets of l equally spaced points of arbitrary distance (length l arithmetic progressions), and R the inclusion relation.

THEOREM 2 (van der Waerden [9]). *R has the Ramsey property for all r.*

(Actually, van der Waerden's Theorem is stronger, and says that if B consists only of the first n integer points, where n depends on l and r, then R satisfies the Ramsey property for r colors.)

Van der Waerden's Theorem was generalized by Gallai [6] and others [3, 1]. The generalization is as follows: Let K be a set of k points in Euclidean m-space, E^m. Let B be the set of points E^m and A the set of k-sets in E^m similar (in fact homothetic, that is, similar without rotations) to K. Let R be the inclusion relation.

THEOREM 3 (Gallai). *R has the Ramsey property for all r.*

Again, as in van der Waerden's Theorem, B need only consist of a finite set of appropriately chosen points. This is due to the "compactness argument" (see [8], p. 69) which, when applied to the Ramsey property, becomes the following:

PROPOSITION 4. *For sets A and B suppose R satisfies the Ramsey property for r colors with $R(a)$ finite for all $a \in A$. Then there are finite subsets $A' \subseteq A$, $B' \subseteq B$ with $R(a') \subseteq B'$ for all $a' \subseteq A'$ such that the induced relation (defined for $A' \times B'$ by $(a', b') \in R'$ iff $(a', b') \in R$) satisfies the Ramsey property for r colors.*

Theorem 3 is like the case above of the unit equilateral triangle except that similarity replaces congruence. In general, we can consider a property $R_H(K, n, r)$, where K is a finite set of points in E^n, r is an integer, and H is a group of transformations on E^n as follows:

$R_H(K, n, r)$: For any r-coloring of the points of E^n there is a monochromatic configuration K' which is the image of K under some element of H. (This, of course, is the statement that, if B is the set of points of E^n, A the set of images of K under H, and R the inclusion relation, then R satisfies the Ramsey property for r colors.)

We are interested in whether for a given K, r, and H there is an n for which $R_H(K, n, r)$ is true. In particular, we are primarily concerned with the group of Euclidean motions (congruences), and we will drop the subscript H in $R_H(K, n, r)$ when this group is considered if this causes no confusion. (In our example, where K was a unit equilateral triangle, we saw that $R(K, 2, 2)$ was false, but $R(K, 4, 2)$ was true.) We remark that, if $R(K, n, r)$ is true, then so is $R(K', n, r)$ for any K' similar to K.

In Sections 3, 4, and 5 we will investigate configurations K such that for each r there is an n such that $R(K, n, r)$ is true. These will be called *Ramsey configurations*. Not all finite configurations are Ramsey, as we shall see later. We begin first with some special cases.

2. Examples

Certain special cases of $R(K, n, r)$ are already known. For instance:

THEOREM 5. *Let P be a pair of points distance d apart. Then $R(P, 2, 7)$ is false, while $R(P, 2, 3)$ is true.*

Proof. We refer the reader to [4] and [2] for proofs. However, we include the proof for $R(P, 2, 3)$ since it consists only of Figure 1, to which we shall refer later. In it there are seven points of which at most two can simultaneously not be distance d apart.

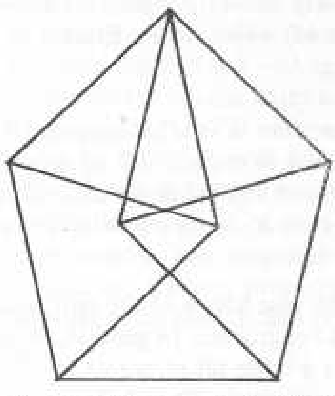

FIG. 1. All edges have length d.

As promised in the introduction, we show that $R(S_3, 3, 2)$ is true, where S_3 is the equilateral triangle of side 1 (or equivalently of side d).

THEOREM 6. *$R(S_3, 3, 2)$ is true.*

Proof. Let E^3 be 2-colored, say red and blue. Then choose any pair of points distance 1 part and both the same color, say red (Theorem 5). Now either there is a third red point at distance 1 from both of these, and we are done, or else there is an entire circle of blue points at distance 1 from both. This circle has radius $\sqrt{3}/2$. Now choose any two points on the circle distance 1 apart. If there is a third point at distance 1 from both which is blue, we are done. Otherwise there is an entire circle of red points (in a plane perpendicular to the plane of the blue circle). If this second alternative holds for each pair of points on the blue circle distance 1 apart, then, as we move around the blue circle, we obtain a whole family of red circles which define a degenerate torus (no hole in the center, due to self intersection). The equatorial radius of this torus is $(\sqrt{2} + \sqrt{3})/2$. Thus we can find three points on the equator mutually $(\sqrt{6} + 3)/4 > 1$ apart. Moving symmetrically from these three points along the surface of the torus toward the middle, we can find three points mutually 1 apart. Since they are on the torus, all three are red, and the proof is complete.

We next consider the unit square C_2. The argument used below was suggested by S. Burr.

THEOREM 7. *$R(C_2, 6, 2)$ is true.*

Proof. Consider the 15 points $(X_1, X_2, ..., X_6)$ in E^6 defined by having four entries equal to 0 and two entries equal to $1/\sqrt{2}$. These 15 points can be represented by edges in the complete graph on 6 vertices, where the edge between v_i and v_j corresponds to the point with $1/\sqrt{2}$ in the i and j coordinates, $1 \leqslant i < j \leqslant 6$. Any 2-coloring of E^6 determines, in particular, a 2-coloring of the 15 points. This determines a 2-coloring of the edges of the complete graph on 6 vertices.

It is well known that in any 2-colored complete 6-graph there exists a monochromatic quadrilateral. That is, there must be four vertices, v_1, v_2, v_3, and v_4 for instance, such that the four edges (v_1v_2), (v_2v_3), (v_3v_4), and (v_4v_1) all have the same color. But this means that the corresponding points in E^6 all have the same color, $1/\sqrt{2}(110000)$, $1/\sqrt{2}(011000)$, $1/\sqrt{2}(001100)$, $1/\sqrt{2}(100100)$. Since these form the vertices of a unit square, the theorem is proved.

We note that $R(C_2, 2, 2)$ is false, as we see by coloring (x, y) according to the parity of $[y]$. Whether it is true for $n = 3, 4, 5$ is undecided.

THEOREM 8. *If T is any set of three points, $R(T, 3, 2)$ is true.*

Proof. Let T be a triangle with sides a, b, and c (where $a + b$ may equal c in the degenerate case). Let E^3 be 2-colored, say with red and blue. Then by Theorem 6, we can find some equilateral monochromatic (say red) triangle ABC of side a. Consider Figure 2 in the plane of ABC. The triangles ABE, DBC, GFC, EFH, ACH, and DEG are all congruent. Then, by choosing the angle EBC properly, we can let them all be congruent to the original triangle T.

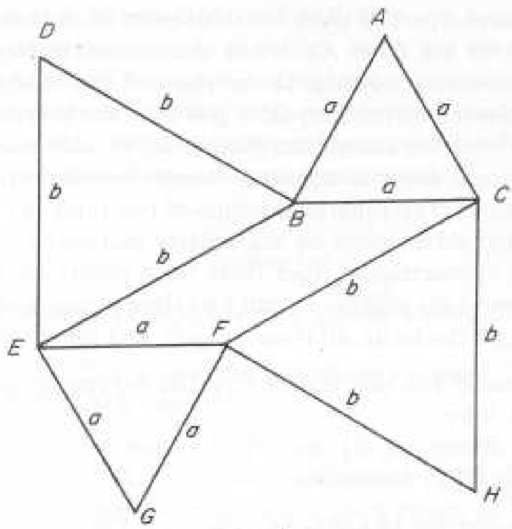

FIG. 2.

Now A, B, and C are all red. Thus, if there are no monochromatic triangles congruent to T, by considering triangles ABE, DBC, and ACH, we see that E, D, and H are blue. Then triangle DEG forces G to be red. But triangles CFG and EFH force F to be blue and red, respectively, a contradiction. Thus one of the six triangles must be monochromatic.

We note that for some triangles (e.g., S_3) $R(T, 2, 2)$ is false. In at least one case, the 30°–60° right triangle, it is true, as we see below.

THEOREM 9. *Let $d > 0$, and let T_1, T_2, T_3 be any three triangles such that T_1 has a side of length d, T_2 a side of length $\sqrt{3}d$, and T_3 a side of length $2d$. Then for any 2-coloring of E^2, there is a triangle T which is congruent to one of T_1, T_2, T_3 and which is monochromatic.*

Proof. By the proof of Theorem 8 above, it is sufficient to show that we must have a monochromatic equilateral triangle with one of the three side lengths d, $\sqrt{3}d$, $2d$. Let E^2 be colored red and blue, and let $u = d(1, 0)$, $v = d(1/2, \sqrt{3}/2)$. By Theorem 5 we may assume $(0, 0)$ and u are both red.

Suppose none of the three kinds of equilateral triangles occurs. Then v and $u - v$ must both be blue. This forces $2u$ to be red, which in turn forces $2v$ to be blue. But then $u + v$ can't be red or blue (because of triangles $(u, 2u, u + v)$ and $(v, 2v, u + v)$), a contradiction.

COROLLARY 10. *Let T be a 30°–60° right triangle. Then $R(T, 2, 2)$ is true.*

The only triangle for which $R(T, 2, 2)$ is known to be true is the 30°–60° right triangle, and the only one for which it is known to be false is the equilateral triangle. We conjecture that $R(T, 2, 2)$ holds unless T is equilateral, and, moreover, that any 2-coloring of E^2 with no monochromatic equilateral triangle of side d in fact has monochromatic equilateral triangles of side d' for all $d' \neq d$.

THEOREM 11. *Let L be the configuration of points in E^2 given by $(-1, 0)$, $(0, 0)$, $(1, 0)$, and $(1, 1)$. Then $R(L, 3, 2)$ is true.*

Proof. Color E^3 red and blue. Then by Theorem 8 there are three points, A, B, C, in a line distance 1 apart and all the same color, say red. Suppose there is no monochromatic L' congruent to L.

Consider the two circles of radius 1, C_A and C_C with centers A and C, respectively, and perpendicular to the line ABC. Both circles must be completely blue, or else we have a red L'. Now consider the circle C_B of radius 1, centered at B and also perpendicular to ABC. This circle must be entirely red, or together with two points on C_A and one on C_C we get a blue L'.

Let S be the sphere of radius $\sqrt{2}$ centered at B, and let S' be the set of points on S which are at most distance 1 from C_B. S' is just S truncated by the planes of C_A and C_C. All points of S' must be blue. For each such point s is distance 1 from some point x on C_B. Let y be the point on C_B diametrically opposite x. Then yBx is perpendicular to sx, since sxB is a right triangle. Thus $sxBy$ is congruent to L, and s must be blue.

Consider a point p in the plane of C_B and distance 2 from B. Then p must be blue, or together with B and two points on C_B we get a red L'. Now consider a point q on S', in the plane of C_B and distance 1 from p. The line joining p and q meets S' in another point r, which must be distance 1 from q. p, q, and r are all blue. Thus the circle of radius 1, center r and perpendicular to the line pqr must be red, or we get a blue L'. But this is a contradiction since this circle meets S', which is all blue.

3. CONFIGURATIONS THAT ARE NOT RAMSEY

We recall that a configuration (set) K in Euclidean space is Ramsey if for each r there is an n for which $R(K, n, r)$ is true. For instance, if K is the equilateral triangle of side length 1, then $R(K, 2r, r)$ holds (since the unit simplex in E^{2r} has $2r + 1$ points, and thus any r-coloring yields three points with the same color.)

We next consider a class of configurations which are not Ramsey. We illustrate first with some special cases.

THEOREM 12. *Let L_k denote the configuration of k collinear points separated by unit distance. Then $R(L_3, n, 4)$, $R(L_4, n, 3)$, and $R(L_6, n, 2)$ are false for all n.*

Proof. For the case of L_3, let each $\mathbf{x} \in E^n$ be colored according to the residue of $[|\mathbf{x}|^2]$ (mod 4). Now suppose we have three points $\mathbf{x}$, $\mathbf{x}+\mathbf{u}$ and $\mathbf{x}-\mathbf{u}$, where $\mathbf{u}$ has length 1. If all three have the same color, there must be integers a_1, a_2, a_3, an integer r, $0 \leqslant r < 4$, and numbers θ_i, $0 \leqslant \theta_i < 1$, $i = 1, 2, 3$, so that $|\mathbf{x}|^2 = 4a_1 + r + \theta_1$, $|\mathbf{x}-\mathbf{u}|^2 = 4a_2 + r + \theta_2$, and $|\mathbf{x}+\mathbf{u}|^2 = 4a_3 + r + \theta_3$. This implies that $1 + 2\mathbf{x}\cdot\mathbf{u} = 4(a_3 - a_1) + \theta_3 - \theta_1$, and $1 - 2\mathbf{x}\cdot\mathbf{u} = 4(a_2 - a_1) + \theta_2 - \theta_1$. Hence $4(a_2 + a_3 - 2a_1) - 2 + (\theta_3 + \theta_2 - 2\theta_1) = 0$, a contradiction since $\theta_i < 1$. Thus $R(L_3, n, 4)$ is false.

For the case of L_4, we color the points $\mathbf{x} \in E^n$ according to the residue $[2|\mathbf{x}|^2]$ (mod 3). Suppose $\mathbf{x} + i\mathbf{u}$, $1 \leqslant i \leqslant 4$, $\mathbf{u}$ a unit vector, are the same color. Let $a_i = |\mathbf{x} + i\mathbf{u}|^2$. Then we have $2a_1 + 2a_3 = 4a_2 + 4$, and $2a_2 + 2a_4 = 4a_3 + 4$. Since all four points are the same color, if we let f_i be the fractional part of $2a_i$, $1 \leqslant i \leqslant 4$, we get, by reduction modulo 3 to reduced residues, $f_1 + f_3 = 2f_2 + 1$ and $f_2 + f_4 = 2f_3 + 1$. Adding these, we get $f_1 + f_4 = f_2 + f_3 + 2$, an impossibility.

For the L_6 case we color the $\mathbf{x}$ in E^n according to the parity of $[|\mathbf{x}|^2/6]$. Let $\mathbf{x} + i\mathbf{u}$, $1 \leqslant i \leqslant 6$, be the same color, where $\mathbf{u}$ is a unit vector. Let $a_i = \frac{1}{6}|\mathbf{x} + i\mathbf{u}|^2$, $1 \leqslant i \leqslant 6$. Then $a_{i+1} + a_{i-1} = 2a_i + 1/3$, for $i = 2, 3, 4, 5$, and all $[a_i]$ have the same parity. We claim that this is impossible.

By replacing each a_i by $a_i + (i-4)[a_3] + (3-i)[a_4]$, we may assume $[a_3] = [a_4] = 0$, and thus that each $[a_i]$ is an even integer. The identities

$$\begin{aligned}
a_2 &= 2a_3 - a_4 + 1/3,\\
a_5 &= 2a_4 - a_3 + 1/3,\\
a_1 &= 2a_2 - a_3 + 1/3,\\
a_6 &= 2a_5 - a_4 + 1/3,\\
a_1 + a_6 &= a_3 + a_4 + 2
\end{aligned}$$

are easily verified.

Using the first two equations, we find a_2 and a_5 are contained in the interval $(-2/3, 7/3)$. But $[a_2]$ and $[a_5]$ are even, so a_2 and a_5 are in $I \cup [2, 7/3)$. If $a_2 \geqslant 2$, then

$$4 \leqslant 2a_2 + a_5 = 3a_3 + 1 < 4,$$

a contradiction. Hence $a_2 \in I$, similarly $a_5 \in I$. By a similar process we get $a_1 \in I$ and $a_6 \in I$. But then $2 \leqslant a_3 + a_4 + 2 = a_1 + a_6 < 2$, a contradiction.

We say that a configuration $K = \{\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of points in E^m is *spherical* if it is imbeddable in the surface of a sphere, that is, if there is a center $\mathbf{x}$ and a radius r so that $|\mathbf{v}_i - \mathbf{x}| = r$ for all $\mathbf{v}_i \in K$.

THEOREM 13. *If K is not spherical, then K is not Ramsey.*

To prove this we require two lemmas.

LEMMA 14. *The set $K = \{\mathbf{v}_0, ..., \mathbf{v}_k\}$ is not spherical if and only if there exist scalars $c_1, ..., c_k$ not all 0 such that:*

$$\sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0) = 0, \tag{1}$$

$$\sum_{i=1}^{k} c_i(|\mathbf{v}_i|^2 - |\mathbf{v}_0|^2) = b \neq 0. \tag{2}$$

Proof. (We use $\mathbf{v}_i^2$ to mean $|\mathbf{v}_i|^2$.) Assume K is spherical, with center $\mathbf{w}$ and radius r, and suppose (1) holds. Then

$$\mathbf{v}_i^2 - \mathbf{v}_0^2 = (\mathbf{v}_i - \mathbf{w})^2 - (\mathbf{v}_0 - \mathbf{w})^2 + 2(\mathbf{v}_i - \mathbf{v}_0) \cdot \mathbf{w} = 2(\mathbf{v}_i - \mathbf{v}_0) \cdot \mathbf{w},$$

and

$$\sum_{i=1}^{k} c_i(\mathbf{v}_i^2 - \mathbf{v}_0^2) = 2\mathbf{w} \cdot \sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0).$$

Hence (2) does not hold.

Now suppose K is not spherical. It is sufficient to assume that K is a minimal non-spherical set. That is, all subsets are spherical. Since every non-degenerate simplex is spherical, it follows that the vectors $\mathbf{v}_i - \mathbf{v}_0$, $1 \leqslant i \leqslant k$, are linearly dependent. There exist c_i, $1 \leqslant i \leqslant k$, not all 0, satisfying (1). Assume $c_k \neq 0$ and that $\{\mathbf{v}_0, ..., \mathbf{v}_{k-1}\}$ is on a sphere with center $\mathbf{w}$ and radius r. Then

$$\begin{aligned}\sum_{i=1}^{k} c_i(\mathbf{v}_i^2 - \mathbf{v}_0^2) &= \sum_{i=1}^{k} c_i[(\mathbf{v}_i - \mathbf{w})^2 - (\mathbf{v}_0 - \mathbf{w})^2] \\ &= c_k(\mathbf{v}_k^2 - \mathbf{v}_0^2) \neq 0,\end{aligned}$$

and (2) holds. This proves Lemma 14.

LEMMA 15. *Let $c_1, ..., c_k$, b be real numbers, $b \neq 0$. Then there exists an integer r, and some r-coloring of the real numbers, such that the equation*

$$\sum_{i=1}^{k} c_i(x_i - x_0) = b \neq 0 \tag{3}$$

has no solution $x_0, x_1, \ldots, x_k$ where all the x_i have the same color (monochromatic solution).

The proof of this lemma, which some may consider of greater interest than Theorem 13, we defer until Section 4 below. It extends the fundamental work of R. Rado [6].

Proof of Theorem 13. Let K be a non-spherical set $\{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$. For an arbitrary n we exhibit a coloring of E^n avoiding any monochromatic set K' congruent to K.

By Lemma 14, there are real numbers, $c_1, c_2, \ldots, c_k$, not all 0, such that equations (1) and (2) of Lemma 14 hold. By Lemma 15 there is some integer r and some r-coloring χ of the real numbers such that equation (3) of Lemma 15 has no monochromatic solution. (That is, χ is a function from the real numbers to $\{1, 2, \ldots, r\}$, where the r colors, or classes, are the $\chi^{-1}(j)$, $1 \leqslant j \leqslant r$.) We now color E^n by the coloring χ^* given by $\chi^*(\mathbf{v}) = \chi(\mathbf{v}^2)$. Thus the colors form spherical "shells" around the origin.

Now we observe that equations (1) and (2) remain valid if K is replaced by any $k+1$-tuple of points congruent to K (using the same choice of c_i). For (1) is clearly invariant under any affine transformation and thus certainly under isometries, while (2) is invariant under isometries fixing the origin, since then the $\mathbf{v}_i^2$ remain unchanged. Furthermore, (2) remains valid after translations as well, since if we translate by $\mathbf{z}$ we get

$$\sum_{i=1}^{k} c_i[(\mathbf{v}_i + \mathbf{z})^2 - (\mathbf{v}_0 + \mathbf{z})^2] = \sum_{i=1}^{k} c_i(\mathbf{v}_i^2 - \mathbf{v}_0^2) + 2\mathbf{z} \cdot \sum_{i=1}^{k} c_i(\mathbf{v}_i - \mathbf{v}_0)$$

$$= \sum_{i=1}^{k} c_i(\mathbf{v}_i^2 - \mathbf{v}_0^2) = b.$$

Thus (1) and (2) both hold for any $\{\mathbf{v}_0', \ldots, \mathbf{v}_k'\}$ congruent to K.

In particular, if we have a monochromatic $\{\mathbf{v}_0', \ldots, \mathbf{v}_k'\}$ congruent to K, then letting $x_i = (\mathbf{v}_i')^2 - (\mathbf{v}_0')^2$ we obtain a monochromatic solution to (3), contrary to the choice of the coloring χ. This completes the proof of Theorem 13 for finite sets. The case in which K is infinite is immediate by considering an appropriate finite subset.

Theorem 13 establishes the necessity of a set being spherical if it is to be Ramsey. The sufficiency of this condition remains undecided. The sufficiency of a stronger condition is established in Section 5 below. We note that the number of colors used depends on the c_i, which in turn depend on the configuration K. The dependence on the c_i appears explicitly in the proof of Theorem 16 (Lemma 15) below.

The coloring used in the proof of Theorem 13 was spherical. That is, any sphere centered at the origin has points of only one color. We might ask whether other kinds of colorings could be used to show sets other than non-spherical sets to be non-Ramsey. In particular, suppose S is a "nice" surface (closed, bounded, separating E^n into two disconnected regions) which is entirely visible from the origin. That is for each point $\mathbf{s} \in S$, the line segment joining the origin and $\mathbf{s}$ meets S only at $\mathbf{s}$). Then E^n can be decomposed into "concentric" surfaces $S_\alpha = \{\alpha\mathbf{s} \mid \mathbf{s} \in S\}$, α a non-negative real number. An *S-coloring* is a coloring which is constant on S_α for each α. We might hope that for some S an S-coloring could be used to show some configuration to be non-Ramsey. Any such configuration would, of course, not be imbeddable in any S_α. However, any non-degenerate simplex which is imbeddable in a sphere is also imbeddable in some S_α if n is large enough, depending on the configuration (see Lesley O'Connor's thesis [5] for a discussion of this and related problems). Thus no non-degenerate simplex can be shown to be non-Ramsey by an S-coloring.

4. Extension of Rado's Results on Monochromatic Solutions of Non-homogeneous Equations

Our object here is to prove Lemma 15 above. Actually, we prove a somewhat stronger result that will be useful later in Section 6 to get a generalization of Theorem 13.

Theorem 16. *Let $c_1, c_2, \ldots, c_k$, $b \neq 0$ be elements of a field F. Then there exists a finite coloring χ of F so that*

$$\sum_{i=1}^{k} c_i(x_i - x_i') = b \tag{4}$$

has no solution $x_1, x_1', x_2, x_2', \ldots, x_k, x_k' \in F$ with $\chi(x_i) = \chi(x_i')$, $1 \leqslant i \leqslant k$.

Proof. Following Rado, we observe that it is sufficient to prove this theorem for the field $F_0 = \Pi(c_1, \ldots, c_k)$, where Π is the prime field of F. To see this choose a Hamel basis B with $b \in B$ for F over F_0 and assume that we have a coloring χ of F_0 for which Theorem 16 holds when $x_i, x_i' \in F_0$, $1 \leqslant i \leqslant k$, and b is replaced by 1. Now color $x \in F$ by $\chi^*(x) = \chi(\mathbf{x})$, where $x = \mathbf{x}b + \cdots$ is the B-expansion of x.

Then

$$\sum_{i=1}^{k} c_i(x_i - x_i') = b \qquad \text{with} \quad \chi^*(x_i) = \chi^*(x_i')$$

leads to

$$\sum_{i=1}^{k} c_i(\mathbf{x}_i - \mathbf{x}_i') = 1 \qquad \text{with} \qquad \chi(\mathbf{x}_i) = \chi(\mathbf{x}_i'),$$

a contradiction. We can therefore prove the theorem by proving it first for prime fields, then for pure transcendental extensions, and then for finite extensions.

Case 0. *$F = \Pi$, the prime field.* This case is essentially given by Rado. If Π is finite, we color all elements with distinct colors so that $\chi(x_i) = \chi(x_i')$ implies $x_i = x_i'$, and (4) has no solution with $\chi(x_i) = \chi(x_i')$.

If $\Pi = Q$, the rational numbers, assume without loss of generality that the c_i and b are integers. Let p be a prime not dividing b, and let M be an integer satisfying $M \geqslant \sum_{i=1}^{k} | c_i |$. Now let χ be a coloring of the rationals given by $\chi(x) = \chi(x')$ if and only if $[x] \equiv [x'] \pmod{p}$ and $[M\{x\}] = [M\{x'\}]$, where $[x]$ is the integer part of x and $\{x\}$ is the fractional part. Thus χ is an Mp-coloring.

Now if $\chi(x_i) = \chi(x_i')$ and

$$b = \sum_{i=1}^{k} c_i(x_i - x_i') = \sum_{i=1}^{k} c_i([x_i] - [x_i']) + \sum_{i=1}^{k} c_i(\{x_i\} - \{x_i'\}),$$

then the first sum on the right is an integer divisible by p which differs from b by at least 1, since b is not divisible by p. The second sum satisfies

$$\Big| \sum_{i=1}^{k} c_i(\{x_i\} - \{x_i'\})\Big| \leq \sum_{i=1}^{k} \frac{| c_i | \, | M\{x_i\} - M\{x_i'\}|}{M}$$

$$< \sum_{i=1}^{k} \frac{| c_i |}{M} = \frac{\sum_{i=1}^{k} | c_i |}{M} \leqslant 1,$$

a contradiction. This completes Case 0.

Case 1. *Purely trancendental extensions.* That is, we assume that the theorem holds for the field F and show that it also holds for $F(y)$, where y is transcendental over F. Multiplying by a suitable polynomial we can assume that all c_i and b are in $F[y]$. We may also assume $b(0) \neq 0$; for, if F is infinite, we may replace y be $y - a$ and $b(0)$ by $b(a)$ for any $a \in F$ if necessary; if F is finite, we first make a finite extension F' of F (for which the theorem holds trivially) such that $b(y)$ does not vanish identically on F', and again replace y by $y - a$.

Now let $m = \max_{1 \leqslant i \leqslant k} \deg c_i(y)$ and write $c_i = \sum_{j=0}^{m} c_{ij} y^j$. For each x_i, $x_i' \in F(y)$ we have Laurent series expansions

$$x_i = yA_i(y) + \sum_{j=0}^{\infty} a_{ij} y^{-j},$$

$$x_i' = yA_i'(y) + \sum_{j=0}^{\infty} a_{ij}' y^{-j},$$

where the sums on the right have only a finite number of non-zero terms, a_{ij}, $a_{ij}' \in F$, $A_i(y)$ and $A_i'(y)$ are in $F(y)$, and $A_i(0)$ and $A_i'(0)$ are in F. Comparing the constant terms on both sides of (4) gives

$$\sum_{i=1}^{k} \sum_{j=0}^{m} c_{ij}(a_{ij} - a_{ij}') = b(0) \neq 0.$$

By hypothesis we can find a coloring χ of F so that this has no solutions with $\chi(a_{ij}) = \chi(a_{ij}')$, $1 \leqslant i \leqslant k$, $0 \leqslant j \leqslant m$. If we now color $x = yA(y) + \sum_{j=0}^{\infty} a_j y^{-j}$ by the "product color" $\chi^*(x) = (\chi(a_0), \ldots, \chi(a_m))$ (that is, $\chi^*(x) = \chi^*(x')$ if and only if $\chi(a_j) = \chi(a_j')$ for all $j = 0, 1, \ldots, m$), then there is no solution of (4) in $F(y)$ with $\chi^*(x_i) = \chi^*(x_i')$, $1 \leqslant i \leqslant k$.

Case 2. *Finite extensions*. We now assume that the theorem holds for F and prove it for a finite extension L of F. Let $[L : F] = d$, and let $\omega_1, \ldots, \omega_d$ be a basis for L over F. We can then write:

$$c_i = \sum_{\delta=1}^{d} c_{i\delta}\omega_\delta,$$

$$x_i = \sum_{\delta=1}^{d} a_{i\delta}\omega_\delta,$$

$$x_i' = \sum_{\delta=1}^{d} a_{i\delta}'\omega_\delta,$$

$$b = \sum_{\delta=1}^{d} b_\delta\omega_\delta, \qquad b_1 \neq 0,$$

and

$$\omega_\alpha\omega_\beta = \sum_{\gamma=1}^{d} \lambda_{\alpha\beta\gamma}\omega_\gamma, \qquad \lambda_{\alpha\beta\gamma} \in F.$$

Comparing coefficients of ω_1, in (4) we obtain

$$\sum_{i=1}^{k} \sum_{\alpha=1}^{d} \sum_{\beta=1}^{d} \lambda_{\alpha\beta 1} c_{i\alpha}(a_{i\beta} - a'_{i\beta}) = b_1 \neq 0.$$

By hypothesis, we can find a coloring χ of F so that this has no solution with $\chi(a_{i\beta}) = \chi(a'_{i\beta})$, $1 \leqslant i \leqslant k$, $1 \leqslant \beta \leqslant d$. If we color each $x = \sum_{\beta=1}^{d} a_\beta \omega_\beta$ by the product coloring $\chi^*(x) = (\chi(a_1),..., \chi(a_d))$, as above in Case 1, then we see that (4) can have no solution in L with $\chi^*(x_i) = \chi^*(x_i')$ for all i. This completes Case 2 and the proof of Theorem 16. We note that in both Cases 1 and 2 the number of colors was dependent on the degrees of the c_i (over the appropriate field). In Case 0, where $\Pi = Q$, the number of colors depended on the magnitudes of the c_i and the prime divisors of b.

It is natural to ask whether Theorem 16 can be extended to expressions in which the linear forms on the left-hand side of (4) are replaced by a homogeneous form of higher degree. This question is settled negatively below.

THEOREM 17. *If Q is colored with k colors then the equation $(x_1 - y_1)(x_2 - y_2) = 1$ always has solutions with color $x_i =$* color y_i $(i = 1, 2)$.

Proof. By van der Waerden's Theorem [9] there is an arithmetic progression with $k!\,(2k+1)^2$ elements all of which are colored alike so $x_1 - y_1 = dn$ has monochromatic solutions with $n = 1, 2,..., k!\,(2k+1)^2$ for some $d > 0$.

Now consider the numbers

$$\frac{1}{d(k+1)!}, \frac{1}{dk!\,(k+2)}, ..., \frac{1}{dk!\,(2k+1)}$$

two of them, say

$$x_2 = \frac{1}{dk!\,(k+i)} \qquad \text{and} \qquad y_2 = \frac{1}{dk!\,(k+j)},$$

have the same color and

$$x_2 - y_2 = \frac{1}{d} \frac{j-i}{k!\,(k+i)(k+j)} = \frac{1}{dn},$$

where $n < k!\,(2k+1)^2$.

By the proof of Theorem 16 we see that in Theorem 13 the number of colors needed to color E^n and to avoid a monochromatic K' congruent to K depends on the number theoretic properties of the distances between points of K. In certain special cases, however, we can obtain an upper bound on the number of colors depending only on the number of points in K. The following is essentially the iteration of Case 1 in Theorem 16, followed by an application of Case 0.

COROLLARY 18. *If $K = \{\mathbf{v}_0, \ldots, \mathbf{v}_k\}$ is a minimal non-spherical set (all subsets are spherical), and the constants c_i in* (1) *and* (2) *of Lemma* 14 *are such that $c_2/c_1, c_3/c_1, \ldots, c_k/c_1$ are algebraically independent over Q, then every E^n has a coloring (in spherical shells) with $(2k)^k$ colors so that there is no monochromatic K' congruent to K in E^n.*

Proof. It suffices to show that there is a $(2k)^k$-coloring of R, the real numbers, so that equation (3) has no real solutions $x_0, x_1, \ldots, x_k$ which are monochromatic.

As in the proof of Theorem 16, Case 1, we may assume that $b \neq 0$. Thus we may assume $b = 1$, $c_1 = 1$, and $c_2, \ldots, c_k$ are algebraically independent transcendentals. Proceeding as in Case 1 we expand the x_i in Laurent series in c_2

$$x_i = \cdots + a_{i2}c_2^{-2} + a_{i1}c_2^{-1} + a_{i0} + a_{i,-1}c_2 + \cdots$$

so that comparing the constant terms in (3) we get

$$(a_{10} - a_{00}) + (a_{21} - a_{01}) + c_3(a_{30} - a_{00}) + \cdots + c_k(a_{k0} - a_{00}) = 1.$$

Expanding the a_{ij} Laurent series in c_3 we get

$$a_{ij} = \cdots + a_{ij2}c_3^{-2} + a_{ij1}c_3^{-1} + a_{ij0} + a_{ij-1}c_3 + \cdots$$

and

$$(a_{100} - a_{000}) + (a_{210} - a_{010}) + (a_{301} - a_{001}) + \cdots + c_k(a_{k00} - a_{000}) = 1.$$

Repeating this process we finally get

$$\begin{aligned}(a_{100\cdots 0} - a_{00\cdots 0}) + (a_{210\cdots 0} - a_{010\cdots 0}) + (a_{3010\cdots 0} - a_{0010\cdots 0})\\ + \cdots + (a_{k0\cdots 01} - a_{00\cdots 01}) = 1,\end{aligned}$$

where the $a_{i_1,\ldots,i_k}$ are rational numbers.

Now we color the rationals with $2k$ colors as follows: Two rationals a

and a' have the same color if and only if $[a] \equiv [a'] \pmod 2$ and $[k\{a\}] = [k\{a'\}]$. It is then clear that the equation in the $a_{i,i_2 \cdots i_2}$ above has no monochromatic solution, since the left side would equal an even integer plus k fractions each less than $1/k$ in absolute value.

The product coloring

$$\chi^*(x_i) = (\chi(a_{i0\cdots 0}), \chi(a_{i10\cdots 0}), \ldots, \chi(a_{i0\cdots 01}))$$

has $(2k)^k$ colors and yields no monochromatic solution to (3).

For three collinear points we have a slightly better result.

Corollary 19. *If $K = \{\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2\}$, where $(\mathbf{v}_1 - \mathbf{v}_0) + \alpha(\mathbf{v}_2 - \mathbf{v}_0) = 0$ and $\alpha \notin Q$ (the rationals), then for every E^n there is a spherical coloring with 16 colors avoiding monochromatic sets congruent to K.*

Proof. If α is transcendental, we can apply the previous corollary, obtaining $4^2 = 16$ colors. If α is algebraic, as in the proof in Theorem 11, it suffices to 16-color the reals, $\mathbb{R}$, so that

$$(x_1 - x_0) + \alpha(x_2 - x_0) = b \neq 0 \tag{5}$$

has no monochromatic real solution. As above, we may assume $b = 1$. It is sufficient to 16-color $Q(\alpha)$, as in the proof of Theorem 16. Assume the minimal polynomial of α is

$$x^n - a_{n-1}x^{n-1} - a_{n-2}x^{n-2} - \cdots - a_0 \in Q[x].$$

Setting $x_i = \sum_{j=0}^{n-1} x_{ij}\alpha^j$ and equating constant terms in (5) yields

$$(x_{10} - x_{00}) + a_0(x_{2,n-1} - x_{0,n-1}) = 1. \tag{6}$$

Now define $\chi(c) = [2c] \pmod 4$. Then the product coloring $\chi^*(x_i) = (\chi(x_{i0}), \chi(a_0 x_{i,n-1}))$ is a 16-coloring of $Q(\alpha)$. If $\chi^*(x_0) = \chi^*(x_1) = \chi^*(x_2)$, then $x_{10} - x_{00} = 2K + \epsilon$, K an integer, and $0 \leqslant \epsilon < 1/2$, and $a_0(x_{2,n-1} - x_{0,n-1}) = 2L + \delta$, L an integer and $0 \leqslant \delta < 1/2$. This contradicts (6), completing the proof.

We observed at the beginning of Section 3 that if $\alpha = 1$ then 4 colors suffice. It remains open whether in fact there is some r such that r colors suffice for all α, rational or irrational. More generally, it is unknown whether for any k there is a number r of colors depending only on k such that r colors suffice to prevent a monochromatic K for any non-Ramsey K with $k + 1$ points.

5. Configurations That Are Ramsey

We observed at the beginning of Section 3 the obvious fact that the equilateral triangle is Ramsey. Similarly, if K is a regular simplex of $k+1$ points, then $R(K, kr, r)$ always holds, and thus K is Ramsey. These and the configurations derived from them by the theorems below are the only ones that are presently known to be Ramsey.

If $K_1 \subseteq E^n$, $K_2 \subseteq E^m$ we define $K_1 \times K_2$ in E^{n+m} to be the set of points

$$\{(x_1, ..., x_n, y_1, ..., y_m) \mid (x_1, ..., x_n) \in K_1, (y_1, ..., y_m) \in K_2\}.$$

Theorem 20. *If K_1 and K_2 are Ramsey, then so is $K_1 \times K_2$.*

Proof. By the compactness principle (Proposition 4 in Section 1), for any integer r there is an integer n_1 and a finite set $T \subseteq E^{n_1}$ such that every r-coloring of T yields a monochromatic K_1' congruent to K_1. Let $|T| = t$. Similarly, for K_2 there is some n_2 and some finite set S in E^{n_2} such that every r^t-coloring of S yields a monochromatic K_2' congruent to K_2.

Consider the set $T \times S$ in $E^{n_1+n_2}$. Let $T \times S$ be r-colored by χ. Now define a coloring χ^* on S by letting $\chi^*(\mathbf{u}) = \chi^*(\mathbf{u}')$, $\mathbf{u}, \mathbf{u}' \in S$, if and only if $\chi(\mathbf{v} \times \mathbf{u}) = \chi(\mathbf{v} \times \mathbf{u}')$ for all $\mathbf{v} \in T$. This is an r^t-coloring of S. Hence there is some K_2' congruent to K_2 in S on which χ^* is constant. Let $\mathbf{u}_0 \in K_2'$. Define a coloring χ^{**} on T by $\chi^{**}(\mathbf{v}) = \chi(\mathbf{v} \times \mathbf{u}_0)$. This is an r-coloring of T. Hence there is a K_1' monochromatic and congruent to K_1. But then χ is monochromatic on $K_1' \times K_2'$, since, by choice of χ^*, χ remains constant as we vary the points in K_2', and, by the choice of χ^{**}, χ remains constant as we vary the points of K_1'. This completes the proof. We obtain a (probably very weak) bound on the size of n for which $R(K_1 \times K_2, n, r)$ holds. Namely, if $R(K_1, n_1, r)$ and $R(K_2, n_2, r^{n_1})$ hold, then $R(K_1 \times K_2, n_1 + n_2, r)$ holds.

We use Theorem 14 to obtain a class of Ramsey configurations. We say that a *brick* in E^n is any set congruent to a set

$$B = \{(x_1, ..., x_n) \mid x_i = 0, a_i; a_i \geqslant 0; 1 \leqslant i \leqslant n\}.$$

That is, B is the set of vertices of a rectangular parallelepiped.

Corollary 21. *Any brick is Ramsey.*

Proof. Since the sets $K_i = \{0, a_i\}$ are simplices, this is a direct result of iterating Theorem 20, as $B = K_1 \times K_2 \times \cdots \times K_n$. The bounds obtained from Theorem 20 on the dimension as a function of the number of colors are colossal. However, better and more explicit bounds are obtained in Part II of this paper (to appear).

COROLLARY 22. *Any subset of the vertices of a brick is Ramsey.*

We remark that the proof of Theorem 20 does not necessarily yield the best bounds for the dimension required for the Ramsey property to hold. For example, the argument in Theorem 20 gives a bound of $n = 10$ for $R(C_2, n, 2)$, in contrast to $n = 6$ from Theorem 7. Similarly, for any rectangle, Theorem 20 gives $n = 10$, whereas a similar but more careful argument will yield $n = 8$. In particular, we could have replaced $n_2 = 8$ in Theorem 20 (for this case) with $n_2 = 6$, since the monochromatic edge of the triangle T (needed to assure the existence of a monochromatic pair with given distance) can occur only in 6 different ways.

The regular unit simplex of k points is itself a subset of a brick, namely, in the cube in E^k with side length $1/\sqrt{2}$. Let an *l-dual* of a simplex of n points be the set obtained by taking the centroids of each of the $\binom{n}{l}$ l-point subsimplices. The 1-dual is the simplex itself, and the $(n-1)$-dual is the usual dual. We see then that any l-dual of a regular simplex is Ramsey (by Theorem 1 [Ramsey's]). Among the sets obtained this way is the regular octahedron, the 2-dual of the tetrahedron. We can realize the l-duals of regular simplices as subsets of bricks as well. For taking the regular simplex of n points to be, for instance, $\{(1, 0,..., 0), (0, 1, 0,..., 0),..., (0,..., 0, 1)\}$, the points of the l-dual are all points $(x_1, x_2, ..., x_n)$ where all but l of the x_i are 0, and these l are equal to $1/\sqrt{l}$. These are clearly vertices of a cube of side $1/\sqrt{l}$.

Some simplices are not realizable as subsets of bricks. For instance, any simplex such that three points in it determine a triangle containing an obtuse angle cannot be so realized. One can ask whether having all angles non-obtuse is sufficient for a simplex to be imbeddable in a brick. In the case of the tetrahedron, we can answer the question in the affirmative, but for the 5-point simplex the answer is negative.

The condition that no angle be obtuse is equivalent to the following property: For any three vertices v_1, v_2, v_3 the distances between them d_{12}, d_{13}, d_{23} satisfy $d_{12}^2 + d_{23}^2 - d_{13}^2 \geqslant 0$, the triangle inequality for the squares of the sides. For the case of five points, the following set of distances are the distances of a simplex which cannot be imbedded in a brick: $d_{12} = d_{23} = d_{13} = \sqrt{2}$, $d_{14} = d_{24} = d_{34} = d_{15} = d_{25} = d_{35} = 1$, $d_{45} = 2/\sqrt{3}$. We can see this by observing that, since bricks are spherical, any imbedding of the simplex in a brick would determine a center $\bar{x}$ equidistant from all points of the simplex. This is clearly impossible.

THEOREM 23. *Let d_{ij}, $1 \leqslant i < j \leqslant 4$ be six distances satisfying $d_{ij}^2 + d_{jk}^2 \geqslant d_{ik}^2$ for each i, j, k. Then there is a 6-dimensional brick such that a subset of four of its vertices realize these six distances.*

Proof. Let v_1, v_2, v_3, v_4 be the vertices we are going to choose, and let them first be vertices of a 7-dimensional brick as follows:

$$v_1 = (0, 0, 0, 0, 0, 0, 0), \qquad v_2 = (0, 0, 0, a_4, a_5, a_6, a_7),$$
$$v_3 = (0, a_2, a_3, 0, 0, a_6, a_7), \qquad v_4 = (a_1, 0, a_3, 0, a_5, 0, a_7).$$

They are vertices of an $a_1 \times a_2 \times \cdots \times a_7$ brick. What we must show is that we can choose the a_i nonnegative with one a_i being 0. We have six equations, one for each edge of the tetrahedron:

$$d_{12}^2 = a_4^2 + a_5^2 + a_6^2 + a_7^2,$$
$$d_{13}^2 = a_2^2 + a_3^2 + a_6^2 + a_7^2,$$
$$d_{23}^2 = a_2^2 + a_3^2 + a_4^2 + a_5^2,$$
$$d_{14}^2 = a_1^2 + a_3^2 + a_5^2 + a_7^2,$$
$$d_{24} = a_1^2 + a_3^2 + a_4^2 + a_6^2,$$
$$d_{34}^2 = a_1^2 + a_2^2 + a_5^2 + a_6^2.$$

Now considering the three equations corresponding to the edges of a triangular face, say the 12-, 23-, 13-edges, we can solve for certain pairwise sums of the a_i^2. For instance, from the 1, 2, 3 triangle we get

$$a_2^2 + a_3^2 = \frac{d_{23}^2 + d_{13}^2 - d_{12}^2}{2},$$
$$a_4^2 + a_5^2 = \frac{d_{12}^2 + d_{23}^2 - d_{13}^2}{2},$$
$$a_6^2 + a_7^2 = \frac{d_{12}^2 + d_{13}^2 - d_{23}^2}{2}.$$

These are all nonnegative by the condition on the d_{ij}^2 (non-obtuse). Doing this for each of the four faces, we eventually get nonnegative solutions for each pair $a_i^2 + a_j^2$ where $i \in \{3, 5, 6\}$, $j \in \{1, 2, 4, 7\}$.

Now choose the smallest $a_i^2 + a_j^2$, say $a_1^2 + a_3^2$. Looking at the equations for $a_i^2 + a_j^2$ with $i = 3$ or $j = 1$ we see that there is a unique solution with $a_1 = 0$ and $a_j \geqslant 0$ for all j. Since the other equations must be consistent with these six, they are also satisfied. Thus we have solved for $a_2, a_3, \ldots, a_7$ so that the given tetrahedron is a subset of the vertices of an $a_2 \times \cdots \times a_7$ brick.

We showed in Section 3 that for a configuration to be Ramsey it is necessary that it be spherical. In this section we see that a sufficient condition is that it be a subset of a brick. It is an open question whether either of these conditions is both necessary and sufficient. In particular, the simplest configuration which the Ramsey property is undecided is three points forming an obtuse triangle. (If they form an non-obtuse triangle, it is Ramsey by Theorem 23, and if they are collinear it is not Ramsey by Theorem 13.)

We point out one more relation between bricks and spheres. We say that a configuration is sphere-Ramsey if for each r there is an integer n and a real number d such that every r-coloring of a sphere S of dimension at least n and radius at least d yields a monochromatic configuration $K' \subseteq S$ congruent to K.

THEOREM 24. *Every brick is sphere-Ramsey.*

Proof. The proof is just like that for Theorem 20 and Corollary 21. We first observe that a sphere of radius at least $\sqrt{k/2(k+1)}$ and dimension at least $k+1$ contains the $k+1$ vertices of a regular unit simplex. Letting $k = r$ and $d = a\sqrt{k/2(k+1)}$ we see that the theorem is true for the one-dimensional brick of length a.

If S_n is the $(n-1)$-sphere of radius d_n, and S_m is the $(m-1)$-sphere of radius d_m, then $S_n \times S_m$ is contained in the $(m+n-1)$-sphere of radius $\sqrt{d_n^2 + d_m^2}$. Using this fact we can argue exactly as in the proof of Theorem 14 to show that, if K_1 is sphere-Ramsey and K_2 is sphere-Ramsey, then $K_1 \times K_2$ is sphere-Ramsey. This shows, then, that all bricks are sphere-Ramsey.

6. GENERALIZATIONS: l-RAMSEY CONFIGURATIONS

A set K in E^n is *l-Ramsey* if for every r there is an N, depending only on r, l and K, such that every r-coloring of E^N yields a set K' congruent to S such that the points of K' are colored with at most l colors. We see that the previous notion of Ramsey is just 1 Ramsey by this definition. Theorem 13 can now be generalized as follows:

THEOREM 25. *If K cannot be imbedded in $l-1$ concentric spheres, then K is not m-Ramsey for $m < l$.*

We use two lemmas, as in the proof of Theorem 13.

LEMMA 26. *Let $\mathbf{x}_1, \ldots, \mathbf{x}_l, \mathbf{y}_1, \ldots, \mathbf{y}_l$ be (not necessarily distinct) points of E^n. Then there exists a point $\mathbf{a} \in E^n$ such that $|\mathbf{x}_i - \mathbf{a}| = |\mathbf{y}_i - \mathbf{a}|$, $1 \leqslant i \leqslant l$, if and only if for all scalars $c_1, c_2, \ldots, c_l$ with $\sum_{i=1}^{l} c_i(\mathbf{x}_i - \mathbf{y}_i) = 0$ we have $\sum_{i=1}^{l} c_i(\mathbf{x}_i^2 - \mathbf{y}_i^2) = 0$.*

Proof. Assume that there exists an $\mathbf{a}$ in E^n so that $|\mathbf{x}_i - \mathbf{a}| = |\mathbf{y}_i - \mathbf{a}|$ for $i = 1, 2, \ldots, l$. Then

$$0 = \sum_{i=1}^{l} c_i(|\mathbf{x}_i - \mathbf{a}|^2 - |\mathbf{y}_i - \mathbf{a}|^2)$$

$$= \sum_{i=1}^{l} c_i(\mathbf{x}_i^2 - \mathbf{y}_i^2) - 2\mathbf{a} \cdot \sum_{i=1}^{l} c_i(\mathbf{x}_i - \mathbf{y}_i).$$

Thus $\sum_{i=1}^{l} c_i(\mathbf{x}_i^2 - \mathbf{y}_i^2) = 0$ whenever $\sum_{i=1}^{l} c_i(\mathbf{x}_i - \mathbf{y}_i) = 0$.

Conversely, the existence of a point $\mathbf{a} \in E^n$ with $|\mathbf{x}_i - \mathbf{a}| = |\mathbf{y}_i - \mathbf{a}|$, $i = 1, \ldots, l$, is equivalent to the consistency of the set of equations $2(\mathbf{x}_i - \mathbf{y}_i) \cdot \mathbf{a} = \mathbf{x}_i^2 - \mathbf{y}_i^2$, $1 \leqslant i \leqslant l$, where the variables are the coordinates of $\mathbf{a}$. This system is consistent if and only if every linear combination annihilating the left-hand side also annihilates the right-hand side. That is $\sum_{i=1}^{l} c_i(\mathbf{x}_i - \mathbf{y}_i) = 0$ implies $\sum_{i=1}^{l} c_i(\mathbf{x}_i^2 - \mathbf{y}_i^2) = 0$.

LEMMA 27. *Let $K = \{\mathbf{x}_1, \ldots, \mathbf{x}_l, \mathbf{y}_1, \ldots, \mathbf{y}_l\}$ be a set of $2l$ not necessarily distinct points of E^n so that there exists no point $\mathbf{a} \in E^n$ with $|\mathbf{x}_i - \mathbf{a}| = |\mathbf{y}_i - \mathbf{a}|$ for all i, $1 \leqslant i \leqslant l$. Then there is a number $r = r(K)$ of colors so that every E^n can be r-colored such that for every K' congruent to K in E^n the colors of $\mathbf{x}_i'$ and $\mathbf{y}_i'$ are not all the same, $i = 1, 2, \ldots, l$.*

Proof. According to Lemma 20 there exist constants $c_1, \ldots, c_l$ so that $\sum_{i=1}^{l} c_i(\mathbf{x}_i - \mathbf{y}_i) = 0$ and $\sum_{i=1}^{l} c_i(\mathbf{x}_i^2 - \mathbf{y}_i^2) = b \neq 0$. Now by Theorem 16 there exists a finite coloring χ of the reals so that the equation $\sum_{i=1}^{l} c_i(u_i - v_i) = b$ has no solution with $\chi(u_i) = \chi(v_i)$, $1 \leqslant i \leqslant l$. Thus, if we use the spherical coloring $\chi^*(\mathbf{x}) = \chi(\mathbf{x}^2)$, the equation $\sum_{i=1}^{l} c_i(\mathbf{x}_i^2 - \mathbf{y}_i^2) = b$ has no solutions with $\chi(\mathbf{x}_i^2) = \chi(\mathbf{y}_i^2)$, $1 \leqslant i \leqslant l$ (or $\chi^*(\mathbf{x}_i) = \chi^*(\mathbf{y}_i)$ for all i).

Proof of Theorem 25. Assume for a finite K that for every sphere-coloring of E^n there exists a set K' congruent to K colored in $m \leqslant l - 1$ colors. Each such coloring gives a partition P of K in the disjoint union $K_1 \cup K_2 \cup \cdots \cup K_m = K$ with the K_i congruent to distinct K_i' each of which is monochromatic.

For each finite K there is only a finite number M of such partitions P. If for each P there is a spherical coloring χ_P of E^n that prevents the existence of a set K' congruent to K with each K_i' monochromatic, then, using the product coloring $\chi = (\chi_{P_1}, ..., \chi_{P_m})$, we get a finite coloring of E^n preventing any K' congruent to K with fewer than l colors.

Now, by assumption, the sets $K_1, ..., K_m$ do not lie on the union of m concentric spheres. Therefore, for each $|K_i| > 1$ we can label the points of K_i as $\mathbf{x}_i, \mathbf{y}_{i1}, ..., \mathbf{y}_{ik_i}$, and there can be no point $\mathbf{a}$ so that $|\mathbf{x}_i - \mathbf{a}| = |\mathbf{y}_{ij} - \mathbf{a}|$ for all pairs $\mathbf{x}_i, \mathbf{y}_{ij}$, $1 \leqslant j \leqslant k_i$, $1 \leqslant i \leqslant m$.

By Lemma 27 it is possible to color E^n with a finite spherical coloring χ in such a way that for no K' congruent to K do we have $\chi(\mathbf{x}_i') = \chi(\mathbf{y}_{ij}')$ for all i, j. In other words, not all K_i' can be monochromatic. This proves Theorem 25 for finite K. The infinite case follows immediately.

THEOREM 28. *If $K = K_1 \times K_2 \times \cdots \times K_t$ and for each i, $1 \leqslant i \leqslant t$, K_i is finite and l_i-Ramsey, then K is $l_1 l_2 \cdots l_t$ Ramsey.*

Proof. We clearly need only to prove this for $t = 2$. So let K_i be l_i-Ramsey, $i = 1, 2$. By the compactness argument (Proposition 4 in Section 1), for any r we can find finite sets A_1 and A_2 such that whenever A_2 is $r^{|K_1|}$-colored it contains an l_2-chromatic K_2' congruent to K_2, and whenever A_1 is $r^{|A_2|}$-colored it contains an l_1-chromatic K_1' congruent to K_1.

Now $A_1 \times A_2$ is contained in some E^n, for n large enough. Any r-coloring χ of E^n induces the r-coloring χ of $A_1 \times A_2$. Each of the points $\mathbf{x} \in A_1$ can be associated with the $|A_2|$-tuple of colors determined by the $\chi(\mathbf{x} \times \mathbf{y})$, $\mathbf{y} \in A_2$. This is, then, an $r^{|A_2|}$-coloring χ^* of A_1. Now, by choice of A_1, there is $K_1' \subseteq A_1$ such that χ^* has only l_1 different values on K_1'.

Now define a coloring χ^{**} on A_2 by letting $\chi^{**}(\mathbf{y}) = \chi^{**}(\mathbf{y}')$ if and only if $\chi(\mathbf{x} \times \mathbf{y}) = \chi(\mathbf{x} \times \mathbf{y}')$ for all $\mathbf{x} \in K_1'$. This is an $r^{|K_1|}$-coloring of A_2 and thus there is a K_2' congruent to K_2 such that $\chi^{**}(\mathbf{y})$ has at most l_2 different values for $\mathbf{y} \in K_2'$.

By definition of the colorings χ^* and χ^{**} we see that $\chi(\mathbf{x} \times \mathbf{y})$ takes only $l_1 l_2$ distinct values on $K_1' \times K_2'$. This establishes the theorem.

Among the open questions that remain are whether Theorem 28 is valid if K is infinite. Also, generalizing from the $l = 1$ case, it is undecided whether any set which is in the union of l concentric spheres must be l-Ramsey.

REFERENCES

1. R. L. GRAHAM AND B. L. ROTHSCHILD, Ramsey's Theorem for n-parameter sets, *Trans. Amer. Math. Soc.* **159** (1971), 257–292.
2. H. HADWIGGER, H. DEBRUNNER, AND V. KLEE, "Combinatorial Geometry in the Plane," Holt, Rinehart and Winston, New York, 1964.

3. A. W. Hales and R. I. Jewett, Regularity and positional games, *Trans. Amer. Math. Soc.* **106** (1963), 222–229.
4. L. Moser and W. Moser, Problem 10, *Canad. Math. Bull.* **4** (1961), 187–189.
5. L. O'Connor, Ph.D. Thesis, University of California at Los Angeles, 1972.
6. R. Rado, Note on combinatorial analysis, *Proc. London. Math. Soc.* (2) **48** (1943), 122–160.
7. F. P. Ramsey, On a problem of formal logic, *Proc. London. Math. Soc.* (2) **30** (1930), 264–286.
8. J. R. Schoenfield, "Mathematical Logic," Addison-Wesley, Reading, 1967.
9. B. L. van der Waerden, Beweis einer Baudetschen Vermutung, *Nieuw Arch. Wisk.* **15** (1927), 212–216.

Items quoted in the first two entries are not included in this list.