by a well-ordered index set. Since, for a given element g of G, only finitely many of the exponents $\alpha_i(g)$ will be non-zero, there will only be finitely many non-identity factors in the right-hand side of (2) and so m is well-defined.

Case 2. The group G/Z is a torsion group.

This case is more difficult but may be reduced to the previous one by first decomposing G/Z into its p-components and then considering, in each of these components, a basic subgroup, which by definition is a direct product of cyclic groups (Fuchs [**1**, p. 98]).

Case 3. Every element of Z has a unique square root.

Here we set $m(g, h) = [g, h]^{\frac{1}{2}}$ and it is not difficult to verify that this satisfies conditions (i) to (iv). The ring so obtained is essentially the same as the one discussed by Kaloujnine [**2**].

This case includes the case when Z has odd exponent.

Whether the conjecture is true in general remains an open question.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

CROSSING NUMBER PROBLEMS

P. ERDÖS, Hungarian Academy of Science, and R. K. GUY, University of Calgary

A **graph**, $G(V, E)$, is a set V of **vertices** and a subset E of the unordered pairs of vertices, called **edges**. A **drawing** is a mapping of a graph into a surface. The vertices go into distinct points, **nodes**. An edge and its incident vertices map into a homeomorphic image of the closed interval $[0, 1]$ with the relevant nodes as endpoints and the interior, an **arc**, containing no node. A **good** drawing is one in which no two arcs incident with a common node have a common point; and no two arcs have more than one point in common. A common point of two arcs is a **crossing**. An **optimal** drawing in a given surface is one which exhibits the least possible number of crossings. Optimal drawings are good. This least number is the **crossing number** of the graph for the surface. We denote the crossing number of G for the plane (or sphere) by $\nu(G)$.

Almost all questions that one can ask about crossing numbers remain unsolved. For the **complete graph**, K_n, with n vertices and all $\binom{n}{2}$ possible edges, it has been conjectured [7] that

$$(1) \qquad (?) \qquad \nu(K_n) = \tfrac{1}{4}[\tfrac{1}{2}n][\tfrac{1}{2}(n-1)][\tfrac{1}{2}(n-2)][\tfrac{1}{2}(n-3)],$$

where brackets denote greatest integer not greater than. For $n \leqq 10$, this has been verified [**10**]:

n	2	3	4	5	6	7	8	9	10
$\nu(K_n)$	0	0	0	1	3	9	18	36	60

Blažek and Koman [**1**] and others [e.g., **7**, **12**] have given constructions which show that (1) is an upper bound. Kleitman's result [**15**, and see below] for the complete bipartite graph implies that for n sufficiently large,

$$(2) \qquad \nu(K_n) \geqq \frac{1}{80} n(n-1)(n-2)(n-3).$$

This is a little better than the lower bound given in [**9**]. It is easy to see that $\nu(K_n)/n^4$ is non-decreasing and so tends to a limit (between $\frac{3}{10}$ and $\frac{3}{8}$). A counting argument shows that if (1) is true for n odd, then it is also true for $n+1$. Eggleton and

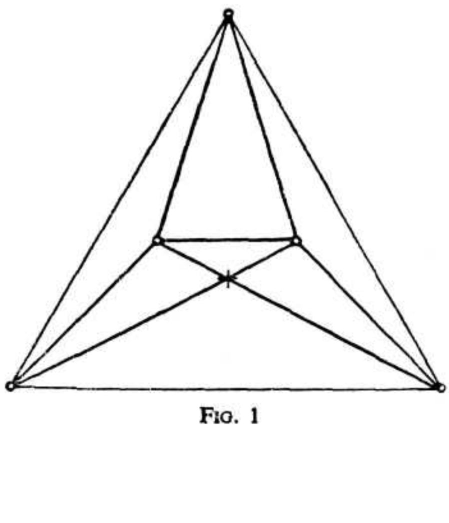


FIG. 1

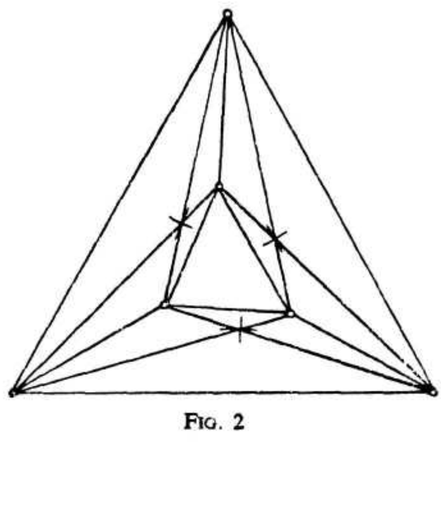


FIG. 2

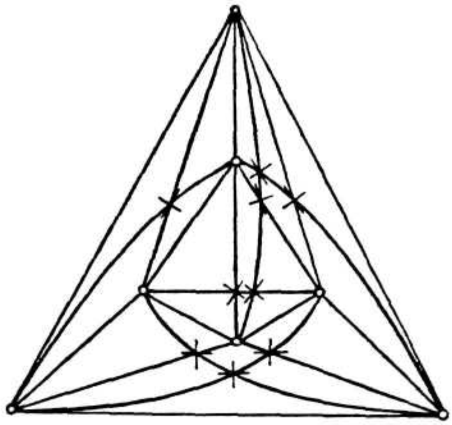


FIG. 3

Guy [3] have also shown that for n odd, $\nu(K_n)$ and $\binom{n}{4}$ have the same parity. Call two drawings **isomorphic** when there is a one-to-one correspondence between the nodes so that if any pair of arcs crosses, the corresponding pair also crosses. Optimal drawings of K_n for $n = 5, 6, 7, 8$ are shown in Figures 1, 2, 3, 4. For $n = 5,6$ these are unique, but for $n = 7$ there are five which are non-isomorphic and for $n = 8$ there are three [10]. For $n = 9$ the number is about 200.

An attempt to put the theory of crossing numbers into algebraic form has been made by Tutte [20].

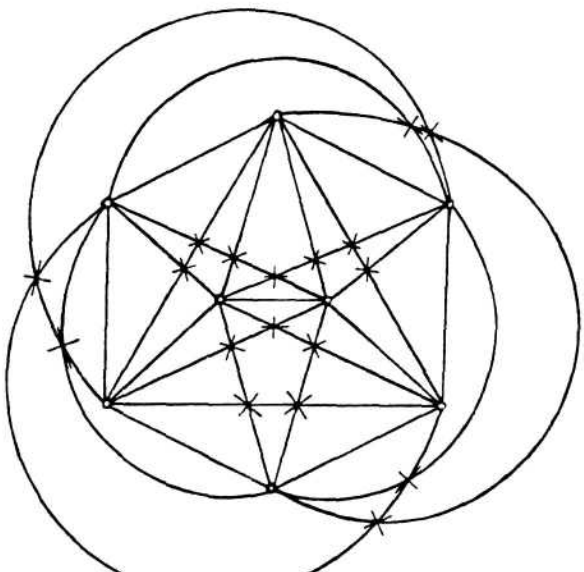


FIG. 4

If the arcs are restricted to be straight line-segments, we have the concept of **rectilinear crossing number**, $\bar{\nu}(G)$, of a graph G. It is clear that $\bar{\nu}(G) \geqq \nu(G)$. A theorem of Fáry [**6**, **19**] may be stated: if a graph can be embedded in the plane, then it can be so drawn using straight line segments. Hence $\nu(G) = 0$ implies $\bar{\nu}(G) = 0$. For $n \leqq 7$ and $n = 9$, $\nu(K_n) = \bar{\nu}(K_n)$. (Figure 3 can be realized with straight line segments.) But Guy [**10**] has confirmed a conjecture of Harary and Hill [**13**] that $\bar{\nu}(K_8) = 19$, in contrast to $\nu(K_8) = 18$. It can also be shown that $\bar{\nu}(K_n) > \nu(K_n)$ for $n \geqq 10$. It is conjectured that $\bar{\nu}(K_{10}) = 63$. Jensen [**14**] and independently Eggleton have shown that

$$\bar{\nu}(K_n) \leqq [(7n^4 - 56n^3 + 128n^2 + 48n[(n-7)/3] + 108)/432] \tag{3}$$

and equality is conjectured. The fact that $\bar{\nu}(K_5) = 1$ gives an immediate proof of Esther Klein's result [**5**] that five points in the plane always include a convex quadrilateral. More generally, there is an exact correspondence between rectilinear crossings and convex quadrilaterals, so the problem of determining the rectilinear crossing number for the complete graph can be restated in the form: what is the least number of convex quadrilaterals determined by n points in the plane? More generally, one can ask for the least number of convex k-gons determined by n points in the plane, for $k > 4$. As before, the ratio of this number to $\binom{n}{k}$ tends to a positive limit as n tends to infinity with k fixed.

The crossing number problem for the **complete bipartite graph**, $K_{m,n}$, on $m + n$ vertices, whose mn edges are just those which join one of the m vertices to one of the n, first appeared as Turán's brick-factory problem. For some years it was thought that Zarankiewicz [**22**] and Urbaník [**21**] had solved this, but a hiatus in the proof was found independently by Ringel and Kainen [see **8**] and the formula

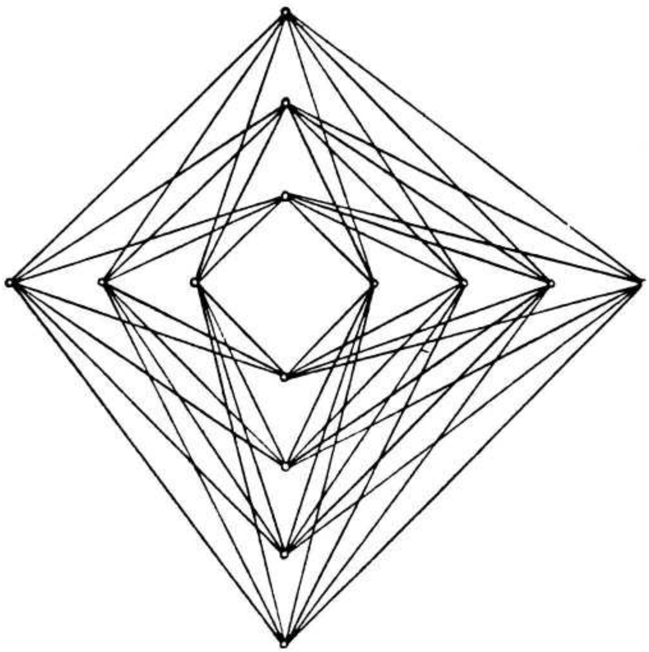
$$(?) \qquad \nu(K_{m,n}) = [\tfrac{1}{2}m][\tfrac{1}{2}(m-1)][\tfrac{1}{2}n][\tfrac{1}{2}(n-1)] \tag{4}$$

is still conjectural. It was established for $\min(m,n) = 3$ by Zarankiewicz and a counting argument again gives the result for each even number if it is known for the preceding odd one. The best result is due to Kleitman [**15**] who established (4) for $\min(m,n) \leqq 6$. The corresponding rectilinear problem may have the same solution (4), since Zarankiewicz's construction uses only straight arcs (Figure 5).

For the 1-skeleton of the **n-cube**, Q_n, whose vertices, the 2^n binary n-tuples, are joined by an edge just if their vectors differ in exactly one component, Eggleton and Guy [**4**] announced that

$$(?) \qquad \nu(Q_n) \leqq \frac{5}{32}4^n - \left[\frac{n^2+1}{2}\right]2^{n-2}, \tag{5}$$

but a gap has been found in the description of the construction, so this must also remain a conjecture. We again conjecture equality in (5).



$$\text{(?)} \qquad \bar{\nu}(K_{7,7}) = \nu(K_{7,7}) = 81.$$

FIG. 5

More generally, let $G(n, k)$ be a graph with n vertices and k edges. Denote by $g(n, k)$ the minimum of $\nu(G)$ taken over all graphs $G(n, k)$. Then we conjecture that

$$(6) \qquad \text{(?)} \qquad \frac{c_1 k^3}{n^2} < g(n, k) < \frac{c_2 k^3}{n^2};$$

in fact, that if $k/n \to \infty$, then $\lim g(n, k)/(k^3/n^2)$ exists. From Euler's theorem, $g(n, 3n-6) = 0$, $g(n, 3n-5) = 1$. The upper bound in (6) is trivial (with $c_2 = 1/8$), for, let l be the least integer with $ln > 2k$ and consider n/l copies of K_l. The lower bound would follow if we could prove that every drawing of a $G(n, k)$ contains an arc with at least $c_3 k^2/n^2$ crossings. In this connexion we can ask the following question: determine or estimate the smallest integer $f(r)$ so that every drawing of a graph $G(n, f(r))$ contains an arc with at least r crossings. Euler's theorem implies that $f(1) = 3n - 5$ and Eggleton and Guy [3] have shown that $f(2) = 4n - 8$ for $n = 6, 7$ and 9, and $4n - 7$ for $n = 8$ or $n \geq 10$. This implies that

$$g(n, k) = k - 3n + 6 \text{ for } 3n - 6 \leq k \leq \min\left(4n - 8, \binom{n}{2}\right),$$

except that $g(7, 20) = 6$ and $g(9, 28) = 8$. But $f(3)$ has not yet been determined.

Another related question is: which graphs $G(n, k)$ have maximal $\nu(G)$ and what is this maximum? We conjecture that the following graph has maximal $\nu(G)$: take l so that

$$\binom{l}{2} \leqq k < \binom{l+1}{2},$$

and the graph consists of K_l with a vertex joined to $k - \binom{l}{2}$ of its vertices (and $n-l-1$ isolated points).

These more general problems can also be posed in the rectilinear case. We can also ask analogous questions for surfaces of higher genus; some results have been obtained for the torus [**11**, **12**], and for the projective plane and Klein bottle [**16**].

We are indebted to R. B. Eggleton for helpful discussions and suggestions, and permission to reproduce his results.

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CLASSROOM NOTES

EDITED BY ROBERT GILMER

Manuscripts for this Department should be sent to Robert Gilmer, Department of Mathematics, Florida State University, Tallahassee, FL 32306. Notes are usually limited to three printed pages.

A PROOF OF UNIQUENESS OF FACTORIZATION IN THE GAUSSIAN INTEGERS

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Let $K(i)$ denote the Gaussian Integers, $K(i) = \{a + bi \mid a, b \text{ are rational integers}\}$. It is well known that $K(i)$ has the unique factorization property. Normally, one shows that $K(i)$ is a Euclidean domain and then uses the fact that every Euclidean domain is a unique factorization domain. We give a direct proof that factorization is unique in $K(i)$ which parallels the proof for the rational integers as given in Niven and Zuckerman (p. 15). We would like to express our appreciation to Professor Ivan Niven for having raised for us the question of the existence of this type of proof.

LEMMA 1. *If z and w are two non-zero complex numbers such that $|w| \leqq |z|$ and $|\arg z - \arg w| < \pi/3$, then $|z - w| < |z|$.*

Proof. The triangle formed by the points 0, z, w in the complex plane has an angle less than $\pi/3$ at the origin, so the side opposite, which is of length $|z - w|$, cannot be the longest side. Further, since $|w| \leqq |z|$, we conclude that $|z - w| < |z|$.

If z is a complex number the *associates* of z are the numbers z, $-z$, iz, $-iz$.

LEMMA 2. *If z and w are complex numbers then there exists an associate w' of w such that $|\arg z - \arg w'| < \pi/3$.*

Proof. The associates of w are at right angles to one another; therefore, there must be one of them in any given sector of angle $2\pi/3$.

If $\alpha \in K(i)$, $\alpha = a + bi$, denote by $N(\alpha)$, the *norm* of α, the non-negative rational integer $a^2 + b^2$. Note that (1) $N(\alpha\beta) = N(\alpha) \cdot N(\beta)$, (2) if ε is a *unit* ($\varepsilon = 1, -1, i$, or $-i$) then $N(\varepsilon) = 1$, and (3) $N(\alpha) = |\alpha|^2$.