Cha in conditions on set mappings and free sets

By PAUL ERDŐS and ANDRÁS HAJNAL in Budapest
and ATTILA MÁTÉ in Szeged

Dedicated to Professor Béla Sz.-Nagy on his 60th birthday

1. Introduction

Given an infinite set E, call a function f a *set mapping* (*on* E) if f maps E into $\mathcal{P}(E)$ (the set of all subsets of E) and is such that $x \notin f(x)$ for any $x \in E$. Call two elements x and y of E *independent* (*with respect to* f) if $x \notin f(y)$ and $y \notin f(x)$. Say that a subset X of E is *free* (*with respect to* f) if any two elements of X are independendent. S. RUZIEVICZ [12] conjectured and A. HAJNAL [5] proved the following: if there is a cardinal $\mu < |E|$ (this latter donetes the cardinality of the set E) such that $|f(x)| < \mu$ holds for any $x \in E$, then there is a free set $X \subseteq E$ of cardinallity $|E|$. A well-known example shows that the weaker assumption $|f(x)| < |E|$ does not even guarantee the existence of an independent couple. Still, one can weaken the cardinality assumption on $f(x)$ while ensuring the existence of a large free set by imposing structural restrictions on the range of f. Before we discuss these restrictions, we need a short review of

Notations and terminology. We work within ZFC, i. e. Zermelo-Fraenkel set theory with the Axiom of Choice. We use the usual notations of set theory, although there is one point to be stressed: $\subset$ *always denotes strict inclusion*, i. e.

$$x \subset y \leftrightarrow x \subseteq y \quad \& \quad x \neq y.$$

As mentioned above, $|x|$ is the cardinallity, and $\mathcal{P}(x)$ is the set of all subsets, of the set x; $\mathrm{dom}(g)$ denotes the domain and $\mathrm{range}(g)$ the range of the function g. The definition of the *full inverse image* $f^{-1}(x)$ of a set X under the set mapping f will be given in Definition 3. 3.

An ordinal is the set of its predecessors, and cardinals are identified with their initial ordinals. A cardinal μ is *inaccessible* if it is a regular cardinal such that for every cardinal $\nu < \mu$ we have $2^\nu < \mu$. Finally, the *weak cardinal power* $\mu^{\underset{\smile}{\nu}}$ is defined as $\bigcup_{\xi < \nu} \mu^{|\xi|}$.

By *Martin's Axiom* we mean, as usual, Proposition **A** in [10, p. 150] (cf. also [16]), i.e. the following proposition:

For any notion C of forcing that satisfies the countable antichain condition (often called countable chain condition), and for any set F of cardinality $<2^{\aleph_0}$ of dense open subsets of C, there exists an F-generic filter.

As is well known, this proposition is consistent with ZFC$+2^{\aleph_0}>\aleph_1$, provided ZFC itself is consistent (see [16]). Furthermore, it is to be noted that Martin's Axiom implies the regularity of $2^{\aleph_0}$ (see [10, Corollary 2 on p. 164]).

The following concept plays a key role in the discussions below.

Definition 1. 1. Given an ordinal η, we say that the set S satisfies the *η-chain condition (with respect to inclusion)* if there is no sequence $\langle s_\alpha : \alpha<\eta\rangle$ of elements of S such that $s_\alpha \subset s_\beta$ whenever $\alpha<\beta<\eta$.

2. Assumptions on the set mapping and results

Throughout this paper $\varkappa$ will denote a regular cardinal and we shall assume that $E=\varkappa$; this amounts to the same as assuming that the cardinality of E is $\varkappa$. We shall consider a subset S of $\mathscr{P}(\varkappa)$ satisfying one of the two conditions below. These are the conditions we shall usually impose upon the set mapping f with $S=$ range (f).

(A) Every element of S has cardinality $<\varkappa$, and for each subset F of $\varkappa$, the set $\{s\cap F : s\in S\}$ satisfies the $\varkappa$-chain condition (see Definition 1. 1).

The other condition is apparently weaker:

(B) Every element of S has cardinality $<\varkappa$, and, moreover, for any $\tau<\varkappa$ and any decomposition $\varkappa=\bigcup_{\alpha<\tau} E_\alpha$ of $\varkappa$ into mutually disjoint sets E_α of cardinality $\varkappa$, there is an ordinal $\gamma<\tau$ and a set $F\subseteqq E_\gamma$ of cardinality $\varkappa$ such that the set $\{s\cap F : s\in S\}$ satisfies the $\varkappa$-chain condition.

As we mentioned just before, it is clear that (A) implies (B). But the converse is not true:

Lemma 2. 1. (B) *does not imply* (A).

Proof. Split $\varkappa$ into two disjoint sets, each of cardinality $\varkappa$: $X=\{\xi_\alpha : \alpha<\varkappa\}$ and $Y=\{\eta_\alpha : \alpha<\varkappa\}$. Take

$$S=\{\{\xi_\alpha\}\cup\{\eta_\beta : \beta<\alpha\} : \alpha<\varkappa\}.$$

Then it is easy to check that (B) holds but (A) does not. In fact, as for (A), the set $\{s\cap Y : s\in S\}$ does not satisfy the $\varkappa$-chain condition. As for (B), take a sequence

$\langle E_\alpha : \alpha < \tau \rangle$ of sets as described, and take a $\gamma < \tau$ such that $|E_\gamma \cap X| = \varkappa$; then (B) is fulfilled with $F = E_\gamma \cap X$. The proof is complete.

The following condition is an alternative form of (B). The slight change is that here $\bigcup_{\alpha<\tau} E_\alpha$ need only be "almost equal" to $\varkappa$, and we do not require that the sets E_α have cardinality $\varkappa$:

(B′) Every element of S has cardinality $<\varkappa$, and, moreover, for any $\tau < \varkappa$ and any sequence of mutually disjoint subsets E_α, $\alpha < \tau$, of $\varkappa$ such that

$$|\varkappa - \bigcup_{\alpha<\tau} E_\alpha| < \varkappa,$$

there is an ordinal $\gamma < \tau$ and a set $F \subseteq E_\gamma$ of cardinality $\varkappa$ such that the set $\{s \cap F : s \in S\}$ satisfies the $\varkappa$-chain condition.

Next we prove

Lemma 2. 2. (B) *and* (B′) *are equivalent.*

Proof. It is clear that (B′) implies (B). We show that the converse is also true. To this end assume that (B) holds and, furthermore, let $\langle E_\alpha : \alpha < \tau \rangle$ be such a sequence as is described in (B′). We may suppose that all the sets E_α have cardinality $\varkappa$, as those of cardinality $<\varkappa$ can simply be omitted. Assume first that

$$(*) \qquad |\varkappa - \bigcup_{\alpha<\tau} E_\alpha| \leqq |\tau|$$

holds. Take mutually disjoint sets E'_α such that $\varkappa = \bigcup_{\alpha<\tau} E'_\alpha$ and such that $E_\alpha \subseteq E'_\alpha$ and $|E'_\alpha - E_\alpha| \leqq 1$ hold for any $\alpha < \tau$. By (B) there is a $\gamma < \varkappa$ and an $F' \subseteq E'_\gamma$ of cardinality $\varkappa$ such that $\{s \cap F' : s \in S\}$ satisfies the $\varkappa$-chain condition. It is then clear that the conclusion of (B′) holds with $F = F' \cap E_\gamma$. This establishes the desired result in case $(*)$ holds. If this is not the case, then start with splitting an arbitrary one of the sets E_α into $|\varkappa - \bigcup_{\alpha<\tau} E_\alpha|$ mutually disjoint sets of cardinality $\varkappa$; then $(*)$ will hold, and the argument above can be used. The proof is complete.

We shall prove that (B) implies the existence of a countably infinite free set. This has essentially been proved by G. FODOR and A. MÁTÉ [3, Theorem 2 on p. 4], although under slightly stronger assumptions (condition (B) of that paper requires somewhat more than condition (B) of ours). If $\varkappa$ is inaccessible and weakly compact, then (B) implies the existence of a free set of cardinality $\varkappa$. (A cardinal is *weakly compact* if it is not *strongly incompact*; for the definition see [6, p. 312] or [14, Definition 1. 11 on p. 61]; cf. also Theorem 1.13 in [14, p. 62].) Not even (A) implies, however, the existence of a free set of cardinality $\varkappa$ in the following cases (in cases (i) and (ii) we actually prove somewhat more): (i) for some cardinal λ, $\varkappa = \lambda^+ = 2^\lambda$; (ii) $\varkappa = 2^{\aleph_0}$ and Martin's Axiom holds (see at the end of the Introduction); and (iii) there exists a Souslin $\varkappa$-tree (the definition of Souslin tree is given in the next section).

A theorem of R. B. JENSEN [8, p. 292] says that, assuming the Axiom of Constructibility (see [4]), there exists a Souslin $\varkappa$-tree if and only if $\varkappa$ is not weakly compact. So, this last result in case (iii) and the result mentioned just before imply that, under the assumption of the Axiom of Constructibility, (A) (or (B)) implies the existence of a free set of cardinality $\varkappa$ if and only if $\varkappa$ is weakly compact (in the constructible universe every weakly compact cardinal is inaccessible — see [6, Theorems 2 and 3 on pp. 315—316]). Finally we mention that the results and problems of this paper are related to Problem 73 in [1, p. 46]. P. ERDŐS and A. HAJNAL have recently solved this problem affirmatively. Their proof has not yet been published, only an announcement was made in [2, p. 16].

3. Existence of "large" free sets

The aim of this section is to establish those of our results which confirm that condition (B) described in the preceding section implies the existence of large free sets. The basic tool of these proofs is trees, so here we recall a few concepts concerning them (we refer to [7] as an excellent expository paper on trees; references to other sources are given there).

A partially ordered set $\langle T, \prec \rangle$ is called a *tree* if for any $x \in T$ the set of predecessors of x, $\mathrm{pr}(x)=\mathrm{pr}(x, \langle T, \prec \rangle)=\{y \in T: y \prec x\}$ is wellordered by $\prec$ (we assume that $\prec$ is irreflexive). We sometimes write T instead of $\langle T, \prec \rangle$. A subset linearly ordered by $\prec$ of T is called a *chain (of* or *in T)*, a maximal chain a *branch*, and, furthermore, a (not necessarily proper) lower segment of a branch is said to be a *path*. An *antichain* is a set of elements mutually incomparable in $\prec$ of T. For any $x \in T$, $\mathrm{o}(x)=\mathrm{o}(x, \langle T, \prec \rangle)$ denotes the order type of $\mathrm{pr}(x)$, and for any ordinal α the set $\{x \in T: \mathrm{o}(x)=\alpha\}$ is called the αth *level* of T. The *length* of a tree T is $\bigcup \{\alpha+1:$ the αth level of T is not empty$\}$. An *α-tree* is a tree with length α.

Assume μ is a cardinal. An *Aronszajn μ-tree* is a μ-tree such that each chain and each level has cardinality $<\mu$. A *Souslin μ-tree* is a μ-tree such that each chain and antichain has cardinality $<\mu$. μ is said to have the *Tree Property* if there exists no Aronszajn μ-tree. It is well known that, assuming μ is inaccessible, μ has the tree property if and only if μ is weakly compact (for a proof, see e.g. [14, Theorem 1. 13 on p. 62]). We need some further notions:

Definition 3. 1. A tree $\langle T', \prec' \rangle$ is called a *loose end-extension* of another one, $\langle T, \prec \rangle$, if $T \subseteqq T'$, the restriction of $\prec'$ to T equals $\prec$, and, furthermore, every branch of T' includes a branch of T as a lower segment.

Assume now that we are given a regular cardinal $\varkappa$ and a set mapping f on $\varkappa$. The following concepts depend on and f, although the terms introduced will not stress this explicitly:

Definition 3. 2. A tree $\langle T, \prec \rangle$ such that $T \subseteq \varkappa$ is called *free* if each of its branches is a free set (with respect to f).

Now, for a tree $\langle T, \prec \rangle$ and for a path p of T denote by $\mathrm{ims}(p, T)$ the set of immediate successors in $\prec$ of p. (Note that the empty set is also a path.)

Definition 3. 3. A free tree T is called *regular* if for every nonmaximal path p of T we have $|\mathrm{ims}(p, T)| < \varkappa$ and

$$\bigcap \{f^{-1}(\{\xi\}) : \xi \in \mathrm{ims}(p, T)\} = 0,$$

where

$$f^{-1}(X) \stackrel{\text{def}}{=} \{\xi < \varkappa : X \cap f(\xi) \neq 0\} \qquad (X \subseteq \varkappa).$$

An important consequence of this definition is given by the next lemma. (We need this lemma only for $p=0$, but it does not require any extra effort to establish it for any p.)

Lemma 3. 4. *Assume T is a regular free tree and p is a path in T. Then, with b running over all branches of T, we have*

$$\bigcap \{f^{-1}(b-p) : \ p \subseteq b\} = 0.$$

Proof. Given any $\xi < \varkappa$, we are going to show that ξ does not belong to the above intersection. To this end, consider those path p' in T for which $p \subseteq p'$ and

$$\xi \notin f^{-1}(p'-p).$$

Note that p itself is such a path, and, by Zorn's lemma, there is a path that is maximal among those having this property. Assume that p' is already such a maximal one. If p' is a branch, then we are ready. If not, then let $\eta \in \mathrm{ims}(p', T)$ be such that $\xi \notin f^{-1}(\{\eta\})$ (there is such an η by the regularity of T). Then

$$\xi \notin f^{-1}(p \cup \{\eta\} - p),$$

which contradicts the maximality of p'. The proof is complete.

Say that a regular free tree T is less than another one, T', if T' is a loose end-extension of T. It follows easily from Zorn's lemma that, under this partial ordering, there is a maximal regular free tree (note that the empty tree is a regular free tree, and so is the union of a linearly ordered set of regular free trees). Our key result in this section says that a maximal regular free tree cannot be too small provided condition (B) (see the preceding section) holds for $S = \mathrm{range}\,(f)$:

Theorem 3. 5. *Assume condition* (B) *holds for $S = \mathrm{range}\,(f)$. Let $\langle T, \prec \rangle$ be a regular free tree having less than $\varkappa$ branches and such that $|T| < \varkappa$. Then T has a proper loose end-extension that is also a regular free tree.*

For the proof we need a simple lemma, which occurs in [3] and [11]. It is important for this lemma that we assumed $\varkappa$ to be a regular cardinal.

Lemma 3. 6. *Let H be a set such that each of its elements has cardinality $<\varkappa$ and such that $|\bigcup H|\geqq\varkappa$, and assume that H satisfies the $\varkappa$-chain condition (with respect to inclusion). Then there is a subset X of cardinality $<\varkappa$ of $\bigcup H$ such that $X\subsetneqq h$ holds for any $h\in H$.*

Proof. H can be considered as a set partially ordered by inclusion. By a well-known theorem of F. Hausdorff, there is a maximal linearly ordered subset of H, say K. By another of his theorems, there is a wellordered subset M of K that is cofinal to K. As H satisfies the $\varkappa$-chain condition, we must have $|M|<\varkappa$. Now take an arbitrary element t of $\bigcup H-\bigcup M$, and put $X=\bigcup M\cup\{t\}$. It is clear that this set satisfies the requirements of the lemma.

Now we establish the announced theorem.

Proof of Theorem 3. 5. Let $\langle b_\alpha:\alpha<\tau\rangle$ $(\tau<\varkappa)$ be an enumeration of the branches in T, and put

$$G_\alpha=\varkappa-f^{-1}(b_\alpha)$$

and

$$E_\alpha=G_\alpha-M-\bigcup\nolimits_{\beta<\alpha}G_\beta \qquad (\alpha<\tau),$$

where

$$M=T\cup\bigcup\{f(\xi):\xi\in T\}.$$

It follows from Lemma 3.4 with $p=0$ that $\bigcup_{\alpha<\tau}E_\alpha=\varkappa-M$. It is clear that here $|M|<\varkappa$, as we assumed both $|T|<\varkappa$ and $|f(\xi)|<\varkappa$ for any $\xi<\varkappa$ (this latter as a part of (B)). So, in view of (B′) (which holds by its equivalence to (B), as established in Lemma 2. 2) we can see that there exists an ordinal $\gamma<\tau$ and a set $F\subseteqq E_\gamma$ of cardinality $\varkappa$ such that

$$\{f(\xi)\cap F:\xi<\varkappa\}$$

satisfies the $\varkappa$-chain condition. So, by the lemma just proved, there is a set $X\subseteqq F$ of cardinality $<\varkappa$ such that $X\subsetneqq f(\xi)\cap F$ holds for any $\xi<\varkappa$, i.e. such that

$$\bigcap\{f^{-1}(\{\delta\}):\delta\in X\}=0.$$

Make the set $T'=T\cup X$ a tree by stipulating that T' is a loose end-extension of T such that $X=\operatorname{ims}(b_\gamma, T')$. It is clear that these stipulations define T' as a tree unambiguously, and, moreover, that T' is a regular free tree. This completes the proof.

As we mentioned above, there exists a maximal regular free tree. By the theorem just proved, such a tree either must have cardinality $\varkappa$ or it must have at least $\varkappa$ branches. In either case, it cannot have only very short branches; as a branch is a free set, we can thus establish the existence of a large free set. We first prove

Theorem 3. 7. *Assume that $\mu<\varkappa$ is a cardinal such that $\nu^{\underline{\mu}}<\varkappa$ holds for any cardinal $\nu<\varkappa$. Then any maximal regular free tree has a branch of cardinality $\geqq\mu$.*

Proof. Take a maximal regular free tree T, and assume that each branch of T has cardinality $<\mu$. Then, in view of Theorem 3. 5., T must have at least $\varkappa$ branches (indeed, if T has less than $\varkappa$ branches, then we also have: $|T|\leqq$the sum of the cardinalities of all branches of $T<\varkappa$). Let $\eta\leqq\mu$ be the least ordinal such that the tree $T|\eta$ has at least $\varkappa$ branches ($T|\eta$ is, by definition, obtained from T by omitting each of its elements in or above the ηth level). Then each level in $T|\eta$ has cardinality $<\varkappa$. In fact, let $\alpha<\eta$. Then $T|\alpha$ must have less than $\varkappa$ branches by the minimality of η. Since for any path p of T we have $|\text{ims}\,(p, T)|<\varkappa$ (this is stipulated in the definition of a regular free tree), we can conclude from here by the regularity of $\varkappa$ that the αth level in T has cardinality $<\varkappa$.

So there is a cardinal $\nu<\varkappa$ such that each level in $T|\eta$ has cardinality $\leqq\nu$. Therefore, noting that each branch in $T|\eta$ has cardinality $<\mu$, the number of branches in $T|\eta$ is at most

$$\bigcup\{\nu^{|\xi|}:\xi\leqq\eta \quad \& \quad \xi<\mu\}\leqq\nu^{\underline{\mu}}<\varkappa,$$

which is a contradiction, proving the theorem.

From this theorem we can immediately conclude

Theorem 3. 8. *Assume that $\varkappa$ is an infinite regular cardinal and condition* (B) *holds with $S=\text{range}(f)$. Then*

(i) *there exists a free set of cardinality $\aleph_0$;*

(ii) *if μ is a cardinal $<\varkappa$ such that for every cardinal $\nu<\varkappa$ we have $\nu^{\underline{\mu}}<\varkappa$, then there exists a free set of cardinality μ;*

(iii) *if $\varkappa$ is inaccessible and weakly compact, then there exists a free set of cardinality $\varkappa$.*

Proof. (ii) directly follows from the preceding theorem. We establish (iii). As $\varkappa$ is inaccessible in this case, the assumptions of the preceding theorem hold for any cardinal $\mu<\varkappa$; so a maximal regular free tree T must have length $\geqq\varkappa$. As $|\text{ims}\,(p, T)|<\varkappa$ holds for any path p in T (cf. Definition 3. 3.), it follows from the inaccessibility of $\varkappa$ that for any $\alpha<\varkappa$ the αth level in T has cardinality $<\varkappa$. As $\varkappa$ has the tree property (cf. e.g. [14, Theorem 1. 13 on *p.* 62]; note that although not mentioned there, this is also true in case $\varkappa=\aleph_0$ — see [9]), T must have a branch of cardinality $\varkappa$. This being a free set, (iii) is proved. Finally, in case $\varkappa>\aleph_0$ (i) follows from (ii), and in case $\varkappa=\aleph_0$ it follows from (iii) (there is no harm in considering $\aleph_0$ inaccessible). The proof is complete.

4. Nonexistence of "too large" free sets

In many cases we can prove that condition (B) (and even the stronger condition (A)) does not ensure the existence of a free set of cardinality $\varkappa$. But we cannot prove even in the simplest case that there is a cardinal $\mu<\varkappa$ such that (B) does not imply the existence of a free set of cardinality μ. We start with the simple

Theorem 4. 1. *Assume that $\varkappa$ is a regular cardinal such that there exists a Souslin $\varkappa$-tree. Then condition* (A) *with $S=\mathrm{range}(f)$ does not imply the existence of a free set of cardinality $\varkappa$.*

Proof. Assume $\langle\varkappa, \precsim\rangle$ is a Souslin $\varkappa$-tree, and for any $\xi<\varkappa$ put

$$f(\xi)=\{\alpha<\varkappa\colon \alpha\prec\xi\}(=\mathrm{pr}(\xi)).$$

A subset of $\varkappa$ is free with respect to this f exactly if it is an antichain in $\langle\varkappa, \prec\rangle$; so there is no free set of cardinality $\varkappa$. We are going to show that $S=\mathrm{range}(f)$ satisfies condition (A). Assume the contrary, and let F be a subset of $\varkappa$ and $\langle\xi_\alpha\colon\alpha<\varkappa\rangle$ a sequence of ordinals $<\varkappa$ such that

$$f(\xi_\alpha)\cap F\subset f(\xi_\beta)\cap F$$

holds for any $\alpha<\beta<\varkappa$ ($\subset$ indicates strict inclusion). Then it is easy to see that

$$\bigcup_{\alpha<\varkappa}(f(\xi_\alpha)\cap F)$$

is a chain of cardinality $\varkappa$ of $\langle\varkappa, \prec\rangle$. This contradicts the fact that the latter is a Souslin $\varkappa$-tree. The proof is complete.

Next we show that, under the assumption of the Generalized Continuum Hypothesis, condition (A) does not guarantee the existence of a free set of cardinality $\varkappa$ if $\varkappa$ is a successor cardinal. Actually, we prove more:

Theorem 4. 2. *Assume $\varkappa$ and λ are infinite cardinals such that $\varkappa=2^\lambda$ and either* (i) *$\varkappa=\lambda^+$, or* (ii) *$\lambda=\aleph_0$ and Martin's Axiom holds. Then there is a set $S\subseteqq\mathcal{P}(\varkappa)$ of cardinality $\varkappa$ satisfying condition* (A) *of Section* 2 *such that for any set $S'\subseteqq S$ of cardinality $\varkappa$ we have $|\varkappa-\bigcup S'|<\lambda$.*

An obvious consequence of this is

Corollary 4. 3. *Assume that either* (i) *or* (ii) *of the preceding theorem holds. Then condition* (A) *with $S=\mathrm{range}\,(f)$ does not imply the existence of a free set of cardinality $\varkappa$.*

For the proof of the above theorem we need the following

Lemma 4. 4. *Assume that either* (i) *or* (ii) *of the preceding theorem holds. Let $\eta<\varkappa$ be an ordinal and $\langle A_\xi\colon\xi<\eta\rangle$ a sequence of sets of cardinality λ. Then there is* *a set* $B_\eta \subseteq \bigcup_{\xi<\eta} A_\xi$ *such that* B_η *meets each* A_ξ, $\xi<\eta$, *but does not include any of them.*

Proof. *Ad* (i). This case, due to F. BERNSTEIN, is well known and simple. We may assume that $\eta \leqq \lambda$; indeed, if this is not the case, the we can rearrange the sequence $\langle A_\xi : \xi<\eta \rangle$. Now define x_ξ and y_ξ by transfinite recursion so that $x_\xi \neq y_\xi$ and

$$x_\xi, y_\xi \in A_\xi - \{x_\alpha, y_\alpha \colon \alpha < \xi\} \qquad (\xi < \eta),$$

and take $B_\eta = \{x_\xi : \xi<\eta\}$.

Ad (ii). Put

$$C = H(\bigcup_{\xi<\eta} A_\xi, 2),$$

that is, let C be the set of all functions with values 0 or 1 the domains of which are finite subsets of $\bigcup_{\xi<\eta} A_\xi$. Consider C as partially ordered by inclusion; then, as is well known, C is a notion of forcing satisfying the countable antichain condition (often called countable chain condition; cf. [13, Lemma 10. 3 on p. 372] — Shoenfield's terminology differs from ours, so that in order to agree with it we should order C by reverse inclusion). The set

$$D_\xi = \{p \in C \colon \exists x, y \in A_\xi [x, y \in \operatorname{dom}(p) \;\&\; p(x)=0 \;\&\; p(y)=1]\}$$

is dense open for any $\xi<\eta$; so, by Martin's Axiom, there exists a $\{D_\xi : \xi<\eta\}$-generic filter G. The set

$$B_\eta = \{x \in \operatorname{dom}(\bigcup G) \colon (\bigcup G)(x) = 1\}$$

satisfies our requirements (note that $\bigcup G$ is a function the domain of which is included in $\bigcup_{\xi<\eta} A_\xi$). The lemma is proved.

Proof of Theorem 4. 2. We deal with cases (i) and (ii) simultaneously. Let $\langle A_\xi : \xi<\varkappa \rangle$ be an enumeration of all subsets of cardinality λ of $\varkappa$, and for each $\eta<\varkappa$ define B_η as described in the lemma just proved. Put $S=\{B_\eta : \eta<\varkappa\}$. We show that S satisfies (A). It is clear that each element of S has cardinality $<\varkappa$; assume that the rest of (A) does not hold, and let F be a subset of $\varkappa$ such that $\{B_\eta \cap F : \eta<\varkappa\}$ does not satisfy the $\varkappa$-chain condition. Then it is easy to see that there exists a set $I \subseteq \varkappa$ of cardinality $\varkappa$ such that

$$B_\alpha \cap F \subset B_\beta \cap F$$

holds for any $\alpha, \beta \in I$ with $\alpha<\beta$. Then for any $\alpha \in I$ with $|\alpha \cap I| \geqq \lambda$ we obviously have $|B_\alpha \cap F| \geqq \lambda$; so, for some $\xi<\varkappa$, we have $A_\xi \subseteq B_\alpha \cap F$. Pick an $\eta \in I$ with $\eta>\alpha, \xi$. Then $A_\xi \not\subseteq B_\eta$, which contradicts the assumption $A_\xi \subseteq B_\alpha \cap F \subset B_\eta \cap F$. Thus we have shown that S satisfies (A).

Now take any subset S' of cardinality $\varkappa$ of S. We are about to show that $|\varkappa - \bigcup S'| < \lambda$. Assume the contrary; then there exists a $\xi<\varkappa$ such that $A_\xi \subseteq \varkappa - \bigcup S'$.

Take a $B_\eta \in S'$ with $\eta > \xi$. Then $A_\xi \cap B_\eta \neq 0$, which is a contradiction. The theorem is proved.

We conclude this paper by pointing out a few problems. As mentioned in Section 2, our discussion is complete as far as the existence of free sets of cardinality $\varkappa$ is concerned in case we assume the Axiom of Constructibility. But without such an assumption many problems remain open. The simples-sounding one is

Problem 1. Assume $\varkappa = \aleph_1$ and $2^{\aleph_0} > \aleph_1$. Does then (A) or (B) with $S = \operatorname{range}(f)$ imply the existence of a free set of cardinality $\varkappa$?

One may try to solve this problem even under the assumption of Martin's Axiom; the answer is unknown to us. Nothing is known about the nonexistence of free sets of a cardinality less than $\varkappa$. E.g. one might ask

Problem 2. Assume $\varkappa = 2^{\aleph_0} > \aleph_1$, and assume that Martin's Axiom holds. Does then (A) or (B) with $S = \operatorname{range}(f)$ imply the existence of a free set of an uncountable cardinality?

It is a well-known result of R. M. Solovay that it is consistent relatively to the existence of a measurable cardinal that $2^{\aleph_0}$ be real-valued measurable (see [15, Theorem 2 and Proposition 1 on pp. 398—399]; cf. also the remark on p. 67 in [14]). The fact that a real-valued measurable cardinal always has the tree property (see [14, Theorem 1. 16 on p. 67]) makes the following problem interesting:

Problem 3. Assume that $\varkappa = 2^{\aleph_0}$, and, furthermore, that $\varkappa$ is real-valued measurable. Does then (A) or (B) with $S = \operatorname{range}(f)$ imply the existence of a free set of cardinality $\varkappa$?

Added in proof. When the paper had already been in print, we obtained the following results, which go a long way in settling Problems 1—3. For an ordinal η, denote by (A_η) the assertion that for the set mapping $f\colon \varkappa \to \mathcal{P}(\varkappa)$ we have $|f(\alpha)| < \varkappa$ whenever $\alpha < \varkappa$, and, for each subset F of $\varkappa$, the set $\{f(\alpha) \cap F\colon \alpha < \varkappa\}$ satisfies the η-chain condition. Then the following propositions are consistent relatively to ZFC: (i) $2^{\aleph_0} = \varkappa =$ anything reasonable, (A_{ω_1}) holds for f, and there is no free set of cardinality $\aleph_1$; (ii) $2^{\aleph_0} = \varkappa$ is real-valued measurable, (A_{ω_1}) holds for f, and there is no free set of cardinality $\aleph_1$. The following propositions are theorems of ZFC: (iii) If $\varkappa = 2^{\aleph_0} = \aleph_2$ and Martin's Axiom holds, then there is an f satisfying $(A_{\omega+1})$ (in fact, $\forall \xi, \eta[\xi < \eta < \omega_2 \to |f(\xi) \cap f(\eta)| < \aleph_0]$) such that there is no free set of cardinality $\aleph_2$; (iv) If $\varkappa = \lambda^+ = 2^\lambda$ and $\operatorname{cf}(\lambda) > \omega$, then there is an f satisfying $(A_\varkappa)$ such that there is no free set of order type $\lambda + \omega$; (v) If $\varkappa = \lambda^+ = 2^\lambda$ and λ is regular, then there is an f such that $(A_{\lambda+1})$ holds (in fact, $\forall \xi, \eta[\xi < \eta < \varkappa \to |f(\xi) \cap f(\eta)| < \lambda)$ and there is no free set of cardinality $\varkappa$.