On Ramsey Like Theorems, Problems and Results

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The well known arrow symbol

$$n \to (k_1, \ldots, k_\ell)^r_\ell \tag{1}$$

means that if we split the r-tuples of a set, $S, |S| = n$ into ℓ classes, than for some $i, 1 \le i \le \ell$ there is a subset $S_i \subseteq S, |S_i| \ge k_i$ all of whose r-tuples are in the i-th class. Denote by $F_r^{(\ell)}(k_1, \ldots, k_\ell)$ the smallest n for which (1) holds. The determination of $F_r^{(\ell)}(k_1, \ldots, k_\ell)$ is probably hopeless and may not be a "reasonable" problem, just as the determination of the n-th prime by a simple explicit formula is not reasonable. Very few exact results are known and all those are for $r = 2$ ($r = 1$ is trivial) [1]. It is perhaps not quite hopeless to try to get asymptotic results but even here our knowledge is meager, or to be more precise non-existent.

Here is a short outline of some of the known results [1],[2] ($c, c_1, \ldots$ will denote positive absolute constants not necessarily the same if they occur in different formulas).

$$F_2^{(2)}(u, v) \le \binom{u+v-2}{u-1} \tag{2}$$

$$2^{\frac{n}{2}} < F_2^{(2)}(n, n) < \frac{c \log \log n}{\log n} \frac{4^n}{n^2} \tag{3}$$

$$c_1 n^2/(\log n)^2 < F_2^{(2)}(3, n) < c_2 n^2 \log \log n/ \log n \tag{4}$$

Put $\exp_r Z = \exp(\exp_{r-1} Z), \exp_1 Z = e^Z$.

$$F_r^{(2)}(n,n) < \exp_{r-1} c_r n. \tag{5}$$

Very likely

$$F_r^{(2)}(n,n) > \exp_{r-1} c_r' n. \tag{6}$$

To prove (6) it would suffice to show that

$$F_3^{(2)}(n,n) > \exp_2 cn. \tag{7}$$

But (7) has so far resisted all attempts. $F_3^{(2)}(n,n) > e^{cn^2}$ can be proved easily by the so-called probabilistic method, but this method cannot give a better result here. Finally $F_2^{(\ell)}(3,\ldots,3) < [e \cdot \ell!] + 1$, but perhaps this bound can be replaced by c^ℓ.
In view of this unsatisfactory state of affairs we will study a modified problem (at least for $\ell = 2$) which may shed some light on these questions.

$$n \to \left(k, \begin{bmatrix} u \\ v \end{bmatrix}\right)^r \tag{8}$$

denotes the truth of the following statement: Split the r-tuples of a set of n elements into two classes. Then either there are k elements all of whose r-tuples are in class I or there is a set of u elements which contains at least v r-tuples of class II. We studied this symbol in our triple paper [4] with Rado if n is an infinite cardinal, but it seems to us that (8) leads to interesting and deep questions for finite n too and we hope to convince the reader of this. $f_r(n;u,v)$ denotes the largest value of k for which (8) holds. We will try to study $f_r(n;u,v)$ as v increases, u is fixed and n is very large; unfortunately we will not be very successful but the problem is really very difficult and we will make some plausible conjectures. The statement $n \to \left(k, \begin{bmatrix} u \\ \binom{u}{r} \end{bmatrix}\right)^r$ is of course the same as the old symbol $n \to (k,u)^r$. We will almost entirely restrict ourselves to finite problems, but will include a short discussion of transfinite tournaments. Often we will not give full details of the proofs, but will give a short outline with appropriate references to the literature.

Our principal conjecture states that as v increases from 1 to $\binom{u}{r}$ at first $f_r(n;u,v)$ grows like a power of n (as v increases), then at a well defined $h_1^{(r)}(u), f_r(n;u,v)$ grows like a power of $\log n$ then at $h_2^{(r)}(u), f_r(n;u,v)$ grows like a power of $\log\log n$ etc. and finally $f_r(n;u,h_{r-2}^{(r)}(u))$ grows like a power of $\log_{r-2} n$ ($\log_t n$ is the t-fold iterated logarithm of n), $h^{(r)}_{r-2}(u) < \binom{u}{r}$; and finally $f_r(n; u, \binom{u}{r})$ also grows like a power of $\log_{r-2} n$. We think we know the value of $h^{(r)}_1(u)$ and can certainly determine it easily for any fixed u. We are much less certain about $h^{(r)}_2(u)$, $h^{(r)}_3(u)$ etc. The first nontrivial case is $u = r + 1$. Trivially $f_r(n; u, 1) = n$ for every $u \geq r$ and we conjecture that $h^{(r)}_i(r + 1) = i + 2$ but can prove this only for $i = 1$.

First we will deal with the case $r = 2$ and $r > 2$ if n is small (the trivial zone). Here no great mysteries remain, but the exact order of magnitude of $f_2(n; u, v)$ is not in general known. Than we discuss the case $r = 3$ in some detail, and finally we give a short resumé of our meagre knowledge for $r > 3$. We discuss the case $r = 3, u = 4$ in some detail, also because of its connection with tournaments.

I. r=2

Assume first $2v \leq u$. Then trivially

$$f_2(n; u, v) = n - v + 1. \tag{9}$$

(9) is a really trivial since $2v \leq u$ implies that if no set of u vertices spans v edges of the second class then there are at most $v - 1$ edges of the second class and (9) follows. Further trivially

$$f_2(2n; 2v, v + 1) = n. \tag{10}$$

and all values of $f_2(n; u, v)$ can easily be determined for $v < u$, we leave the simple (but sometimes unpleasant) details to the reader. Here we do not really use $r = 2$, in the general case the values of $f_r(n; u, v)$ are trivial for $v < \left[\frac{u+r-2}{r-1}\right]$ and again we leave the determination of $f_r(n; u, v)$ in the "trivial zone" to the reader (for $r > 2$ the details will be even more unpleasant than for $r = 2$). The first non-trivial result states

Theorem 1. *Let $n > n_0(u)$, then*
$f_2(n; u, v) < n^{1-\varepsilon_u}, f_r\left(n; u, \left[\frac{u+r-2}{r-1}\right]\right) < n^{1-\varepsilon^{(r)}_u}.$

Both inequalities of Theorem 1 easily follow by the so-called probability method - in fact the first one is contained in [8] and the second one follows by the method used in [8] and [9]. We only state the idea of the method the interested reader can easily reconstruct the details by consulting the two papers quoted above.

Consider the complete r-graph of n vertices. Choose in all possible ways $\ell = [n^{1+n}]$ of its r-tuples where $n = n(u)$ is sufficiently small. Thus we get $\left[\begin{matrix}\binom{n}{r}\\ \ell\end{matrix}\right]$ r-graphs on n vertices. A simple computation shows that if we neglect $\sigma\left(\left[\begin{matrix}\binom{n}{r}\\ \ell\end{matrix}\right]\right)$ of these r-graphs then the remaining r-graphs have the following two properties: There is an ε_u so that every set of $[n^{1-\varepsilon_u}]$ vertices contains at least $t = t(n)$ of our chosen r-tuples but that there are only $\sigma(t)$ u-tuples which contain at least $\left[\frac{u+r-2}{r-1}\right]$ of our r-tuples. Omitting all the r-tuples from these "bad" u-tuples, we obtain an r-graphs on n vertices every u-tuple of which has fewer than $\left[\frac{u+r-2}{r-1}\right]$ r-tuples, but every set of $[n^{1-\varepsilon_v}]$ vertices contains at least one of our r-tuples. This completes the outline of the proof of Theorem 1. This simple minded method is often surprisingly successful (for further applications see the forthcoming book of P. Erdős and J. Spencer "On applications of probability methods to combinatorial problems").

It is easy to see that

$$f_r\left(n; u, \left[\frac{u+r-2}{r-1}\right]\right) > n^c$$

for some $c = c(u, r)$ but it would be of interest to try to get as good inequalities for $f_r\left(n; u, \left[\frac{n+r-2}{r-1}\right]\right)$ as possible. We will only discuss the case $r = 2$. The case $u = 3$ has already been stated in (4), (4) can be written as

$$c_1 n^{1/2}\left(\frac{\log\log n}{\log n}\right)^{1/2} < f_2(n; 3, 3) < c_2 n^{1/2}/\log n.$$

We would guess that

$$c_3^{n^{1-\frac{1}{u}}}/(\log n)^{c_4} < f_2(n; 2s+1, 2s+1) < c_3^{n^{1-\frac{1}{u}}}/(\log n)^{c_6} \tag{11}$$

for even u we do not hazard a guess.

It is likely that

$$\lim_{n\Rightarrow\infty} \log f_2(n; u, v)/\log n = c_{u,v}$$

exists for every u and v, as stated previously $c_{u,v} = 1$ for $v < u$. $c_{u,v}$ is of course a decreasing function of v and we do not know if it is strictly decreasing. We feel sure that

$$\lim_{n \Rightarrow \infty} \log f_2 \left(n; u, \binom{u}{2} \right) / \log n = \frac{1}{u-1}$$

but this has been proved only for $u = 3$ (see [10]).

II. r=3

First we investigate $u = 4$. $v = 1$ is of course in the trivial zone and $f_3(n; 4, 1) = n$. $v = 2$ is not quite trivial.

Theorem 2. $(2n)^{1/2} \leq f_3(n; 4, 2) < c_1 n^{c_2}$ $(c_2 < 1)$.

First we prove the lower bound. Let $X_1, \ldots, X_t$ be a maximal independent set of our graph i.e. no triple $(X_i, X_j, X_\ell), 1 \leq i < j < \ell \leq t$ occurs in our three-graph but for every other vertex J_j $1 \leq j \leq n-t$ there is a triple (X_u, X_v, X_j) in our graph. No two triples of our graph can have a pair in common (for otherwise we have four vertices containing two triples) thus $\binom{t}{2} \geq n - t$, which proves the lower bound. $c_3 n^{1/2}$ with a smaller c_3 than $2^{1/2}$ cold also have been obtained by a result of J. Spencer [10]. The upper bound can be obtained by the probability method. It seems certain that

$$\lim_{n \to \infty} \log f_3(n; 4, 2) / \log n \tag{12}$$

exist, but we have not proved this and do not know if the value of the limit in (12) is 1/2. The probability method does not seem to give this.

Theorem 3. $c_1 \log n / \log \log n < f_3(n; 4, 3) \leq (2 \log n / \log 2) + 1$.

First we prove the lower bound. Denote again by $X_1, \ldots, X_t$ the vertices of a maximal independent set. For every other vertex $X_j, 1 \leq j \leq n-t$ our graph must contain a triple (X_u, X_v, X_j), $1 \leq u < v \leq n-t$. Thus we can assume that the same pair say (X_1, X_2) occurs in at least n/t^2 triples (X_1, X_2, X_j), $1 \leq j \leq n/t^2$. Now none of the triples (X_i, X_{j_1}, X_{j_2}), $i = 1, 2$ $1 \leq j_1 < j_2 \leq n/t^2$ can occur in our 3-graph for otherwise the quadruple

$(X_1, X_2, X_{j_1}, X_{j_2})$ would contain three triples of our graph. Thus say X_1 is independent of $\{X_1, \ldots, X_{[\frac{n}{t^2}]}\}$.
Repeating the same argument for the 3-graph spanned by $\{X_1, \ldots, X_{[\frac{n}{t^2}]}\}$ we obtain the lower bound of Theorem 3. We unfortunately could not decide if $f_3(n; 4, 3) > c \log n$ is true or not.

Now we prove the upper bound of Theorem 3. Erdős and Moser [11] proved that there is a tournament T_n (i.e. a directed complete graph) of n vertices which does not contain a transitive subtournament of more than $2 \log n / \log 2$ vertices. Further it is easy to see that a tournament of four vertices contains at least two transitive subtournaments of 3 elements. Consider now the set of all triples of our T_n which form a non transitive triple. Clearly this family does not contain $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$, but since T_n does not contain a transitive subtournament of size $[2 \log n / \log 2] + 1$, every subset of size $[2 \log n / \log 2] + 1$ must contain one of our triples, this completes the proof of Theorem 3.

Now we give short discussion of infinite tournaments. Ramsey's theorem easily implies that every infinite tournament contains an infinite transitive subtournament. A few years ago Laver constructed a tournament of a size $\aleph_1$ which does not contain an uncountable transitive subtournament. Using the continuum hypothesis one can construct for every $m \geq \aleph_0$ a tournament of size m^+ which does not contain a transitive subtournament of size m^+.

On the other hand if m is weakly compact then every tournament of size m contains a transitive subtournament of size m.

Finally it follows from results of Jensen and Shore that if $V = L$ is assumed and $cf(m)$ is not weakly compact then there is a tournament of size m which does not contain a transitive subtournament of size m. Now we prove

Theorem 4. *Let m be strong limit, and $cf(m)$ weakly compact. Then every tournament of size m contains a transitive subtournament of size m.*

We split the triples of our tournament into two classes. In the first class are the transitive triples and in the second class the non transitive ones. As stated previously every quadruple contains at most two triples of the second class. A well-known theorem of Rado and ourselves [4] states: Let

m be strong limit, $cf(m)$ weakly compact. Then $m \to (m, \begin{bmatrix} 4 \\ 3 \end{bmatrix})^3$. Thus our tournament contains a subtournament of size m all of whose triples are transitive which of course implies that it is itself transitive, which completes the proof of Theorem 4.

Theorem 5. $f_3(n; k, \binom{k}{3}) > 1/2(\log n)^{\frac{1}{2(k-1)}}$

Theorem 5 is essentially contained in older results of Erdős, Hajnal and Rado [4] and Erdős and Rado [3] but for completeness we give the simple proof. Theorem 5 is clearly equivalent to

$$n \to ([(\log n)^{\frac{1}{2(k-1)}}], k)^3. \tag{13}$$

Let $|S| = n$ be a set of n elements. Split the triples of S into two classes. Assume that we already found elements $X_1, \dots, X_\ell$ of S and a subset

$$S_1^{(\ell)} \subset S, X_i \notin S_1^{(\ell)} \quad 1 \le i \le \ell, \quad |S_1^{(\ell)}| > \frac{n}{2^{\ell^2}},$$

so that all the triples (X_{i_1}, X_{i_2}, X_i), where $i_1 < i \le \ell$ and (X_{i_1}, X_{i_2}, Z), $Z \in S_1^{(\ell)}$ belong to the same class (in other words the class of (X_{i_1}, X_{i_2}, X_i) and (X_{i_1}, X_{i_2}, Z) only depends on the pair (X_{i_1}, X_{i_2})). Let $X_{\ell+1}$ be any element of $S_1^{(j)}$. We now divide $S_1^{(\ell)}$ into 2^ℓ classes, $Z_1 \in S_1^{(\ell)}$ and $Z_2 \in S_2^{(\ell)}$ belong to the same class if all the triples

$$(X_i, X_{\ell+1}, Z_1) \,\text{and}\, (X_i, X_{\ell+1}, Z_2),\ i = 1, \dots, \ell$$

belong to the same class. Clearly at least one of these classes has at least

$$\frac{1}{2^\ell}\left(|S_1^{(\ell)}| - 1\right) > \frac{n}{2^{(\ell+1)^2}}$$

elements. This class we call $S_1^{(\ell+1)}$.

We continue this construction as log as possible and obtain a sequence $X_1, \dots, X_t$, $t > (\log n)^{1/2}$, so that the class of the triple (X_i, X_j, X_k), $1 \le i < j < k \le t$ only depends on the pair (X_i, X_j). Put the pair (X_i, X_j), $1 \le i < j \le t-1$ in the same class as the triple (X_i, X_j, X_t). Then by (2) there either is a set of elements $k-1$, $X_{i_1}, \dots, X_{i_{k-1}}$ $1 \le i_1 < \cdots < i_{k-1} \le t-1$ all of whose pairs are in class II or a set S_1 of $\left[t^{1/k-1}\right]$ elements all whose pairs are in class I. In the first case all triples of $(X_{i_1}, \dots, X_{i_{k-1}}, X_t)$ are in II and in the second all triples of $S_1 \cup \{X_t\}$ are in I. This proves (13) and Theorem 4.

It seems certain that to every $\varepsilon > 0$ there is a $k_0 = k_0(\varepsilon)$, so that for every $k > k_0$

$$n \nrightarrow \left(\left[(\log n)^{\varepsilon} \right], k \right)^3 . \tag{14}$$

We could not even prove

$$\lim_{n \Rightarrow \infty} f_3 \left(n; k, \binom{k}{3} \right) / \log n = 0. \tag{15}$$

(15) probably holds already for $k = 4$.

Theorems 2 and 3 give $h_1^{(3)}(4) = 3$. It is not difficult to show that $h_1^{(3)}(5) = 5$, $h_1^{(3)}(6) = 9$, $h_1^{(3)}(7) = 14$, $h_1^{(3)}(8) = 21$, $h_1^{(3)}(9) = 31$. Now we try to explain all that we know about $h_1^{(3)}(u)$. First we define a function $g_1^{(3)}(u)$ by recursion. Put $g_1^{(3)}(1) = g_1^{(3)}(2) = 0$. Assume that $g_1^{(3)}(m)$ has already been defined for all $m < n$. Then

$$g_1^{(3)}(n) \;=\; \max_{a+b+c=n} (g_1^{(3)}(a) + g_1^{(3)}(c) + abc). \tag{16}$$

In fact it is easy to see that

$$g_1^{(3)}(n) = X + Y + Z + XYZ \tag{17}$$

where X+Y+Z=n and X,Y,Z are as nearly equal as possible.

Theorem 6. $f_2(n; k, g_1^{(3)}(k)) > n^{\varepsilon_k}$.

The proof of Theorem 6 will be by induction with respect to k. We will not give an explicit estimation for ε_k, though this could be done without much difficulty. Before giving the details we state our principal.

Conjecture I. $h_1^{(3)}(k) \;=\; g_1^{(3)}(k) + 1.$ In other words

$$f_3(n; k, g_1^{(3)}(k) + 1) < c_k^{'} \log n. \tag{18}$$

Unfortunately we cannot prove (18), but before we discuss this we prove Theorem 6. In fact we will prove a stronger statement. We define by induction a class of 3-graphs $G^{(3)}(k; \ell_k)$ of k vertices and $\ell_k \leq g_1^{(3)}(k)$ triples as follows: $G^{(3)}(3,\, 1)$ is a triple (i.e. a 3-graph of three vertices and one triple). Assume that the graphs $G^{(3)}(m;\, \ell_m)$ have already been defined for all $m < k$. The graphs $G^{(3)}(k;\, \ell_k)$ are defined as follows. Consider all partitions $k = u_1 + u_2 + u_3$ and consider $G^{(3)}(u_1;\, \ell_{u_1}) \cup G^{(3)}(u_2;\, \ell_{u_2}) \cup G^{(3)}(u_3;\, \ell_{u_3})$ with the vertices $X_1, \ldots, X_{u_1}; Y_1, \ldots, Y_{u_2}; Z_1, \ldots, Z_{u_3}$ and add further all the triples (X_i, Y_ℓ, Z_r), $1 \le i \le u_1$, $1 \le j \le u_2$, $1 \le r \le u_3$. Thus we obtain our graphs $G^{(3)}(k;\, \ell_k)$.

Instead of Theorem 6. we prove the following stronger structural Theorem 7.

Theorem 7. *Let $\varepsilon_k > 0$ be sufficiently small and $n > n_0(\varepsilon_k)$ sufficiently large. Then every $G^{(3)}(n)$ either contains all the graphs $G^{(3)}(k;\, \ell_k)$ as subgraphs or $G^{(3)}(n)$ contains an independent set of size n^{ε_k}.*

First we remark that if t is fixed and n_t is small enough and $m > m_0(t, n_t)$, $|S| = m, A_1, \ldots, A_j, A_i \subset S, |A_i| = t$ a family of t-tuples so that every subset $S_1 \subset S, |S_1| > m^{n_t}$ contains at least one of the A's then $j > m^{t-1/2}$.

The remark follows immediately from the result of Spencer [10] (a direct proof is also easy). By the way the result remains true with $j > m^{t-\delta_t}$ for every $\delta_t > 0$ if ε_t is small enough.

To prove Theorem 7 we use induction with respect to k. Theorem 7 trivially holds for $k \le 3$. Assume that it holds for every $k' < k$ and we will prove it for k. Let $G^{(3)}(k;\, \ell_k)$ be defined by the partition $u_1 + u_2 + u_3 = k$, $1 \le u_1 \le u_2 \le u_3$. Consider now the graph $G^{(3)}(u_1 + u_2 + 1)$ having the vertices $X_1, \ldots, X_{u_1}; Y_1, \ldots, Y_{u_2}; Z_1$. On the X's it coincides with $G^{(3)}(u_1;\, \ell_{u_1})$ on the G's with $G^{(3)}(u_2;\, \ell_{u_2})$ and further it contains all the triples (X_i, Y_j, Z_1); $1 \le i \le u_1$, $1 \le j \le u_2$, $\big(G^{(3)}(u_1 + u_2 + 1)\big)$ is a subgraph of $G^{(3)}(k;\, \ell_k)$.

We can of course assume that $G^3(n)$ does not contain an independent set of size n^{ε_k}. But then by our induction hypothesis it contains our $G^{(3)}(u_1 + u_2 + 1)$ as a subgraph and in fact we can assume that if ε_k is small enough then every set of vertices of size $n^{\delta_k}, \delta_k = \varepsilon_k^{1/2}$ contains our $G^{(3)}(u_1 + u_2 + 1)$ as a subgraph. But then by our remark $G^{(3)}(n)$ contains at least $n^{u_1+u_2+1/2}$ copies of our $G^{(3)}(u_1 + u_2 + 1)$. But then a simple computation shows that there are vertices

$$X_1, \ldots, X_{u_1}; Y_1, \ldots, Y_{u_2}; Z_1, \ldots Z_s,\ s \ge n^{1/2}$$

so that our $G^{(3)}(n)$ contains on $X_1, \ldots, X_{u_1}$ a 3-graph isomorphic to $G^{(3)}(u_1;\, \ell_{u_1})$ on $Y_1, \ldots, Y_{u_2}$ a subgraph isomorphic to $G^{(3)}(u_2;\, \ell_{u_2})$ and finally $G^{(3)}(n)$ contains all the triples

$$(X_i, Y_j, Z_r);\ 1 \le i \le u_1,\ 1 \le j \le u_2,\ 1 \le r \le s.$$

Now consider the subgraph of $G^{(3)}(n)$ spanned by $Z_1, \ldots Z_s$. We can of course again assume that it does not contain an independent set of size n^{ε_k}. Thus by our induction hypothesis it contains a subgraph $G^{(3)}(u_3;\, \ell_{u_3})$ having the vertices $Z_1, \ldots Z_{u_3}$. Now the graph spanned by $X_1, \ldots, X_{u_1}; Y_1, \ldots, Y_{u_2}; Z_1, \ldots Z_{u_3}$ contains our $G^{(3)}(k;\, \ell_k)$ as required - this completes the proof of Theorem 7.

Having proved Theorem 7 we now explain our reasons for believing conjecture I. Let us colour the edges of the complete graph whose vertices are the integers by three colours. We wish to maximize the number of triangles (a, b, c), $1 \le a < b < c \le k$ for which the edge (a, b) has colour I, (b, c) colour II and (a, c) colour III. Denote this maximum by $F_1(k)$. It is immediate that $F_1(k) \ge g_1^{(3)}(k)$.

Conjecture II. $F_1(k) = g_1^{(3)}(k)$.

Unfortunately we have no real evidence for conjecture II except that we easily proved it for small values of k.

Theorem 8. $h_1^{(3)}(k) \le F_1(k) + 1$.

Observe that Theorem 8 and Conjecture II implies by Theorem 6 Conjecture I, thus the only missing link is the proof of Conjecture II.
Theorem 8 is clearly equivalent to

$$n \not\to \left(c_k \log n,\ \left[\begin{array}{c} k \\ F_1(k)+1 \end{array}\right]\right)^3 \tag{19}$$

where c_k is a sufficiently large constant.

We prove (19) by the so called probability method.

Colour the edges of a graph of n labelled vertices by 3 colours in all possible ways. The number of distinct colourings is $3^{\binom{n}{2}}$. The triples

(a,b,c), $1 \le a < b < c$ for which (a,b) has colour I, (b,c) has colour II and (a,c) has colour III we put in class II the other triples we put in class I. By the definition of $F_1^{(3)}(k)$ no k-tuple can contain more than $F_1^{(3)}(k)$ triples of class II. On the other hand a simple computation shows that all but $\sigma(3^{\binom{n}{2}})$ of the colourings have the property that every set of $[c_k \log n]$ contains a triple of the second class if c_k is a sufficiently large constant. To see this put $[c_k \log n] = T$. It is well known that a set of size T contains $(1+\sigma(1))\frac{T^2}{6} = L$ triples any two of which have at most one element in common. Thus by a simple argument [12] there are for sufficiently large c_k at most

$$3^{\binom{n}{2}} \binom{n}{T} (1-1/8)^L < 2^{\binom{n}{2}} n^T \, e^{-T^2/100} = \sigma(3^{\binom{n}{2}})$$

colourings for which there is a set of size T not containing a triple of the second class. This proves (19) and Theorem 8.

Before closing this chapter we mention a few related questions. Let us colour the edges of the complete graph whose vertices are the integers $\le k$ by two colours so that the number of triangles (a,b,c), $a < b < c$, (a,b) and (b,c) are coloured I and (a,c) is coloured II is maximal. Denote this maximum by $F_2(k)$. Perhaps $F_1(k) = F_2(k)$. Trivially $F_2(k) \ge F_1(k)$.
An older problem of V.T. Sós and P. Erdős states: Colour the edges of a complete graph of n vertices by three colours so that the number of tirangles all whose edges get a different colour is maximal. Denote this maximum by $F_3(k)$. They conjectured that $F_3(k)$ is obtained as follows: Clearly $F_3(1) = F_3(2) = 0$, $F_3(3) = 1$, $F_3(4) = 4$. Suppose $F_3(k_1)$ has already been determined for every $k_1 < k$. Then

$$\begin{aligned} F_3(k) &= F_3(u_1) + F_3(u_2) + F_3(u_3) + F_3(u_4) \\ &\quad + u_1u_2u_3 + u_1u_2u_4 + u_1u_3u_4 + u_2u_3u_4, \end{aligned} \tag{20}$$

where $u_1 + u_2 + u_3 + u_4 = k$ and the u's are as nearly equal as possible. We made no progress with (20).

III. $r > 3$

The first non-trivial case is $u = r+1$. As stated already in the introduction we conjecture

$$h_i^{(r)}(r+1) = i + 2 \tag{21}$$

We can only prove this for $i = 1$.

$$f_r(n; r+1, 2) > n^{1/r-1} \tag{22}$$

follows easily like the proof of Theorem 2. Thus to prove (21) for $i = 1$ we only have to show

Theorem 9. $c'_r \log n / \log\log n < f_r(n; r+1, 3) < c_r \log n$.

For $r = 3$ Theorem 9 follows from Theorem 3, but we give an outline of a slightly different proof which works also for $r = 3$. We again use methods of probabilistic combinatorial analysis. Colour the edges of a K_n (a complete graph of n labelled vertices) by $\binom{r}{2}$ colours. We can do this in $\binom{r}{2}^{\binom{n}{2}}$ ways. Colour the edges of K_r of r labelled vertices $1, \ldots, r$ by $\binom{r}{2}$ different colours from 1 to $\binom{r}{2}$, say lexicographically. A simple computation shows that for all but $\sigma\left(\binom{r}{2}^{\binom{n}{2}}\right)$ of the colourings of our K_n every $K_{[c_r \log n]}$ contains at least one K_r coloured in this way, but every $K_{r=1}$ contains at most two such complete graphs K_r. Put now the r-tuples coloured lexicographically in class II and the others in class I and we obtain the upper bound in Theorem 9.
The lower bound can be obtained as in Theorem 3.
Unfortunately we have no idea how to prove (21) for $i > 1$. By the methods of Theorem 5 it is quite easy to prove

$$f_r(n; r+1, i+2) > c_{r,i}(\log_i n)^{\varepsilon_{r,i}}. \tag{23}$$

The great difficulty is to prove an inequality in the opposite direction.
Now we investigate $h_1^{(r)}(k)$. Define $g_1^{(r)}(k)$ as follows: $g_1^{(r)}(k) = 0$ for $k < r$, $g_1^{(r)}(r) = 1$. Assume that $g_1^{(r)}(k_1)$ has already been defined for all $k_1 < k$. Put

$$g_1^{(r)}(k) = \sum_{i=1}^{r} g_1^{(r)}(u_1) + \prod_{i=1}^{r} u_i$$

where $\sum_{i=1}^{r} u_i = k$ and the u's are as nearly equal as possible. We have

Theorem 6.' $f_r(n; k, g_1^{(r)}(k)) > n^{\varepsilon_{k,r}}$.

The proof is similar to that of Theorem 6 and we suppress it.

Conjecture I.' $h_1^{(r)}(k) = g_1^{(r)}(k) + 1$.

We are of course much less certain about conjecture I' than about conjecture I. This and many other remaining questions we hope to investigate (if we live) in the future, though we should add that it is our sincere hope that others we do this before us.

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