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ON A PROBLEM OF GRÜNBAUM

BY

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In memory of my friend and collaborator, Leo Moser

P_n will denote a set of n points in the plane. A well known theorem of Gallai–Sylvester (see e.g. [4]) states that if the points of P_n do not all lie on a line then they always determine an ordinary line, i.e. a line which goes through precisely two of the points of P_n.

Using this theorem I proved that if the points do not all lie on a line, they determine at least n lines. I conjectured that if $n>n_0$ and no $n-1$ points of P_n are on a line, they determine at least $2n-4$ lines. This conjecture was proved by Kelly and Moser [3], who, in fact, proved the following more general result:

Let P_n be such that at most $n-k$ of its points are collinear. Assume

(1) $$n \geq \tfrac{1}{2}(3(3k-2)^2+3k-1).$$

Then P_n determines at least

(2) $$kn-\tfrac{1}{2}(3k+2)(k-1)$$

lines. They also observed that (2) is best possible.

B. Grünbaum asked the following question: Determine the sequence of integers $m_1^{(n)}<m_2^{(n)}<\cdots$ so that for every i there is a P_n which determines exactly $m_i^{(n)}$ lines. $m_1^{(n)}=1$, $m_2^{(n)}=n$, $m_3^{(n)}=2n-4$ if $n\geq 27$ (see [3]). Clearly the largest value of $m_i^{(n)}$ is $\binom{n}{2}$. Grünbaum observed that $\binom{n}{2}-1$ and $\binom{n}{2}-3$ cannot be values of $m_i^{(n)}$. The proof is easy. If the points are not in general position at least three must be on a line, thus $m_i^{(n)}=\binom{n}{2}-1$ is impossible. If 4 points are on a line or there are two lines containing three points we get at most $\binom{n}{2}-5$ or $\binom{n}{2}-4$ lines, thus $m_i^{(n)}=\binom{n}{2}-3$ is also impossible.

The problem of characterizing the sequence $\{m_i^{(n)}\}$ seems to be very difficult. We prove the following

THEOREM. *There exists c_1 such that for each m satisfying $c_1n^{3/2}<m\leq\binom{n}{2}$, $m\neq\binom{n}{2}-1$, $m\neq\binom{n}{2}-3$, there is a P_n which determines exactly m lines.*

We also show that our theorem is best possible in the following sense: There is a c_2 (c_1 and c_2 are absolute positive constants) so that there is an $m>c_2n^{3/2}$ for which there is no P_n which determines exactly m lines. To determine the largest

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such m, seems to be a difficult problem; I doubt that the methods of this paper can solve it. In view of this we do not attempt to get the best values for c_1 and c_2.

First we show that there is an $m > c_2 n^{3/2}$ so that no P_n determines m lines. Let k_0 be the largest integer for which

$$n > \tfrac{1}{2}(3(3k_0-2)^2+3k_0-1), \quad \text{i.e.} \quad k_0 = (1+o(1))\left(\frac{2n}{27}\right)^{1/2}. \tag{3}$$

Put

$$m = k_0 n - \tfrac{1}{2}(3k_0+2)(k_0-1)-1. \tag{4}$$

It is easy to see that no P_n determines exactly m lines. If at most $n-k_0$ of the points lie on a line then by (2) P_n determines at least $m+1$ lines. Assume next that $n-l$, $l<k_0$ points of P_n are on a line. Then clearly P_n determines at most

$$1+\binom{l}{2}+l(n-l), \quad l < k_0$$

lines which by (3) and (4) is clearly less than m if $n > n_0$.

Now we prove our theorem. First we note the following

LEMMA. *Let c_1 be sufficiently large. Then every integer*

$$t < \binom{n}{2} - c_1 n^{3/2}, \qquad t \neq 1, \quad t \neq 3 \tag{5}$$

can be written in the form

$$t = \sum_i \alpha_i \left(\binom{n_i}{2}-1\right), \qquad \sum_i \alpha_i n_i \leq n, \quad n_i \geq 3 \tag{6}$$

where the α_i are positive integers.

Assume that our lemma has already been proved then we deduce our Theorem as follows:

Put $m=\binom{n}{2}-t$. Our P_n which determines exactly m lines is constructed in the following way: P_n has α_i lines $i=1,\dots$ each of which has n_i points, otherwise the points are in general position, i.e. no three of them are on a line. It is clear by (6) that such a configuration exists and by (6) it determines

$$\binom{n}{2} - \sum_i \alpha_i \left(\binom{n_i}{2}-1\right) = m$$

lines. Thus we only have to prove our lemma.

Let n_1 be the largest integer for which $\binom{n_1}{2} < t-4$. Clearly $n_1 \leq \sqrt{2t}+1 < n-10\sqrt{n}$ for sufficiently large c_1, also

$$t - \binom{n_1}{2} < 3n_1 < 3n.$$

Let now n_2 be the largest integer for which

$$\binom{n_2}{2} \le t - \binom{n_1}{2} - 4.$$

Clearly $n_2 < 3\sqrt{n}$ and

$$4 \le t - \binom{n_1}{2} - \binom{n_2}{2} < 6\sqrt{n}. \tag{7}$$

By (7) we can write

$$t = \binom{n_1}{2} + \binom{n_2}{2} + \alpha_3\left(\binom{4}{2} - 1\right) + \alpha_4\left(\binom{3}{2} - 1\right)$$

where $\alpha_3 + \alpha_4 < 3\sqrt{n}$. Thus (5) and (6) are satisfied and the proof of our lemma is complete.

It might be possible to determine the smallest t which cannot be written in the form (6), but we do not discuss this question here.

I would like to say a few words about possible generalizations of our theorem. The following result is well known [2]:

Let S be a set of n elements $x_1, \ldots, x_n$. Suppose $A_i \subset S$, $2 \le |A_i| < n$ $(1 \le i \le k)$ and each pair (x_r, x_s) $(1 \le r, s \le n)$ is contained in exactly one A_i. Then $k \ge n$. Here I can prove that if

$$n + cn^{3/4} < m \le \binom{n}{2}, \quad m \ne \binom{n}{2} - 1, \quad m \ne \binom{n}{2} - 3$$

then there are m sets $A_i \subset S$, $2 \le |A_i|$, so that every pair (x_r, x_s) is contained in one and only one A_k. Probably $cn^{3/4}$ is best possible.

A straightforward application of our method leads to the following

THEOREM. *Let* $cn^2 < m \le \binom{n}{3}$, $m \ne \binom{n}{3} - a_i$ *where* a_i *runs through a finite set of numbers which could easily be determined explicitly. Then there is a* P_n *which determines exactly* m *circles. A recent result of Elliott* [1] *shows that the order of magnitude* cn^2 *is best possible.*

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