A CHARACTERIZATION OF FINITELY MONOTONIC ADDITIVE FUNCTIONS

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Let $f(m)$ be a real-valued, number theoretic function. We say that $f(m)$ is *additive* if $f(mn) = f(m)+f(n)$ whenever $(m, n) = 1$. If $f(m)$ satisfies the additional restriction that $f(p) = f(p^2) = f(p^3) = \ldots$, then we say that $f(m)$ is *strongly additive*. We denote the class of additive functions by $\mathscr{A}$.

A function $f \in \mathscr{A}$ is called *finitely monotonic* if there exists an infinite sequence $x_k \to \infty$ and a positive constant λ, so that for each x_k there are integers

$$1 \leqslant a_1 < a_2 < \ldots < a_n \leqslant x_k$$

satisfying $n \geqslant \lambda x_k$ and $f(a_1) \leqslant f(a_2) \leqslant \ldots \leqslant f(a_n)$. In other words, $f(m)$ is said to be finitely monotonic if, infinitely often, $f(m)$ is non-decreasing on a positive proportion of the integers between 1 and x_k. Let $\mathscr{M}$ denote the class of finitely monotonic functions.

Approximately 25 years ago, Erdös [**3**] proved that a monotonic, additive function is a constant multiple of the logarithm. In the same paper Erdös conjectured that even when an additive function is monotonic on a sequence of integers with density 1, then the conclusion still holds. This was later proved by Kátai [**4**]. At about the same time Kátai's result appeared, B. J. Birch proved the following theorem, which may be found in [**1**].

THEOREM (Birch). *Let $f(m)$ be an additive function, and let $g(m)$ be any monotonic non-decreasing function. Suppose that for every $\varepsilon > 0$, $|f(m)-g(m)| < \varepsilon$ for all but $o(x)$ of the integers $1 \leqslant m \leqslant x$, as $x \to \infty$. Then $f(m) = c \log m$.*

In the present paper, we shall show that if f is finitely monotonic, then f approximates a constant multiple of the logarithm. Thus, we prove the

THEOREM. *Let $f \in \mathscr{A}$. A necessary and sufficient condition that $f \in \mathscr{M}$ is that there exist a positive constant c and an additive function g so that*

$$f(m) = c \log m + g(m), \tag{1}$$

where

$$\sum_{g(p) \neq 0} \frac{1}{p} < \infty. \tag{2}$$

This theorem was first stated as Theorem XII in [**3**], although without proof. We include all of the details here.

Proof of Theorem (sufficiency). Suppose that $f(m)$ satisfies (1) and (2). Then $g(m)$ must vanish on a sequence of integers of positive density. On this sequence, $f(m)$ is non-decreasing.

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To prove that the conditions (1) and (2) are necessary will be much more difficult. We shall first deduce from Lemma 1 and Lemma 2 that if $f \in \mathcal{M}$, then f has the form

$$f(m) = c \log m + g(m), \tag{3}$$

where

$$\sum_p \frac{(g'(p))^2}{p} < \infty, \tag{4}$$

and where $g'(p) = g(p)$ if $|g(p)| \leqslant 1$ and $g'(p) = 1$ otherwise. Next, we employ Lemma 3 and Lemma 4 to prove that the condition (4) can be strengthened to the condition (2). This will prove the theorem.

Definition. Let $f \in \mathcal{A}$. Then f is said to be *finitely distributed* if there exists an infinite sequence $x_k \to \infty$ and positive constants c_1 and c_2 so that for each x_k there exist integers $1 \leqslant a_1 < \ldots < a_n \leqslant x_k$ for which $|f(a_i) - f(a_j)| \leqslant c_2$, $1 \leqslant i, j \leqslant n$, and $n \geqslant c_1 x_k$.

It is seen from this definition that finitely distributed functions are distinguished by the fact that, infinitely often, a positive proportion of their values, defined on $[1, x_k]$, lie in a strip of constant width. (The functions $c \log n$, for example, are finitely distributed for each constant c.)

The study of finitely distributed functions was begun by Erdös in **[3]**. One of the results of his work there is the

LEMMA 1 (Erdös). *A necessary and sufficient condition that f be finitely distributed is that f satisfy conditions* (3) *and* (4).

Proof of Lemma 1. Erdös' original proof may be found in Theorem V of **[3]**. Another proof, based on analytic methods is given in **[5]**.

LEMMA 2. *Suppose that $f \in \mathcal{M}$. Then f satisfies conditions* (3) *and* (4).

Proof of Lemma 2. We suppose that for each $x_k \to \infty$ there are sets of integers $\mathscr{C}_k = \mathscr{C}(f, x_k) = \{a_j \leqslant x_k : 1 \leqslant j \leqslant n;\ n \geqslant \lambda x_k\}$ for which

$$f(a_1) \leqslant f(a_2) \leqslant \ldots \leqslant f(a_n).$$

We shall deduce that $f(m)$ is finitely distributed. The conclusion of Lemma 2 will then follow immediately from Lemma 1.

Thus, choose $\varepsilon > 0$. Choose primes q and r so that

$$\prod_{q \leqslant p \leqslant r} (1 - p^{-1}) < \varepsilon,$$

where the product is over primes p in the indicated range. Also, put

$$P = \prod_{q \leqslant p \leqslant r} p.$$

Then the number of $a_i \in \mathscr{C}_k$ for which $(a_i, P) = 1$ does not exceed $2\varepsilon x_k$, for all sufficiently large x_k.

Define numbers a_i' by $a_i = a_i' \pi_i$, where π_i is the largest factor of a_i dividing P. It is possible that a_i' and π_i are not relatively prime. But if we choose q so large that

$$\sum_{q \leqslant n} n^{-2} < \varepsilon, \tag{5}$$

then there are at most εx_k of the a_i for which $(a_i', \pi_i) > 1$. Hence, we add the requirement that the prime q satisfies (5). Thus, at least $(\lambda - 3\varepsilon)\, x_k$ of the $a_i \in \mathscr{C}_k$ satisfy the conditions $a_i = a_i' \pi_i$, $\pi_i \mid P$, $\pi_i > 1$, $(a_i', \pi_i) = 1$. Denote this subset of $\mathscr{C}_k$ by $\mathscr{D}_k$.

Now suppose that for infinitely many x_k there are two numbers $a_j > a_i$ of $\mathscr{D}_k$ for which $a_j' = a_i'$, and that there are at least δx_k numbers $a_l \in \mathscr{D}_k$ which satisfy $a_j > a_l > a_i$ (i.e., $j - i \geqslant \delta x_k$), where $\delta > 0$ is independent of k. Then f is finitely distributed. To see this, recall that $a_j' = a_i'$ means that

$$\frac{a_j}{\pi_j} = \frac{a_i}{\pi_i},$$

from which it follows that

$$f(a_j) - f(a_i) = f(\pi_j) - f(\pi_i),$$

since $(\pi_i, a_i) = 1$. Moreover, since $a_j > a_l > a_i$, we have

$$|f(a_l) - f(a_i)| \leqslant |f(\pi_j) - f(\pi_i)|;$$

and so f is finitely distributed.

Therefore, we assume that between any two numbers a_j and a_i of $\mathscr{D}_k$ such that $a_j' = a_i'$, there are $o(x_k)$ numbers a_l of $\mathscr{D}_k$, as $x_k \to \infty$. We shall arrive at a contradiction.

Put

$$\mu = \min_{\pi_j, \pi_i \in P} \left\{ \left| \frac{\pi_j}{\pi_i} - 1 \right| : \pi_j > \pi_i \right\}.$$

Then $\mu > 0$ and independent of x_k.

Choose the largest number $a_j \in \mathscr{D}_k$ for which $a_j' = a_i'$ for some $i \neq j$. Denote this largest number by a_{j_1}. Then let a_{i_1} be the smallest number such that $a_{j_1}' = a_{i_1}'$. Between a_{j_1} and a_{i_1} there are at most $o(x_k)$ numbers of $\mathscr{D}_k$. Also,

$$a_{j_1} = a_{i_1} \pi_{j_1} \pi_{i_1}^{-1} \geqslant a_{i_1}(1+\mu).$$

Next, let a_{j_2} be the largest number of $\mathscr{D}_k$ less than a_{i_1} and for which $a_{j_2'} = a_i$ for some $i \neq j_2$. Let a_{i_2} be the smallest number for which $a_{j_2}' = a_{i_2}'$. As before, $a_{j_2} \geqslant a_{i_2}(1+\mu)$.

Continuing in this way, we obtain a sequence of numbers

$$a_{j_1} > a_{i_1} > a_{j_2} > a_{i_2} > \ldots > a_{j_h} > a_{i_h},$$

where h is chosen so that $(1+\mu)^h \geqslant q > (1+\mu)^{h-1}$. With h chosen in this way, there are at most x_k/q numbers of $\mathscr{D}_k$ less than a_{i_h}. We note, also, that the number of a_i for which a_i' can equal a given a_j' is at most the number of distinct π_i, a bounded number (certainly less than e^r). Finally, note that the number of a_i for which a_i' is never equal to another a_j', is at most x_k/q.

Hence, in the above procedure, we have accounted for a total of at most

$$\left(1/q + 3\varepsilon + o(h) + 1/q\right) x_k + 2he^r$$

numbers in $\mathscr{C}_k$, which contradicts $|\mathscr{C}_k| \geqslant \lambda x_k$, if ε is chosen sufficiently small.

It follows that $f(m)$ is finitely distributed. A direct application of Lemma 1 shows that f must satisfy conditions (3) and (4).

LEMMA 3. *Suppose that $f \in \mathscr{A}$ is finitely monotonic. Then the strongly additive function f^*, defined by $f^*(p^r) = f(p)$, is also finitely monotonic.*

Proof of Lemma 3. The hypotheses of Lemma 3 state that there exists an infinite sequence $x_k \to \infty$ and a positive constant λ so that for each x_k there are integers $1 \leqslant a_1 < a_2 < \ldots < a_n \leqslant x_k$ with $n \geqslant \lambda x_k$ and $f(a_1) \leqslant f(a_2) \leqslant \ldots \leqslant f(a_n)$.

Choose $N = N(\lambda)$ so large that

$$\sum_{\substack{p^r > N \\ r \geqslant 2}} p^{-r} < \lambda/2.$$

With this choice of N, at least $\lambda x_k/2$ of the $a_i \leqslant x_k$ have no prime power divisor p^r ($r \geqslant 2$) satisfying $p^r > N$. Hence, the order of the set $S_k = S_k(N)$, defined by

$$S_k = \{a_i \leqslant x_k : p^r \mid a_i,\ r \geqslant 2 \Rightarrow p^r \leqslant N\},$$

is at least $\lambda x_k/2$.

Let $\mathscr{D}$ consist of those integers whose prime power divisors p^r satisfy $p^r \leqslant N$ (where we now allow the possibility $r = 1$), and let D denote the product of all of the integers $d \in \mathscr{D}$. For each $d \in \mathscr{D}$, put

$$S_k^{(d)} = \{a_i \in S_k : (a_i, D) = d\}.$$

Then some set $S_k^{(d)}$ has order at least $\lambda x_k/2D$; and for each a_i in this set, we see that a_i/d is square-free. In addition, if $a_i < a_j$ are in this set, then $f(a_i/d) \leqslant f(a_j/d)$. It follows that the strongly additive f^*, defined by $f^*(p^r) = f(p)$, is finitely monotonic.

Henceforth, without loss of generality, we will assume that the finitely monotonic function f, given in the statement of the theorem of this paper, is strongly additive. This assumption is justified by Lemma 3.

LEMMA 4. *Suppose that f is a strongly additive function which satisfies* (3) *and* (4). *Then the finite frequencies $n^{-1}\sum_m 1$, where summation is over values of m such that $m \leqslant n, f(m) - c \log m - \alpha(n) < x$, have a limiting distribution function $F(x)$ as $n \to \infty$, where*

$$\alpha(n) = \sum_{p \leqslant n} \frac{g'(p)}{p}.$$

Moreover, $F(x)$ will be continuous if and only if

$$\sum_{g(p) \neq 0} \frac{1}{p} = \infty.$$

Proof of Lemma 4. The statement of Lemma 4 was first enunciated by Erdös as Theorem II of [**3**]; and a proof was given there in the case when $|g(p)|$ is bounded. A complete proof of Lemma 4 may be found in Theorem 2 of [**2**].

Proof of Theorem (*Necessity*). From Lemma 4, we may find a constant A so that the number of $m \leqslant x_k$ for which $-A \leqslant f(m) - c \log m - \alpha(x_k) \leqslant A$ exceeds $(1 - \lambda/4)\, x_k$. Since there are at least λx_k elements of $\mathscr{C}_k$ ($\mathscr{C}_k$ is defined in the proof of Lemma 2), there are at least $(\lambda - 2(\lambda/4))\, x_k = \lambda x_k/2$ elements of $\mathscr{C}_k$ which satisfy $\lambda x_k/4 \leqslant a_i \leqslant x_k$ and $-A \leqslant f(a_i) - c \log a_i - \alpha(x_k) \leqslant A$. Denote the set of these a_i in $\mathscr{C}_k$ by $\mathscr{S}_k$, where $|\mathscr{S}_k| \geqslant \lambda x_k/2$.

Divide the interval $[\lambda x_k/4, x_k)$ into T equal parts, where T is a large, but fixed, positive integer. Then, we have

$$[\lambda x_k/4, x_k) = \bigcup_{l=0}^{T-1} [\delta_l x_k, \delta_{l+1} x_k)$$

$$= \bigcup_{l=0}^{T-1} I_l,$$

where

$$\delta_l = \frac{(\lambda/4)(T-l)+l}{T}.$$

An interval I_l will be called *good* if it contains at least $\lambda x_k/4T$ of the numbers of $\mathscr{S}_k$. Clearly, the number of elements of $\mathscr{S}_k$, which do not lie in good intervals, is not more than $T(\lambda x_k/4T) = \lambda x_k/4$. Hence, there are at least $\lambda x_k/4$ numbers of $\mathscr{S}_k$ in good intervals; and, so, there are at least

$$\frac{\lambda x_k/4}{(1-\lambda/4)\, x_k/T} = \frac{\lambda T}{4-\lambda} = \nu T$$

good intervals. It follows that on one of these good intervals, say on I_L, $0 \leqslant L \leqslant T-1$, the total variation of $f(a_i) - c\log a_i - \alpha(x_k)$ does not exceed $2A/\nu T$, since f is monotonic on the $a_i \in \mathscr{S}_k$. Moreover, since I_L is a good interval,

$$|\mathscr{S}_k \cap I_L| \geqslant \lambda x_k/4T.$$

Therefore, if we let $\sum'_m 1$ denote the summation over those natural numbers m satisfying

$$\delta_L x_k < m \leqslant \delta_{L+1} x_k,$$

and

$$\eta - \frac{2A}{\nu T} < f(m) - c\log m - \alpha(x_k) < \eta + \frac{2A}{\nu T},$$

then, for some real number η, we have

$$\begin{aligned}(\delta_{L+1} - \delta_L)^{-1} x_k^{-1} \sum_m{}' 1 &= (1-\lambda/4)^{-1}\, T x_k^{-1} \sum_m{}' 1 \\ &\geqslant (1-\lambda/4)^{-1}\, T x_k^{-1} (\lambda x_k/4T) \\ &= \nu > 0. \end{aligned} \tag{6}$$

Suppose, now, that $F(x)$ is a continuous function. Let $\sum''_m 1$ denote the summation over those natural numbers m satisfying

$$1 \leqslant m \leqslant \delta_{l+1} x_k,$$

and

$$\eta - \frac{2A}{\nu T} < f(m) - c\log m - \alpha(\delta_{l+1} x_k) < \eta + \frac{2A}{\nu T}.$$

Then

$$\delta_{l+1}^{-1} x_k^{-1} \sum_m{}'' 1 = F\left(\eta + \frac{2A}{\nu T}\right) - F\left(\eta + \frac{2A}{\nu T}\right) + o(1),$$

as $x_k \to \infty$. Since $\alpha(x_k)-\alpha(\delta_{l+1} x_k) = o(1)$ as $x_k \to \infty$, we see that

$$\delta_{l+1}\left[F\left(\eta+\frac{2A}{vT}\right)-F\left(\eta-\frac{2A}{vT}\right)\right] = x_k^{-1} \sum_m{}''' 1+o(1), \quad x_k \to \infty, \tag{7}$$

where the symbol $\sum'''_m 1$ denotes summation over integers m satisfying

$$1 \leqslant m \leqslant \delta_{l+1} x_k,$$

and

$$\eta-\frac{2A}{vT} < f(m)-c\log m-\alpha(x_k) < \eta+\frac{2A}{vT}.$$

Subtracting equation (7) with $l = L-1$ from equation (7) with $l = L$, and dividing the difference by $\delta_{L+1}-\delta_L$, yields

$$F\left(\eta+\frac{2A}{vT}\right)-F\left(\eta-\frac{2A}{vT}\right) = (\delta_{L+1}-\delta_L)^{-1} x_k^{-1} \sum_m{}' 1+o(1), \quad x_k \to \infty. \tag{8}$$

Combining equations (6) and (8), we obtain

$$F\left(\eta+\frac{2A}{vT}\right)-F\left(\eta-\frac{2A}{vT}\right)+o(1) \geqslant v$$

as $x_k \to \infty$. Since T can be chosen as large as we like (but fixed with respect to x_k) we see that F cannot be continuous. Hence, by Lemma 4,

$$\sum_{g(p)\neq 0} \frac{1}{p} < \infty,$$

which proves the theorem.

References

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