# TOPICS IN COMBINATORIAL ANALYSIS

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In the present paper I will discuss some combinatorial problems which my colleagues and I considered in the recent past. I will restrict myself to finite problems and will try to discuss as much as possible new problems. It might of course turn out that the answer to some of the questions is simple.

1. A few weeks ago I posed the following question: Let

$$|G| = n, A_k \subset G, 1 \leq k \leq 2^{n-1} + n+1.$$

Then there are three $A$'s every two of which intersect but all of them do not intersect. The empty set, the singleton and all sets containing a given element show that this result is best possible.

E. Milner recently found a simple proof by induction with respect to $n$. He proved the result in the following form: Let

$$|G| = n, A_k \subset G, |A_k| \geq 2, 1 \leq k \leq 2^{n-1}$$

then there are three A's every two of which intersect but all of them do not intersect.

Let now $f(\ell,n)$ be the smallest integer such that if

$$|G| = n, A_k \subset G, |A_k| = \ell, 1 \leq k \leq f(\ell,n)$$

there are always three A's every two of which have common element, but all of them do not have common element.

A well known theorem of Turan implies $f(2,n) = [\frac{n^2}{4}] + 1$ but perhaps for $\ell > 2, f(k,n) = \binom{n-1}{\ell-1} + 1$.

A related question is the following one: Determine the smallest integer $f(n)$ so that if $|G| = n, A_k \subset G, 1 \leq k \leq f(n)$ then there are three elements $x,y,z,$ and three A's say $A_i$, $A_j$, $A_\ell$ so that $x,y \in A_i$, $z \notin A_i$, $x,z \in A_j$, $y \notin A_j$, $y,z \in A_k$, $x \notin A_k$. I have not succeeded in determining or estimating

$f(n)$. Sauer asked how many such sets can be given if we assume $|A_k| = 3$. He conjectured that the answer is $[(\frac{n-1}{2})^2] + 1$. The $[(\frac{n-1}{2})^2]$ sets $x_1\ x_i\ x_j$, $2 \leq i \leq \frac{n}{2} < j \leq n$, do not have this property. Hajnal and I observed that if $A_i \subset G$, $|A_i| = 3$, $1 \leq i < Cn^2$ where $C$ is a sufficiently large constant, then there are 6 distinct elements $a,b,c,x,y,z$ so that $(abx)$, $(acy)$, $(bcz)$ are all $A$'s. We have not succeeded in determining the best value of $C$.

2. Denote by $f(k,n)$ the smallest integer such that if we split the k-tuples of a set of $f(k,n)$ elements into two classes there always is a set of $n$ elements all of whose k-tuples are in the same class. The fact that $f(k,n)$ is finite for every $k$ and $n$ is of course Ramsey's theorem. It is known that (the upper bound is due to Yackel).

(1) $c_1 n 2^{n/2} < f(2,n) < c_2\ 4^n \log\log n / n^{1/2} \log n$

The proof of the lower bound is probabilistic and non-constructive. It would be very desirable to obtain a constructive proof of the lower bound especially in view of the following circumstances.

$$(2) \qquad 2^{c_1 n^2} < f(3,n) < 2^{2^{c_2 n}}$$

The lower bound is obtained by probabilistic considerations, and it seems impossible to obtain more by these methods. (2) was proved by Hajnal, Rado and myself, and we believe that the upper bound gives the right order of magnitude.

For $k > 3$ we know that

$$2^{c_2 f(k-1,n)} < f(k,n) < 2^{c_1 f(k-1,n)}$$

Thus the case $k = 3$ is crucial.

The basic elements of an r graph (for $r = 2$ we get the ordinary graphs) are its r-tuples and vertices. $G^{(r)}(n,t)$ denotes an r graph of n vertices and t r-tuples. I proved that every

$$G^{(r)}(n, c_1\tbinom{n}{r}), \ 0 < c_1 < 1,$$

contains a $K^{(r)}(\ell,\cdots,\ell)$ for $\ell < c_2 (\log n)^{1/r}$, where $K^{(r)}(\ell,\cdots,\ell)$ is defined as follows: The vertices of $K^{(r)}(\ell,\cdots,\ell)$ are $x_i^{(j)}$, $1 \leq i \leq \ell$; $1 \leq j \leq r$, and its $k^r$ r-tuples are $\{x_{i_1}^{(1)}, x_{i_2}^{(2)},\cdots,x_{i_r}^{(r)}\}$, $1 \leq i_1,\cdots,i_r \leq \ell$. I also showed that in a certain sense this theorem is best possible, it fails for $c_3(\log n)^{1/r}$ if $c_3 = c_3(c_1)$ is sufficiently large. For $r = 2$ these results are due to Körvari and the Turans.

Define the density of an r graph $G^{(r)}(n;t)$ as

$$\frac{t}{\binom{n}{r}}$$

Our theorem can also be stated in the following form. Let $n$ be sufficiently large, then every r-graph of n vertices and positive density contains a large r-graph of density $\frac{r!}{r^r}$. I conjecture that there is an absolute constant $c_r$ such that if $n$ is sufficiently large then every r graph of r vertices and density

$\geq \frac{r!}{r^r} + \epsilon$ (i.e., every $G^{(r)}(n;\binom{n}{r}(\frac{r!}{r^r} + \epsilon))$ contains a large subgraph of density $\geq \frac{r!}{r^r} + c_r$. For $r = 2$, this and considerably more was proved by Stone and myself.

Recently I proved the following theorem: Split the r-tuples $r \geq 3$ of a set of n elements into two classes, Then there are elements

$$x_i^{(j)}, 1 \leq i \leq c_1(\log n)^{1/r-1}, 1 \leq j \leq r-1$$

such that all the r-tuples

$$(3) \qquad (x_{i_1}^{(1)}, x_{i_2}^{(1)}, x_{i_3}^{(2)}, \ldots, x_{i_r}^{(r-1)}),$$

$$1 \leq i_s \leq c_1(\log n)^{1/r-1}, \; 1 \leq s \leq r$$

belong to the same class. A simple probabilistic argument shows that the theorem fails for $c_2(\log n)^{1/r-1}$ if $c_2$ is sufficiently large. Also it is easy to see that for no $c < 1$ does a $G^{(r)}(n, c\binom{n}{r})$ necessarily contain a configuration of type (3). It is not clear if the theorem can be strengthened, e.g., let $r = 3$. Is it true that there are elements

$$X_i,\ Y_i,\ 1 \le i \le f(n),\ f(n)/_{\log\log n} \to \infty$$

so that all the pairs $(X_{i_1}, X_{i_2}, Y_j)$ and $(X_i, Y_{j_1}, Y_{j_2})$ belong to the same class? Perhaps this even holds with $f(n) > c(\log n)^{1/r-1}$.

The proof of (3) is quite complicated and I was assisted by some suggestion of J. Spencer.

Spencer and I proved the following theorem: Split the r-tuples of a set of n elements into two classes. Then for every $m \le n$ there is a $t \le m$ such that there is a set of t elements with at least

$$\frac{1}{2}\binom{t}{r} + c_1\, m^{\frac{r+1}{2}} \left(\log \frac{c_2\, n}{m}\right)^{\frac{r-1}{2}} \qquad (4)$$

r tuples of the same class.

Apart from the value of $c_1$ and $c_2$, (4) is best possible. The proof of (4) requires tricky combinatorial and and probabilistic bilities considerations. A slightly weaker form of (4) will soon appear in our paper in Networks. For applications to probabilistic methods in combinatorial analysis, see also our forthcoming book with J. Spencer.

1. P. Erdös, A. Hajnal, R. Rado, Partition relations for cardinal numbers, Acta Math. Acad. Sci. Hung. 16(1965), 93-196.

2. P. Erdös and G. Szekeres, On a combinatorial problem in geometry, Comp. Math. Vol. 2(1935).

3. P. Erdös and A. Stone, On the structure of linear graphs, Bull. Am. Math. Soc. 52(1946), 1087-1091.

4. P. Erdös, On some extremal problems in r-graphs, Discrete Math. Vol. 1(1971).

5. P. Erdös, Some remarks on the theory of graphs, Bull. Amer. Math. Soc. 53(1947) 292-294.

3. Rado and I investigated the following

question: Define $f(r,n)$ as the smallest integer such that if $|A_k| = n$, $1 \leq k \leq f(r,n)$, then one can always find $r$ A's which have pairwise the same intersection. We proved

$$(1) \qquad (r-1)^{n+1} < f(r,n) < c_r^n \, n!$$

Both the upper and the lower bound in (1) have been improved by Abbott by factors tending to infinity exponentially, but nobody has yet proved

$$(2) \qquad f(r,n) < c_r^n \, .$$

(2) is open even for $r = 3$. (2) would have many applications in number theory and combinatorial analysis and I several times offered 100 dollars for a proof or disproof of (2).

Denote by $g(r,n)$ the smallest integer such that if

$$|G| = n, \; A_i \subset G, \; 1 \leq i \leq g(r,n)$$

then there are always $r$ A's which have pairwise the same intersection. (1) implies

$$g(r,n) < 2^{n-c_r'\sqrt{n}}$$

and (2 ) would imply

$$g(r,n) < (2-\epsilon_r)^n.$$

Abbott and I observed that $\lim_{n=\infty} g(r,n)^{1/n}$ exists and we obtained some rough lower bounds for $g(3,n)$.

P. Erdös and R. Rado, Intersection theorems for systems of sets, J. London Math. Soc. 35(1960), 85-90. H. L. Abbott, Some remarks on a combinatorial theorem of Erdös and Rado, Canad. Math. Bull. 9(1966), 155-160.

4. A family of sets $\{A_k\}$ is said to have property B if there is a set S which meets all the $A_k$ but does not contain any of them. $m(n)$ is the smallest integer such that there are $m(n)$ sets $\{A_k\}$, $|A_k| = n, 1 \leq k \leq m(n)$; not having property B. It is known that

$$(1) \qquad 2^n(1 + \tfrac{4}{n})^{-1} < m(n) < n^2\, 2^{n+1}$$

$m(2) = 3$, $m(3) = 7$, $m(4)$ is not yet known. (Hanson showed $16 \leq m(4) \leq 29$).

Define $m^*(n)$ as the smallest family of sets $\{A_k\}$, $1 \leq k \leq m^*(n)$, which do not have property B and for which $|A_k| = n$, $|A_i \cap A_j| \leq 1$. $m^*(2) = 3$, $m^*(3) = 7$, $m^*(4)$ is unknown. It is known that $m^*(n)$ is finite for every $n$, and, in fact, Hajnal and I showed that for large $n$, $m^*(n) < 11^n$. As far as I know it is not even known that

$$(2) \qquad \lim_{n=\infty} (m^*(n))^{1/n}$$

exists. I am sure that the answer is affirmative and that the limit is greater than 2.

Abbott and I have made the following simple observation: Clearly all the subsets taken $n$ at a time of a set of $2n-1$ elements do not have property B. On the other hand if the family $\{A_k\}$, $1 \leq k \leq t$, $|A_k| = n$, $|A_i \cap A_j| \leq 1$ does not have property B then $|\bigcup_{k=1}^{t} A_k| = X$ must be very large. To see this, observe that by (1), $t \geq 2^n(1+\frac{4}{n})^{-1}$, thus since $|A_i \cap A_j| \leq 1$, we must have

$$(3) \qquad t\binom{n}{2} < \binom{x}{2} \text{ or } x \geq (1+ o(1))n2^{n/2}.$$

P. Erdös and A. Hajnal, On a property of families of sets, Acta Math. Acad. Sci. Hungar. 12(1961), 87-123; P. Erdös, On a combinatorial problem II, ibid. 15(1964), 445-447; W. Schmidt, On a problem of Erdös and Hajnal, ibid.

5. V. T. Sos and I considered the following question. Color the edges of a $K_n$ (complete graph of n vertices) by three colors so that we get the largest number of triangles all of whose edges get different colors. Denote this number by $f(n)$. It is easy to see that

$$\lim f(n)/\binom{n}{3} = c$$

exists, but we could not determine $c$. $f(3) = 1$, $f(4) = (4)$, $f(5) = 7$. $f(n) \geq (1 + o(1))\frac{5}{64} n^3$ is easy, and perhaps this is best possible, or $c = \frac{15}{32}$.

Clearly many generalizations are possible.

Recently Hajnal and I considered the following modified problem: Let $X_1,\cdots,X_n$ be the vertices of $K_n$. Color the edges by three colors I, II and III. Denote by $g(n)$ the largest number of triangles $(X_i, X_j, X_\ell)$, $i < j < \ell$ so that $(X_i, X_j)$ has the color I, $(X_j, X_\ell)$ color II and $(X_i, X_\ell)$ color III. Perhaps

$$\lim_{n\to\infty} g(n)/\binom{n}{3} = 1/4$$

$g(n) \geq (\frac{1}{24} + o(1))\, n^3$ is easy to see, the upper bound seems more difficult.

6. I now discuss two further questions connected with Ramsey's theorem. It will be useful to introduce the arrow symbol of Rado (which we avoided in 2.): $n \to (a,b)^k$ means that if we split the k-tuples of a set of n-elements into two classes there either is a set of a elements all whose k-tuples are in class I or a set of b elements all whose k-tuples are in class II. $n \not\to (a,b)^k$ means that there is a splitting for which

the above does not hold. $f(k,n)$ of 2. thus satisfies

$$f(k,n) \to (n,n)^k,\ f(k,n)-1 \not\to (n,n)^k.$$

$n \to (a,[{b \atop t}])^3$ means that if we split the triplets of a set of $n$ elements into two classes there either is a set of $a$ elements all whose triplets are in class I or a set of b elements which contain at least t triplets of class II. This symbol was extensively investigated for infinite cardinals in our triple paper with Hajnal and Rado.

Hajnal and I recently showed

$$(1)\ n \to (n^{1/2};[{4 \atop 2}])^3,\ n \not\to (c_1 \log n,\ [{4 \atop 3}])^3,$$
$$n \to (c_2 \log n,\ [{4 \atop 3}])^3,$$

(1) suggested the following conjecture: there is an $h(t)$ so that

$$(2)\ n \to (n^{\alpha_t},\ [{t \atop h(t)}])^3, \text{ but}$$
$$n \not\to (c_t \log n,\ [{t \atop h(t)+1}])^3.$$

We know that $h(4) = 2$, $h(5) = 4$, $h(6) = 8$, $h(7) \geq 13$. It is almost certain that

$h(7) = 13$.

By probabilistic arguments we can show ($g(t)$ is defined in 5.)

$$n \not\rightarrow (c \log n, [{}^{\,t}_{g(t)+1}])^3.$$

Unfortunately, we are very far from being able to show

$$n \rightarrow (n^\alpha, [{}^{\,t}_{g(t)}])^3$$

We can show

$$n \rightarrow (n^\alpha, [{}^{t}_{\ell}])^3, \quad \text{where} \quad \ell = (1 + o(1)) \frac{t^3}{26} .$$

All these questions could of course be investigated for $r > 3$ too, but we have not yet had the time to do this. For $r = 2$ it is known that $n \rightarrow (cn^{1/t}, t+1)^2$.

Bercov and Hobby proved the following Ramsey type theorem. Let $G$ be a set. Two disjoint non-empty classes of r-tuples of $G$ are said to have property $P(r;u,v)$ if every u-tuple of $G$ which contains an r-tuple of class I also contains an r tuple of class II and every v tuple of $G$ which contains an r tuple of class II also contains an r-tuple of class I. Their theorem asserts that there is a smallest integer $F(r;\ u,v)$ so that for $|G| \geq F(r;\ u,v)$ no classes of property $P(r;\ u,v)$ exist.

Clearly $F(r;\ u,v) \to (u,v)^r$, and I thought that perhaps $F(r;\ u,v)$ might be the smallest integer with this property, in other words $F(r;\ u,v)$ coincides with the Ramsey function. Abbott, Milner and I showed this for $r = 2$, $u \leq 4$, $v \leq 4$.

Milner and I, in fact, observed that if $|G| \geq 11$ and one has a system $P(2;4,4)$ on $G$, then there can be no empty quadruple i.e., every quadruple contains an edge of class I and II, and since $11 \nrightarrow (4,9)^2$ our conjecture follows for $r = 2$, $u = 4$, $r = 4$. Perhaps this situation is true generally. Let $m$ be the largest integer for which $m \nrightarrow (u,v)^r$. Then if $|G| = m$ and there are two classes of r-tuples of $G$ having property $P(r;u,v)$, every v-tuple must contain an r-tuple of both classes. This, if true, would imply our conjecture. I can only prove it for $r = 2$, $u = 3$ and every $v \geq 3$. Kleitman just informs me that he showed that $F(2,5,5)$ coincides with the corresponding Ramsey number.

R. D. Bercov and CH. R. Hobby, Permutation groups on unordered sets, Math. Zeitschrift 115(1970), 165-168.

To finish this report I state two problems from combinatorial geometry.

The following question is due to G. Simmons: Let there be given a set of $2n$ points no three on a line, $X_1, \cdots, X_{2n}$. A line $(X_i, X_j)$ is called a bisector of the set if $n-1$ points are on both sides of this line. Simmons asked: What is the largest number of bisectors? Denote this maximum by $f(n)$. Straus proved $f(n) > cn \log n$, and Lovasz proved $f(n) < cn^{3/2}$. Several papers on these and related subjects will appear in the near future.

Straus and I recently asked the following question: Let there be given n points in the plane $X_1, \cdots, X_n$. Join $k\,n$ pairs $(X_i, X_j)$ by a path. Prove that there is a straight line which cuts at least $k(k+1)$ of these paths. We can prove this with $\frac{k^2}{2}$ and can show that $k(k+1)$ is best possible if it is true, also we can show it for $k = 1$ and $k = 2$.

Sylvester asked the following question: Let there be given n points in the plane no four on a line. What is the maximum number of lines which pass through three of our points? Sylvester showed that there can be $\frac{1}{3}\binom{n}{2}-c_1 n$ such lines and a result of Kelly and Moser implies that the number of such lines is less than $\frac{1}{3}\binom{n}{2}-c_2 n$.

More generally let $X_1, \cdots, X_n$ be n points no $r+1$ of them is on a straight line. Denote by $f(r,n)$ the largest number of lines which go through precisely $r$ of the points. I conjectured $f(r,n) = o(n^2)$ but could not even prove $\lim f(r,n)/_n = \infty$. Karteszi proved

(1) $$f(r,n) > c_r \, n\log n$$

by showing $f(r,rn) \geq n + r\, f(r,n)$.

Croft and I considered the function $f^*(r,n)$ where $f^*(r,n)$ denotes the maximum number of lines which pass through precisely $r$ of the points $X_i$, we now no longer assume that no $r+1$ of the points are on a line. The example of the lattice points in the plane, easily shows that

$$f^*(r,n) > c'_r \, n^2$$

$c'_r < \frac{1}{\binom{r}{}}$ is immediate, and we conjectured that $f^*(r,n) < \epsilon_r n^2$ where $\epsilon_r \to 0$ as $r \to \infty$.

W. O. J. Moser and L. M. Kelly, On the number of ordinary lines determined by n-points, Canad. J. Math. 10(1958), 270-279.

The paper of Karteszi appeared in Hungarian, Kozepiskolai Mat. Lapok 1962.