PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

RAMSEY BOUNDS FOR GRAPH PRODUCTS

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Here we show that Ramsey numbers $M(k_1, \cdots, k_n)$ give sharp upper bounds for the independence numbers of product graphs, in terms of the independence numbers of the factors.

The Ramsey number $M(k_1, \cdots, k_n)$ is the smallest integer m with the property that no matter how the $\binom{m}{2}$ edges of the complete graph on m nodes are partitioned into n colors, there will be at least one index i for which a complete subgraph on k_i nodes has all of its edges in the ith color. Ramsey's Theorem tells that these numbers exist but only a few exact values are known.

The complement graph $\bar{G}$ has the same nodes as G and the complementary set of edges.

The independence number $\alpha(G)$ of a graph G, is the largest number of nodes in any complete subgraph of $\bar{G}$.

The product $G_1 \times \cdots \times G_n$ of graphs $G_1, \cdots, G_n$ is the graph whose nodes are all the ordered n-tuples $(a_1, \cdots, a_n)$ in which a_i is a node of G_i for each i from 1 to n, and whose edges are as follows. A set of two nodes $\{(a_1, \cdots, a_n), (b_1, \cdots, b_n)\}$ will be an edge of $G_1 \times \cdots \times G_n$ if and only if the nodes are distinct and for each i from 1 to n, $a_i = b_i$ or $\{a_i, b_i\}$ is an edge of G_i.

THEOREM 1. *For arbitrary graphs $G_1, \cdots, G_n$*

$$\alpha(G_1 \times \cdots \times G_n) < M(\alpha(G_1) + 1, \cdots, \alpha(G_n) + 1) .$$

Proof. We have a complete subgraph of $\overline{G_1 \times \cdots \times G_n}$ on $\alpha(G_1 \times \cdots \times G_n)$ nodes. Its edges can be n colored by the following rule: give $\{(a_1, \cdots, a_n), (x_1, \cdots, x_n)\}$ color i if i is the first index for which $\{a_i, x_i\}$ is an edge of $\bar{G}_i$.

With this coloration any case where all the edges on k nodes have color i requires a complete k subgraph of $\bar{G}_i$ and so requires $k < \alpha(G_i) + 1$. With the definition of the Ramsey number this ensures that

$$\alpha(G_1 \times \cdots \times G_n) < M(\alpha(G_1) + 1, \cdots, \alpha(G_n) + 1) .$$

THEOREM 2. *If $k_1, \cdots, k_n$ are given, there exist graphs $G_1, \cdots, G_n$ such that for each index i from 1 to n, $\alpha(G_i) = k_i$ and*

$$\alpha(G_1 \times \cdots \times G_n) = M(k_1 + 1, \cdots, k_n + 1) - 1 .$$

Proof. From the definition of the Ramsey number there must exist an n color partition of the edges of the complete graph on $M(k_1+1, \cdots, k_n+1) - 1 = m$ modes such that for every i from 1 to n the largest complete subgraph in the ith color is on k_i nodes. For each i let G_i be the graph on the same m nodes having all the edges not of color i. Thus for each i, $\alpha(G_i) = k_i$. These G_i make the diagonal a complete m subgraph of $\overline{G_1 \times \cdots \times G_n}$, and so

$$\alpha(G_1 \times \cdots \times G_n) \geqq m .$$

Applying Theorem 1 we have

$$\alpha(G_1 \times \cdots \times G_n) = M(k_1+1, \cdots, k_n+1) - 1$$

THEOREM 3. *If n and k are given, there exists a graph G such that $\alpha(G) = k$ and putting $k_i = k$ for every i,*

$$\alpha(G^n) = M(k_1+1, \cdots, k_n+1) - 1 .$$

Proof. With $m = M(k_1+1, \cdots, k_n+1) - 1$ and every $k_i = k$, refer to the graphs $G_1, \cdots, G_n$ as specified for Theorem 2. Now construct G as follows. Let the nodes of G be all the ordered pairs (a, i) such that $1 \leqq i \leqq n$ and a is a node of G_i. Let $\{(a, i), (b, j)\}$ be an edge of G if and only if $i \neq j$ or $\{a, b\}$ is an edge of G_i.

Thus constructed $\alpha(G) = k$ because each $\alpha(G_i) = k$. $\overline{G^n}$ will have a subgraph isomorphic to $\overline{G_1 \times \cdots \times G_n}$ and consequently

$$\alpha(G^n) \geqq \alpha(G_1 \times \cdots \times G_n) = m .$$

So again with Theorem 1 we have

$$\alpha(G^n) = m = M(k_1+1, \cdots, k_n+1) - 1 .$$

A question remains whether for every k, n with

$$k^2 \leqq n < M(k+1, k+1)$$

there exists G such that $\alpha(G) = k$ and $\alpha(G^2) = n$. It is known that $M(4, 4) = 18$, and for each n between 9 and 17 we have found a graph G such that $\alpha(G) = 3$ and $\alpha(G^2) = n$. However it is only known that $37 < M(5, 5) < 58$ and for example we have no proof that there exists a graph G such that $\alpha(G) = 4$ and $\alpha(G^2) = M(5, 5) - 2$.

Received May 25, 1970. The work of the latter two authors represents one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NAS 7-100, sponsored by the National Aeronautics and Space Administration.

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