PARTITION RELATIONS FOR η_α-SETS

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1. *Introduction*

For every ordinal number α, an ordered set S is called an *η_α-set* if the following *condition P_α* is satisfied: if A and B are subsets of S, each of cardinal number less than $\aleph_\alpha$, and if $a < b$ whenever $a \in A$ and $b \in B$, then there exists $x \in S$ such that $a < x$ for all $a \in A$ and $x < b$ for all $b \in B$. η_α-sets were first introduced and studied by Hausdorff [**1**] and further properties of such sets can be found in [**2**]. We denote by C_α the set of all sequences $(\varepsilon_\nu)_{\nu<\omega_\alpha}$ such that (i) $\varepsilon_\nu \in \{0, 1\}$ $(\nu < \omega_\alpha)$, (ii) $\varepsilon_\nu \neq 0$ for some $\nu < \omega_\alpha$, (iii) given any $\delta < \omega_\alpha$, there is ν such that $\delta \leqslant \nu < \omega_\alpha$ and $\varepsilon_\nu = 0$. Also, we denote by R_α the set of those elements of C_α which have a last 1, i.e. for which there is $\delta < \omega_\alpha$ such that $\varepsilon_\delta = 1$ and $\varepsilon_\nu = 0$ for $\delta < \nu < \omega_\alpha$. We order C_α and R_α lexicographically and denote the order types of these sets by λ_α and η_α respectively. In [**1**; p. 179], Hausdorff denotes by η_α a somewhat different order type. Let us for the moment denote Hausdorff's type by $\eta_\alpha{}^H$. Then $\eta_\alpha \leqslant \eta_\alpha{}^H \leqslant \eta_\alpha$, and for the purpose we have in mind the two types are not essentially different from each other. λ_α and η_α are generalizations of the types of the linear continuum and the set of rational numbers respectively, ordered by magnitude; the latter two types are λ_0 and η_0. It is well known that R_α is an η_α-set if $\aleph_\alpha$ is regular.†

The cardinal of a set S is denoted by $|S|$, and for every cardinal r we put‡ $[S]^r = \{X : X \subset S;\ |X| = r\}$. The *partition relation*

$$\theta \to (\theta_0, \theta_1)^r \tag{1}$$

connecting order types θ, θ_0, θ_1 and a cardinal r was first introduced in [**4**] and means that the following statement is true. If S is an ordered set of order type $\mathrm{tp}\, S = \theta$ and if $[S]^r = K_0 \cup K_1$ then there are a set $T \subset S$ and a number $i < 2$ such that $\mathrm{tp}\, T = \theta_i$ and $[T]^r \subset K_i$. The negation of (1) is written $\theta \nrightarrow (\theta_0, \theta_1)^r$. The relation (1) has obvious analogues where some or all of the order types are replaced by cardinal numbers (see [**5**]).

Erdös [**6**] and Kurepa [**7**] independently proved that, under the assumption of the generalised continuum hypothesis (GCH)

$$\aleph_{\alpha+2} \to (\aleph_{\alpha+2}, \aleph_{\alpha+1})^2. \tag{2}$$

Partition relations of a more general kind are discussed in [**5**] where a great variety of such relations are established. Erdös, Hajnal and Rado [**8**] have subsequently given an almost complete discussion of partition relations for cardinal numbers.

Received 30 June, 1969. This work was completed when the authors were together at Reading in the sping of 1967. P. Erdös was then in receipt of a grant from the Science Research Council, London.

† [**1**, **3**]. If $\aleph_\alpha$ is singular then R_α is not an η_α-set. It should be noted that if $\aleph_\alpha$ is singular then the condition P_α implies $P_{\alpha+1}$ so that every η_α-set is also an $\eta_{\alpha+1}$-set.

‡ $X \subset S$ denotes inclusion in the wide sense.

[J. LONDON MATH. SOC. (2), 3 (1971), 193–204]

However, there remain many unsettled partition problems for order types or ordinal numbers. Perhaps the most striking problem of this kind is to decide whether

$$\omega^{\omega} \rightarrow (\omega^{\omega}, 3)^2$$

is true or false.*

In this paper we are mainly concerned with partition relations involving the types η_α. Assuming GCH we shall prove (Theorem 1) that

$$\eta_{\alpha+2} \rightarrow (\eta_{\alpha+2}, \aleph_{\alpha+1})^2 \tag{3}$$

which is a strengthening of (2). In fact our Theorem 1 gives the more general result that

$$\eta_{\alpha+1} \rightarrow (\eta_{\alpha+1}, \aleph_{\mathrm{cf}(\alpha)})^2 \tag{4}$$

which corresponds to the cardinal relation

$$\aleph_{\alpha+1} \rightarrow (\aleph_{\alpha+1}, \aleph_{\mathrm{cf}(\alpha)})^2 \tag{5}$$

proved in **[8]**. A very simple argument **[5]** shows that

$$\eta_0 \rightarrow (\eta_0, \aleph_0)^2, \tag{6}$$

and this argument requires only formal generalisation to yield

$$\eta_\alpha \rightarrow (\eta_\alpha, \aleph_0)^2 \tag{7}$$

whenever $\aleph_\alpha$ is regular. We do not give this more general argument here since (7) also follows from Theorem 1 and the known result (6). The cardinal partition relation

$$\aleph_\alpha \rightarrow (\aleph_\alpha, \aleph_0)^2$$

due to Dushnik and Miller **[9]** holds without restriction on α but we are unable to extend (7) to the case of singular $\aleph_\alpha$. Thus we cannot prove

$$\eta_\omega \rightarrow (\eta_\omega, \aleph_0)^2. \tag{8}$$

In fact, we cannot even decide whether the much weaker relation

$$\eta_\omega \rightarrow (\eta_\omega, 3)^2$$

is true. It can be shown, however, that the relation

$$\eta_\omega \rightarrow (\eta_\omega, \mathscr{C}_4)^2$$

holds, where $\mathscr{C}_4$ denotes a circuit of length 4. This last relation means that if the vertices of a combinatorial graph form an ordered set of type η_ω then there is either an independent, i.e. edge-free, set of nodes of type η_ω, or the graph contains a circuit of length 4. The proof of this result is not given here; it can be obtained by methods used in a forthcoming paper by Erdös, Hajnal and Milner **[10]**. By way of contrast to the undecided relation (8) we shall prove (Theorem 2) that

$$\zeta_\omega \rightarrow (\zeta_\omega, \aleph_0)^2,$$

where $\zeta_\omega = \eta_0 + \eta_1 + \eta_2 + \ldots$. We remark finally that

$$\eta_\alpha \rightarrow (\eta_\alpha, \aleph_\alpha)^2$$

holds for $\alpha = 0$ and for those hypothetical "measurable" cardinals $\aleph_\alpha$ for which

* (Added in proof): This relation has in the meantime been proved by C. C. Chang.

in the boolean algebra of the subsets of an $\aleph_\alpha$-set every $\aleph_\alpha$-additive ideal can be extended to an $\aleph_\alpha$-additive prime ideal.

Erdös and Hajnal [11] observed that if $2^{\aleph_0} = \aleph_1$ then every graph on the set of real numbers either contains an independent set of the second category or has an infinite complete subgraph. An argument similar to theirs gives the following result (Theorem 4). If $\aleph_\alpha$ is regular and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ then every graph on C_α which does not contain a complete subgraph of power $\aleph_{\mathrm{cf}(\alpha)}$ contains an independent set which is not the union of $\aleph_\alpha$ nowhere dense sets.

For the partitioning of triplets, i.e. the "exponent" $r = 3$, we give only negative results. Let ϕ denote any arbitrary order type. We shall show (Theorem 5) that

$$\phi \nrightarrow (\omega+\omega^*,\ 4)^3$$

and

$$\phi \nrightarrow (\omega^*+\omega,\ 4)^3.$$

These two relations were first proved, by a method which differs from ours, by A. H. Kruse [12] who obtained the stronger result: if $\omega, \omega^* \leqslant \psi$ and $r \geqslant 3$ then $\phi \nrightarrow (\psi, r+1)^r$. Also, we shall prove that, for all ϕ,

$$\phi \nrightarrow (\omega+\omega^* \vee \omega^*+\omega,\ 5)^3,$$

a relation which means that if $\mathrm{tp}\, S = \phi$ then there is a partition $[S]^3 = K_0 \cup K_1$ with the property that (i) $[T]^3 \not\subset K_0$ for all $T \subset S$ such that either $\mathrm{tp}\, T = \omega+\omega^*$ or $\mathrm{tp}\, T = \omega^*+\omega$, (ii) $[U]^3 \not\subset K_1$ for all $U \in [S]^5$. At present we cannot prove the stronger relation

$$\phi \nrightarrow (\omega+\omega^* \vee \omega^*+\omega,\ 4)^3.$$

We can easily give a partition of $[R_0]^3$ into two classes K_0 and K_1 such that K_1 contains at most three of the four elements of $[X]^3$ for every $X \in [R_0]^4$, and such that in addition $[T]^3 \not\subset K_0$ whenever T is a subset of R_0 which is dense in some interval. However, we cannot exclude the possibility that $[U]^3 \subset K_0$ for some subset U of R_0 of type η_0. Thus, in the notation used in [8; p. 157], which is explained at the end of the present note, we cannot decide whether*

$$\eta_0 \to \left(\eta_0, \begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)^3.$$

In fact, we cannot answer the following apparently easier question. If $E \subset [R_0]^3$ and $|[X]^3 \cap E| < 3$ whenever $X \in [R_0]^4$, does it follow that there are an element x of R_0 and subsets L and U of R_0, each of type η_0, such that $l < x < u$ and $\{l, x, u\} \notin E$ whenever $l \in L$ and $u \in V$?

2. *Additional notation*

The *obliteration sign* $\wedge$ placed above any symbol means that that symbol is to be removed. Thus $\{x_0, \ldots, \hat{x}_\alpha\}$ denotes the set $\{x_\nu : \nu < \alpha\}$. If ρ is a binary relation then $\{x_0, \ldots, \hat{x}_\alpha\}_\rho$ denotes the set $\{x_0, \ldots, \hat{x}_\alpha\}$ and at the same time expresses the condition that $x_\mu \rho x_\nu$ for $\mu < \nu < \rho$.

The *cofinality cardinal* of $\aleph_\alpha$, written $\aleph_{\mathrm{cf}(\alpha)}$, is the least cardinal $\aleph_\beta$ such that $\aleph_\alpha$ can be expressed as the sum of $\aleph_\beta$ cardinals each less than $\aleph_\alpha$. The cardinal $\aleph_\alpha$ is *regular* if $a = \mathrm{cf}(\alpha)$ and *singular* if $\alpha > \mathrm{cf}(\alpha)$.

* (Added in proof): F. Galvin has now proved this relation.

Suppose that S is totally ordered by $<$. The order type of S under this order is denoted by $\mathrm{tp}_< S$ or simply by $\mathrm{tp}\, S$ if there is no confusion about the intended ordering. For $X \subset S$ we write $L(X) = \{y : y \in S;\ y < x \textit{ for all } x \in X\}$. In this paper we use the term "interval of S" in a special sense. An *interval* of an ordered set S is a non-empty set of the form $(a, b) = \{x : a < x < b\}$, where $\{a, b\}_< \subset S$.† If $|S| \geqslant \aleph_0$ then there are only $|S|^2 = |S|$ intervals of S. A set D is *dense* in S if every interval of S contains an element of D. A set N is *nowhere dense* in S if every interval I of S contains a subinterval I' such that $I' \cap N = \phi$. A set $B \subset C_\alpha$ is of the *first category* (in C_α) if B is the union of $\aleph_\alpha$ nowhere dense subsets of C_α; otherwise B is of *second category*. If $x = (\varepsilon_\nu)_{\nu<\omega_\alpha} \in R_\alpha$ we put $\delta(x) = \nu_0$ if $\varepsilon_{\nu_0} = 1$ and $\varepsilon_\nu = 0$ for $\nu_0 < \nu < \omega_\alpha$.

We shall require a property of systems of intervals in R_α. Let $\aleph_\alpha$ be regular and $n < \omega_\alpha$, and consider a decreasing nest of intervals $I_0, \ldots, \hat{I}_n$ in R_α, so that $I_0 \supset \ldots \supset \hat{I}_n$. We now show that there is an interval

$$I \subset \bigcap (\nu < n) I_\nu. \tag{9}$$

Let $I_\nu = (a_\nu, b_\nu)$ $(\nu < n)$. By condition P_α, there is $x_\nu \in I_\nu$ $(\nu < n)$. Then, for $\mu < \nu < n$, we have $x_\nu \in I_\nu \subset I_\mu$; $a_\nu < x_\nu < b_\mu$; $a_\mu < x_\nu < b_\nu$. Thus every member of $A = \{a_0, \ldots, \hat{a}_n\}$ precedes every member of $B = \{b_0, \ldots, \hat{b}_n\}$, and since $|n| < \aleph_\alpha$ two applications of P_α yield elements x, y of R_α such that

$$a_0, \ldots, \hat{a}_n < x < y < b_0, \ldots, \hat{b}_n.$$

Then (9) holds for $I = (x, y)$.

It now follows, just as in the case of the real line, that C_α itself is of second category provided $\aleph_\alpha$ is regular. For, let $n = \omega_\alpha$ and let $N_0, \ldots, \hat{N}_n$ be sets each nowhere dense in C_α. Let $m < n$ and $a_0 < \ldots < \hat{a}_m < \hat{b}_m < \ldots < b_0$, where all $a_\nu, b_\nu \in R_\alpha$. Let $(a_\nu, b_\nu)_C \bigcap N_\nu = \varnothing$ for $\nu < m$, where $(a, b)_C = \{x \in C_\alpha : a < x < b\}$. We define a_m, b_m: since $m < n = \omega_\alpha$ it follows from (9) that there are a, $b \in R_\alpha$ such that $(a, b)_R \subset \bigcap (\nu < m)(a_\nu, b_\nu)_R$, where $(u, v)_R = \{x \in R_\alpha : u < x < v\}$. There are $a', b' \in C_\alpha$ with $a < a' < b' < b$ and $(a', b')_C \cap N_m = \varnothing$. There are a_m, $b_m \in R_\alpha$ with $a' < a_m < b_m < b'$. This defines $a_0, \ldots, \hat{a}_n, b_0, \ldots, \hat{b}_n \in R_\alpha$ such that

$$a_0 < \ldots < \hat{a}_n < \hat{b}_n < \ldots < b_0 \quad \text{and} \quad (a_\nu, b_\nu)_C \cap N_\nu = \varnothing \quad \text{for} \quad \nu < n.$$

There is $x = \sup \{a_0, \ldots, \hat{a}_n\} \in C_\alpha$ and we have $a_\nu < x < b_\nu$; $x \notin N_\nu$; $x \notin N_0 \cup \ldots \cup \hat{N}_n$, so that $N_0 \cup \ldots \cup \hat{N}_n \neq C_\alpha$, and C_α is of second category*.

If θ and ϕ are order types then $\theta \geqslant \phi$, also written as $\phi \leqslant \theta$, means that every set of type θ contains a subset of type ϕ. The negation of $\theta \geqslant \phi$ is written $\theta \ngeqslant \phi$. It is easily seen that if $\aleph_\alpha$ is regular and S is dense in some interval of R_α then $\mathrm{tp}\, S \geqslant \eta_\alpha$. Although not quite so obvious this is also true for singular $\aleph_\alpha$ (see **[10]**).

An *r-graph* is an ordered pair $G = (S, E)$ such that $E \subset [S]^r$. S is called the set of *vertices* and E the set of *edges* (r-edges). A *complete a-subgraph* of G is a set $S' \in [S]^a$ such that $[S']^r \subset E$, A set S'' is an *independent* subset if $[S'']^r \cap E = \varnothing$. A *graph* is simply a 2-graph; in this special case we denote, for $x \in S$, by $E(x)$ the

† Usually, an interval is every set $I \subset S$ such that whenever x, $y \in I$ and $x < z < y$ then $z \in I$. With this usual definition there are, for instance, $2^{\aleph_0}$ intervals in the set of rationals, whereas with our definition there are only $\aleph_0$ intervals.

* (Added in proof): If $\aleph_\alpha$ is singular and G.C.H. is assumed then C_α is not of second category.

set $\{y : \{x, y\} \in E\}$ of vertices joined to x by an edge. For $S' \subset S$ we put

$$E(S') = \bigcup (x \in S')\, E(x).$$

If an ordinal π has no immediate predecessor, i.e. if $\pi \neq \mu+1$ for every μ, then we put $\pi^- = \pi$, and we put $(\mu+1)^- = \mu$ for every μ.

3. *The main results*

THEOREM 1. *Let* $\alpha = \mathrm{cf}(\alpha) > \beta$ *and*

$$\aleph_{\alpha_0}^k < \aleph_\alpha \qquad (\alpha_0 < \alpha;\ k < \aleph_\beta). \tag{10}$$

Then
$$\eta_\alpha \to (\eta_\alpha, \aleph_\beta)^2.$$

If the generalised continuum hypothesis is assumed then the theorem gives the results (3) and (4).

The condition (10) is satisfied if $\aleph_\alpha$ is "strongly inaccessible", and in this case we have

$$\eta_\alpha \to (\eta_\alpha, \aleph_\beta)^2 \qquad (\beta < \alpha).$$

The corresponding relation for cardinals was proved in [**8**]. Our proof of Theorem 1 has certain features in common with the proof of the Ramification Lemma in [**8**].

Proof. Let $\mathrm{tp}\, S = \eta_\alpha$; $G = (S, E)$; $E \subset [S]^2$. Suppose that the graph G has no independent set of type η_α. We have to show that G has a complete $\aleph_\beta$-subgraph. We shall define ordinals $n(v_0, \ldots, \hat{v}_\rho) < \omega_\alpha$ for $\rho < \omega_\beta$ and $v_\sigma < n(v_0, \ldots, \hat{v}_\sigma)$ $(\sigma < \rho)$. Put

$$N_\rho = \{(v_0, \ldots, \hat{v}_\rho) : v_\sigma < n(v_0, \ldots, \hat{v}_\sigma) \quad \text{for} \quad \sigma < \rho\} \qquad (\rho \leqslant \omega_\beta).$$

We shall also define intervals $S_\rho \subset S$ $(\rho \leqslant \omega_\beta)$; subsets $Y(v)$ of S $(v \in N_\rho;\ \rho \leqslant \omega_\beta)$; elements $x(v)$ of S $(v \in N_{\rho+1};\ \rho < \omega_\beta)$. These will satisfy the following relations for $\rho \leqslant \omega_\beta$.

$$S_\sigma \supset S_\rho \qquad (\sigma < \rho), \tag{11}$$

$$S_\rho = \bigcup (v \in N_\rho)\, Y(v), \tag{12}$$

$$Y(v') \cap Y(v'') = \varnothing\ (\{v', v''\}_{\neq} \subset N_\rho), \tag{13}$$

$$\{x(v_0, \ldots, v_\sigma)\} \cup Y(v_0, \ldots, \hat{v}_\rho) \subset Y(v_0, \ldots, \hat{v}_\sigma)\big(\sigma < \rho;\ (v_0, \ldots, \hat{v}_\rho) \in N_\rho\big), \tag{14}$$

$$\{x(v), x\} \in E \quad \big(v \in N_{\sigma+1};\ x \in Y(v);\ \sigma < \rho\big). \tag{15}$$

If these ordinals, intervals, sets and elements have been defined so that (11)–(15) hold then we can complete the proof as follows. S_{ω_β} is an interval and hence $S_{\omega_\beta} \neq \varnothing$. By (12), there is $v = (v_0, \ldots, \hat{v}_{\omega_\beta}) \in N_{\omega_\beta}$ such that $Y(v) \neq \varnothing$. Let $\sigma < \rho < \omega_\beta$. By (15), $\{x(v_0, \ldots, v_\sigma), x\} \in E$ for $x \in Y(v_0, \ldots, v_\sigma)$. By (14),

$$x(v_0, \ldots, v_\rho) \in Y(v_0, \ldots, \hat{v}_\rho) \subset Y(v_0, \ldots, v_\sigma).$$

Hence
$$[\{x(v_0, \ldots, v_\rho) : \rho < \omega_\beta\}_{\neq}]^2 \subset E,$$

and the theorem is proved.

We now define the ordinals, intervals, sets and elements. Let $\pi \leqslant \omega_\beta$ and suppose we have already defined: ordinals $n(v_0, \ldots, \hat{v}_\sigma)$ for $\sigma < \rho < \pi$ and $v_\tau < n(v_0, \ldots, \hat{v}_\tau)$ $(\tau < \sigma)$, elements $x(v)$ for $v \in N_{\sigma+1}$ and $\sigma < \rho < \pi$, sets $Y(v)$ for $v \in N_\rho$ and $\rho < \pi$, and intervals S_ρ for $\rho < \pi$, in such a way that (11)–(15) hold for

$\rho < \pi$. If (i) $\pi = \pi^-$ then N_π is already defined—we note that N_0 has a single element *viz.* the sequence $\square$ of length 0—and (15) holds for $\rho = \pi$, and we only have to define (a) $Y(v)$ for $v \in N_\pi$ and (b) S_π so that (11)–(14) hold for $\rho = \pi$. If (ii) $\pi = \mu+1$ then N_μ is defined and we must define (a) $n(v)$ for $v \in N_\mu$ (which then defines $N_{\mu+1}$) and (b) $x(v)$ and $Y(v)$ for $v \in N_{\mu+1}$, and (c) $S_{\mu+1}$, so that (11)–(15) hold for $\rho = \mu+1$.

Case 1. $\pi = \pi^-$. Since $S_0, \ldots, \hat{S}_\pi$ are intervals in S such that $S_0 \supset \ldots \supset \hat{S}_\pi$, and we have $\pi \leqslant \omega_\beta < \omega_\alpha$, there is an interval $S_\pi \subset S \cap S_0 \cap \ldots \cap \hat{S}_\pi$. Put

$$Y(v_0, \ldots, \hat{v}_\pi) = S_\pi \cap \bigcap(\rho < \pi)\, Y(v_0, \ldots, \hat{v}_\rho)$$

for $v_\sigma < n(v_0, \ldots, \hat{v}_\sigma)$ $(\sigma < \pi)$. Thus, if $\pi = 0$ we choose an interval $S_0 \subset S$ and put $Y(\square) = S_0$. From the definitions and the induction hypothesis we deduce that

$$S_\pi \supset \bigcup(v \in N_\pi)\, Y(v) = \bigcup((v_0, \ldots, \hat{v}_\pi) \in N_\pi)\, S_\pi \cap \bigcap(\rho < \pi)\, Y(v_0, \ldots, \hat{v}_\rho)$$
$$= S_\pi \cap \Big(\bigcup((v_0, \ldots, \hat{v}_\pi) \in N_\pi) \bigcap (\rho < \pi)\, Y(v_0, \ldots, \hat{v}_\rho)\Big).$$

Now, by the distributive law,

$$\bigcap(\rho < \pi) \bigcup((v_0, \ldots, \hat{v}_\rho) \in N_\rho)\, Y(v_0, \ldots, \hat{v}_\rho)$$
$$= \bigcup((v_{\sigma 0}, \ldots, \hat{v}_{\sigma\sigma}) \in N_\sigma \;\; \textit{for} \;\; \sigma < \pi) \bigcap(\rho < \pi)\, Y(v_{\rho 0}, \ldots, \hat{v}_{\rho\rho}) = A,$$

say. But $Y(v_{\rho 0}, \ldots, \hat{v}_{\rho\rho}) \cap Y(v_{\tau 0}, \ldots, \hat{v}_{\tau\tau}) = \varnothing$ if $\rho \leqslant \tau$ and $(v_{\rho 0}, \ldots, \hat{v}_{\rho\rho}) \neq (v_{\tau 0}, \ldots, \hat{v}_{\tau\rho})$. Hence $A = \bigcup((v_0, \ldots, \hat{v}_\pi) \in N_\pi) \bigcap(\rho < \pi)\, Y(v_0, \ldots, \hat{v}_\rho)$, and

$$S_\pi \supset \bigcup(v \in N_\pi)\, Y(v) = S_\pi \cap A$$
$$= S_\pi \cap \bigcap(\rho < \pi) \bigcup((v_0, \ldots, \hat{v}_\rho) \in N_\rho)\, Y(v_0, \ldots, \hat{v}_\rho)$$
$$= S_\pi \cap \bigcap(\rho < \pi)\, S_\rho = S_\pi.$$

Hence $S_\pi = \bigcup(v \in N_\pi)\, Y(v)$, and (12) holds for $\rho = \pi$. Clearly, (11), (13), (14) hold for $\rho = \pi$.

Case 2. $\pi = \mu+1$. We begin by showing that if $\sigma \leqslant \mu$ then $|N_\sigma| < \aleph_\alpha$, and there is $\gamma(\sigma) < \alpha$ such that

$$|n(v_0, \ldots, \hat{v}_\sigma)| \leqslant \aleph_{\gamma(\sigma)} \quad \text{for} \quad (v_0, \ldots, \hat{v}_\sigma) \in N_\sigma. \tag{16}$$

Let $\sigma_0 \leqslant \mu$ and suppose that for every $\sigma < \sigma_0$ there is $\gamma(\sigma) < \alpha$ such that (16) holds. We shall use the fact that $\aleph_\alpha$ is regular. We have $|N_{\sigma_0}| \leqslant \Pi(\sigma < \sigma_0)\, \aleph_{\gamma(\sigma)}$ and there is $\delta < \alpha$ such that

$$\Pi(\sigma < \sigma_0)\, \aleph_{\gamma(\sigma)} \leqslant \aleph_\delta^{|\sigma_0|} < \aleph_\alpha,$$

by (10). Hence $|N_{\sigma_0}| < \aleph_\alpha$. Since each $|n(v_0, \ldots, \hat{v}_{\sigma_0})| < \aleph_\alpha$, and there are only $|N_{\sigma_0}| < \aleph_\alpha$ sequences $(v_0, \ldots, \hat{v}_{\sigma_0}) \in N_{\sigma_0}$, there exists $\gamma(\sigma_0) < \alpha$ such that

$$|n(v_0, \ldots, \hat{v}_{\sigma_0})| \leqslant \aleph_{\gamma(\sigma_0)}$$

for all $(v_0, \ldots, \hat{v}_{\sigma_0}) \in N_{\sigma_0}$. We have proved that $|N_\mu| < \aleph_\alpha$.

Let $(N_\mu, \prec)$ be a well-order. Let $v \in N_\mu$. We now define intervals $I_v \subset S_\mu$, ordinals $n(v) < \omega_\alpha$ and elements $x(v, \nu)$ $(\nu < n(v))$. Here we put $(v, \nu) = (v_0, \ldots, \hat{v}_\mu, \nu)$ if $v = (v_0, \ldots, \hat{v}_\mu)$. Suppose we have already defined intervals $I_{v'} \subset S_\mu$ for all $v' \prec v$

in such a way that $I_{v''} \supset I_{v'}$ whenever $v'' \prec v' \prec v$. Since $|\{v' : v' \prec v\}| \leqslant |N_\mu| < \aleph_\alpha$ we can find an interval

$$J_v \subset S_\mu \cap \bigcap (v' \prec v) I_{v'}.$$

Let $J_{v,0}, \ldots, \hat{J}_{v,\omega_\alpha}$ be all subintervals of J_v. We construct $x(v, \nu)$ as follows. Let $\nu_0 < \omega_\alpha$, and let $x(v, \nu)$ be already defined for $\nu < \nu_0$. Put

$$Q_{\nu_0} = J_{v,\nu_0} \cap Y(v) - E(\{x(v,\nu) : \nu < \nu_0\}).$$

Here $E(S')$ is the set defined at the end of §2. If $Q_{\nu_0} \neq \emptyset$ then we select $x(v, \nu_0) \in Q_{\nu_0}$, and if $Q_{\nu_0} = \emptyset$ then we do not define $x(v, \nu_0)$. Then, for every $v \in N_\mu$, there is $\bar\nu \leqslant \omega_\alpha$ such that $x(v, \nu)$ is defined for $\nu < \bar\nu$ and not defined for $\nu = \bar\nu$. If we assume that $\bar\nu = \omega_\alpha$ then $Q_\nu \neq \emptyset$ for $\nu < \omega_\alpha$, and the set $X = \{x(v,\nu) : \nu < \omega_\alpha\}$ is independent and satisfies $X \cap J_{v,\nu} \neq \emptyset$ for $\nu < \omega_\alpha$. Then X is dense in J_v, and therefore tp $X \geqslant \eta_\alpha$ which is a contradiction against our initial assumption about the graph. Hence $\bar\nu < \omega_\alpha$. We now put $n(v) = \bar\nu$ and $X(v) = \{x(v,\nu) : \nu < n(v)\}$. Then

$$J_{v,n(v)} \cap Y(v) \subset E(X(v))$$

and

$$x(v,\nu) \in J_{v,\nu} \cap Y(v) \qquad (\nu < n(v)).$$

If $x \in J_{v,n(v)} \cap Y(v) - X(v)$ then $x \in E(X(v)) - X(v)$ and hence $E(x) \cap X(v) \neq \emptyset$. Thus

$$E(x) \cap X(v) \neq \emptyset \quad \text{if} \quad x \in J_{v,n(v)} \cap Y(v) - X(v). \tag{17}$$

We also note that $X(v)$ is independent and $|X(v)| \leqslant |n(v)| < \aleph_\alpha$. Hence there is an interval

$$I_v \subset J_{v,n(v)} - X(v).$$

Then $I_v \subset J_v \subset I_{v'}$ for $v' \prec v$, so that the I_v form a decreasing nest as v increases in the well-order $(N_\mu, \prec)$. Since $|N_\mu| < \aleph_\alpha$ it follows that there is an interval.

$$S_{\mu+1} \subset S_\mu \cap \bigcap (v \in N_\mu) I_v.$$

Then $S_{\mu+1} \subset S_\mu$ and (11) holds for $\rho = \mu+1$. For $v \in N_\mu$ and $\nu < n(v)$ we put

$$Y(v,\nu) = S_{\mu+1} \cap E(x(v,\nu)) \cap Y(v) - \bigcup((v',\nu') < (v,\nu))\, Y(v',\nu'). \tag{18}$$

Here the relation $(v', \nu') < (v, \nu)$ is meant to express that either $v' \prec v$, or $v' = v$ and $\nu' < \nu$. We have now defined all the entities we set out to define, *viz.* $n(v)$ for $v \in N_\mu$; $N_{\mu+1} = \{(v,\nu) : v \in N_\mu;\ \nu < n(v)\}$; $x(v,\nu)$ and $Y(v,\nu)$ for $(v,\nu) \in N_{\mu+1}$; and $S_{\mu+1}$. We now have to prove that (12)–(15) hold for $\rho = \mu+1$.

Proof of (13): This follows from (18).

Proof of (14): We have to show that $\{x(v,\nu)\} \cup Y(v,\nu) \subset Y(v)$ for $(v,\nu) \in N_{\mu+1}$. This follows from the choice of $x(v,\nu)$ and $Y(v,\nu)$.

Proof of (15): We have to show that $\{x(v,\nu), x\} \in E$ if $(v,\nu) \in N_{\mu+1}$ and $x \in Y(v,\nu)$. This follows from (18).

Proof of (12): We have to show that

$$S_{\mu+1} = \bigcup((v,\nu) \in N_{\mu+1})\, Y(v,\nu). \tag{19}$$

Let $x \in S_{\mu+1}$. Then, by (11), $x \in S_{\mu+1} \subset S_\mu$. By (12) and (13) there is a unique $v' \in N_\mu$ such that $x \in Y(v')$. Then, by definition of $S_{\mu+1}$ and $I_{v'}$,

$$x \in S_{\mu+1} \subset I_{v'} \subset J_{v',n(v')} - X(v').$$

By (17), $E(x) \cap X(v') \neq \emptyset$ and hence, by definition of $X(v')$, there is a least $v' < n(v')$ such that $\{x(v', v'), x\} \in E$. Put $v_1' = (v', v')$. We now show that $x \in Y(v_1')$. We have $x \in S_{\mu+1} \cap E(x(v_1')) \cap Y(v')$. Also, by definition of v' and by (13) and (18),

$$x \notin Y(v', v'') \qquad (v'' < v').$$

Finally, again by (13), $x \notin Y(v'')$ $(v'' \in N_\mu;\ v'' \prec v')$. Thus $x \notin Y(v'', v'')$ whenever $(v'', v'') < (v', v')$. But now (18) shows that $x \in Y(v_1')$. Since x was any element of $S_{\mu+1}$, we have proved that $S_{\mu+1} \subset \bigcup((v, v) \in N_{\mu+1})\, Y(v, v)$. On the other hand we have, by (18), $\bigcup((v, v) \in N_{\mu+1})\, Y(v, v) \subset S_{\mu+1}$. Hence (19) follows, and the proof of Theorem 1 is completed.

It follows from (6) and the fact that (10) holds for $\alpha > \beta = 0$ that

$$\eta_\alpha \to (\eta_\alpha, \aleph_0)^2 \quad \text{if} \quad \alpha = \text{cf}\,(\alpha). \tag{20}$$

As we remarked in §1, this relation can, in fact, be proved by an easy extension of an argument given in **[5]** which deals with the case $\alpha = 0$. The relation (20) may be considered as an analogue of the formula

$$\aleph_\alpha \to (\aleph_\alpha, \aleph_0)^2$$

due to Dushnik and Miller [9] which, however, was proved by these authors not only for regular $\aleph_\alpha$ but for all $\aleph_\alpha$. We are unable to decide whether (20) holds for singular $\aleph_\alpha$, not even in the first case when the problem is to decide if

$$\eta_\omega \to (\eta_\omega, \aleph_0)^2 \tag{21}$$

is true or false. In fact, we cannot even answer the seemingly easier question concerning the truth of the relation

$$\eta_\omega \to (\eta_\omega, 3)^2.$$

In contrast to the unsolved problem relating to (21) it is, however, comparatively easy to prove (Theorem 2), assuming a weak form of G.C.H., that

$$\zeta_\omega \to (\zeta_\omega, \aleph_0)^2$$

where $\zeta_\omega = \eta_0 + \eta_1 + \ldots + \hat{\eta}_\omega$. We first prove a simple lemma. Although (i) is well known **[14]** we give the short proof.

Lemma. (i) *If* cf $(\alpha) > \beta$ *then* $\eta_\alpha \to (\eta_\alpha)^1_{\aleph_\beta}$.

(ii) *If the order type* ϕ *satisfies* $\eta_n \leqslant \phi \leqslant \zeta_\omega$ *for all* $n < \omega$ *then* $\phi \geqslant \zeta_\omega$.

Proof of (i). Let $R_\alpha = \bigcup(v < \omega_\beta)\, S_v$ and $\text{tp}\, S_v \ngeqslant \eta_\alpha$ $(v < \omega_\beta)$. Then S_v is nowhere dense in R_α, and by an obvious recursive definition we can find intervals I_v of R_α such that $I_0 \supset I_1 \supset \ldots \supset \hat{I}_{\omega_\beta}$ and $I_v \cap S_v = \emptyset$ $(v < \omega_\beta)$. Then

$$\bigcap(v < \omega_\beta)\, I_v = J \neq \emptyset,$$

and we have the contradiction $\emptyset \neq R_\alpha \cap J = \bigcup(v < \omega_\beta)\, S_v \cap J = \emptyset$.

Proof of (ii). Let $S = S_0 \cup \ldots \cup \hat{S}_\omega$, where S is ordered, $S_v \subset L(S_{v+1})$ and $\text{tp}\, S_v = \eta_v$ for $v < \omega$. Then $\text{tp}\, S = \zeta_\omega$, and there is $X \subset S$ such that $\text{tp}\, X = \phi$. Then $\text{tp}\, X \geqslant \eta_1$ and, by (i), there is $n_0 < \omega$ such that $\text{tp}\, X \cap S_{n_0} \geqslant \eta_1$. Then $n_0 > 0$. Let $\lambda < \omega$; $n_0 < \ldots < n_\lambda < \omega$; $\text{tp}\, X \cap S_{n_\lambda} \geqslant \eta_{\lambda+1}$. Then $\text{tp}\, X \geqslant \eta_{n_\lambda+1}$, and there is $n_{\lambda+1} < \omega$ such that $\text{tp}\, X \cap S_{n_{\lambda+1}} \geqslant \eta_{n_\lambda+1}$. Then $n_{\lambda+1} > n_\lambda$. We have thus defined n_λ for $\lambda < \omega$ such that $0 < n_0 < \ldots < \hat{n}_\omega < \omega$ and $\mathrm{tp}\, X \cap S_{n_\lambda} \geqslant \eta_{n_\lambda+1} \geqslant \eta_\lambda$ $(\lambda < \omega)$. Then $\phi = \mathrm{tp}\, X \geqslant \Sigma(\lambda < \omega)\, \mathrm{tp}\, X \cap S_{n_\lambda} \geqslant \Sigma\eta_\lambda = \zeta_\omega$.

THEOREM 2. *Suppose that* $2^{\aleph_n} < \aleph_\omega$ *for* $n < \omega$. *Then*

$$\zeta_\omega \to (\zeta_\omega, \aleph_0)^2,$$

where

$$\zeta_\omega = \eta_0 + \ldots + \hat{\eta}_\omega.$$

Proof. The hypothesis implies that, for $n < \omega$,

$$|\eta_n| \leqslant \Sigma(\nu < \omega_n)\, 2^{|\nu|} \leqslant \aleph_n 2^{\aleph_n} = \aleph_{p(n)} < \aleph_\omega.$$

Let $\mathrm{tp}\, S = \zeta_\omega$ and let $G = (S, E)$ be a graph on S. We will assume that G does not contain an infinite complete subgraph and deduce that there is an independent set X with $\mathrm{tp}\, X \geqslant \zeta_\omega$.

Suppose that whenever $S' \subset S$ and $\mathrm{tp}\, S' \geqslant \zeta_\omega$ then there is $x \in S'$ such that $\mathrm{tp}\, E(x) \cap S' \geqslant \zeta_\omega$. Then there are sets S_ν and elements x_ν of S_ν such that $S_0 = S$ and, for $\nu < \omega$, $E(x_\nu) \cap S_\nu = S_{\nu+1}$ and $\mathrm{tp}\, S_\nu \geqslant \zeta_\omega$. Then $\{x_0, \ldots, \hat{x}_\omega\}_{\neq}$ is an infinite complete subgraph of G contrary to our assumption. It follows that there is a set $S' \subset S$ such that $\mathrm{tp}\, S' \geqslant \zeta_\omega$ and $\mathrm{tp}\, E(x) \cap S' \ngeqslant \zeta_\omega$ $(x \in S')$. Therefore, by part (ii) of our lemma, for each $x \in S'$ there is $n(x) < \omega$ such that $\mathrm{tp}\, E(x) \cap S' \ngeqslant \eta_{n(x)}$.

Let $1 \leqslant \lambda < \omega$. There is $T_\lambda' \subset S'$ such that $\mathrm{tp}\, T_\lambda' = \eta_\lambda$. By part (i) of the lemma there are a set $T_\lambda'' \subset T_\lambda'$ and a number $n_\lambda < \omega$ such that $\mathrm{tp}\, T_\lambda'' = \eta_\lambda$ and $n(x) = n_\lambda$ for all $x \in T_\lambda''$. Moreover, since G contains no infinite complete subgraph, it follows from (20) that there is an independent set $T_\lambda \subset T_\lambda''$ such that $\mathrm{tp}\, T_\lambda = \eta_\lambda$. Put $m(\lambda) = \sup\{n_1, \ldots, n_\lambda\}$ $(1 \leqslant \lambda < \omega)$.

We now define integers $\lambda(\rho)$ and sets $I(\rho)$ for $1 \leqslant \rho < \omega$. Put $\lambda(1) = 1$ and $I(1) = T_1$. For some k, where $1 \leqslant k < \omega$, suppose that $\lambda(\kappa)$ and $I(\kappa)$ have been defined for $1 \leqslant \kappa \leqslant k$, and that $\lambda(1) < \ldots < \lambda(k) < \omega$; $I(\kappa) \subset T_{\lambda(\kappa)}$; $\mathrm{tp}\, I(\kappa) = \eta_{\lambda(\kappa)}$ for $1 \leqslant \kappa \leqslant k$. We then define $\lambda(k+1)$ and $I(k+1)$ as follows. Put

$$\sigma = 1 + \sup\{p(\lambda(k)),\ m(\lambda(k)),\ \lambda(k)\}.$$

If we assume that the set $A = E(I(1) \cup \ldots \cup I(k))$ is dense in T_σ then $\mathrm{tp}\, A \geqslant \eta_\sigma$, and since $|I(1) \cup \ldots \cup I(k)| \leqslant |\eta_{\lambda(1)}| + \ldots + |\eta_{\lambda(k)}| \leqslant \aleph_{p(\lambda(k))} < \aleph_{\mathrm{cf}(\sigma)}$, it follows from the lemma that there is a number κ in $1 \leqslant \kappa \leqslant k$ and an element x of $I(\kappa)$ such that $\mathrm{tp}\, E(x) \cap T_\sigma \geqslant \eta_\sigma$. Then $\mathrm{tp}\, E(x) \cap S' \geqslant \eta_\sigma \geqslant \eta_{m(\lambda(k))} \geqslant \eta_{n_{\lambda(\kappa)}} = \eta_{n(x)}$ which contradicts the definition of $n(x)$. Hence A is not dense in T_σ, and there is an interval $I(k+1)$ of T_σ such that $A \cap I(k+1) = \varnothing$. Put $\lambda(k+1) = \sigma$. This completes the definition of $\lambda(\rho)$ and $I(\rho)$ for $1 \leqslant \rho < \omega$. We have $1 = \lambda(1) < \lambda(2) < \ldots$ and, for $1 \leqslant \rho < \omega$, $I(\rho) \subset T_{\lambda(\rho)}$; $\mathrm{tp}\, I(\rho) \geqslant \eta_{\lambda(\rho)} \geqslant \eta_\rho$; $[E(I(1) \cup \ldots \cup I(\rho))] \cap I(\rho+1) = \varnothing$. Then $I(1) \cup \ldots \cup \hat{I}(\omega)$ is an independent set of vertices of G of type $\geqslant \zeta_\omega$, and Theorem 2 follows.

4. *A remark on λ_α-sets*

Lusin [13] proved with the aid of the continuum hypothesis that there is a set of real numbers of power $\aleph_1$ which meets every set of the first category in at most $\aleph_0$ points. Erdös and Hajnal [11] observed that this immediately implies that every graph on the set of real numbers either contains an infinite complete subgraph or an independent set of the second category. Both, Lusin's theorem and the theorem of Erdös and Hajnal, can be generalised.

THEOREM 3. *If $\alpha \geq 0$ and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, and if $\aleph_\alpha$ is regular, then there is a set $C' \subset C_\alpha$ such that $|C'| = \aleph_{\alpha+1}$ and $|C' \cap X| \leq \aleph_\alpha$ for every set X of the first category in C_α.*

Proof. Every open set in C_α is, by definition, the union of intervals in C_α. If (a, b) is one such interval and $x \in (a, b)$ then, by definition of C_α and R_α, there are a', $b' \in R_\alpha$ such that $a < a' < x < b' < b$. Thus every open set is the union of intervals whose endpoints lie in R_α. Hence there are only $2^{|R_\alpha|} = \aleph_{\alpha+1}$ open sets and therefore only $\aleph_{\alpha+1}$ closed sets in C_α. Let $N_0, \ldots, \hat{N}_{\omega_{\alpha+1}}$ be all closed nowhere dense subsets of C_α. For $\nu < \omega_{\alpha+1}$ the set $M_\nu = N_0 \cup \ldots \cup \hat{N}_\nu$ is of the first category, and we can choose elements x_ν such that $x_\nu \in C_\alpha - (M_\nu \cup \{x_0, \ldots, \hat{x}_\nu\})$ $(\nu < \omega_{\alpha+1})$. Then the set $C' = \{x_0, \ldots, \hat{x}_{\omega_{\alpha+1}}\}$ has the required properties. For we have $|C'| = \aleph_{\alpha+1}$. Let $x_\nu \in N_\mu$. Then $\nu \leq \mu$, and hence $C' \cap N_\mu \subset \{x_0, \ldots, x_\mu\}$; $|C' \cap N_\mu| \leq |\mu+1| \leq \aleph_\alpha (\mu < \omega_{\alpha+1})$. Let X be a set of the first category. Then X is the union of $\aleph_\alpha$ nowhere dense sets A_λ and therefore contained in the union of the closures B_λ of the A_λ. But the B_λ are nowhere dense and therefore occur among the N_μ. Hence $|C' \cap X| \leq \aleph_\alpha$.

An immediate deduction from Theorem 3 is the following.

THEOREM 4. *If $\alpha \geq 0$ and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$, and if $\aleph_\alpha$ is regular, then every graph on C_α either contains a complete subgraph of power $\aleph_{\mathrm{cf}(\alpha)}$ or an independent set of vertices of the second category.*

Proof. Let G be a graph on C_α without a complete subgraph of power $\aleph_{\mathrm{cf}(\alpha)}$, and let C' be the set of Theorem 3. Let G' be the subgraph of G spanned by C', consisting of those vertices which belong to C' and of those edges which join points of C'. Since G' has no complete subgraph of power $\aleph_{\mathrm{cf}(\alpha)}$ there is by (5) an independent set X in G' of power $\aleph_{\alpha+1}$. Then X is independent in G and $|X \cap C'| = |X| > \aleph_\alpha$. Therefore X is of second category.

5. *Some negative relations*

The proofs of the negative relations for triplets mentioned in §1 are similar to each other. We state these results as a single theorem.

THEOREM 5. *Let ϕ be any order type. Then*

$$\phi \nrightarrow (\omega+\omega^*, 4)^3, \tag{22}$$

$$\phi \nrightarrow (\omega^*+\omega, 4)^3, \tag{23}$$

$$\phi \nrightarrow (\omega+\omega^* \vee \omega^*+\omega, 5)^3. \tag{24}$$

Here (24) has the following meaning. If $\mathrm{tp}\, S = \phi$ then every 3-graph on S either contains an independent set of type $\omega+\omega^*$ or an independent set of type $\omega^*+\omega$ or a complete subgraph of 5 elements. (22) and (23) were first proved, by a different method, by A. H. Kruse [**12**].

Remark. We cannot decide if the relation

$$\phi \nrightarrow (\omega+\omega^* \vee \omega^*+\omega, 4)^3$$

holds for every ϕ. It would imply (22)–(24).

Proof. Let $\mathrm{tp}_< S = \phi$ and let $\prec$ be a well-ordering of S. We define 3-graphs $G_i = (S, E_i)$ $(i = 1, 2, 3)$ as follows. A set $\{x, y, z\}_< \subset S$ belongs to E_1 if and only if $x \prec y \succ z$ and to E_2 if and only if $x \succ y \prec z$. We put $E_3 = E_1 \cup E_2$.

(i) Let $\{a, b, c, d\}_< \subset S$. If $\{a, b, c\} \in E_1$ then $b \succ c$ and hence $\{b, c, d\} \notin E_1$, and if $\{a, b, c\} \in E_2$ then, similarly, $\{b, c, d\} \notin E_2$. Hence neither G_1 nor G_2 has a complete subgraph of 4 elements.

(ii) Let $\{a_0, a_1, a_2, a_3, a_4\}_< \subset S$. Then there are indices r, s, t such that $0 \leqslant r < s < t \leqslant 4$ and either $a_r \prec a_s \prec a_t$ or $a_r \succ a_s \succ a_t$. In either case $\{a_r, a_s, a_t\} \notin E_3$, and hence G_3 has no complete subgraph of 5 elements.

(iii) Let $X \subset S$ and $\mathrm{tp}_< S = \omega+\omega^*$. Then there are elements a_ν, b_ν of X such that, for $\mu < \nu < \omega$, we have $a_\mu < a_\nu < b_\nu < b_\mu$; $a_\mu \prec a_\nu$; $b_\mu \prec b_\nu$. Then, if $a_0 \prec b_0$, we have $\{a_0, b_1, b_0\} \in E_1 \cap E_3$, and if $a_0 \succ b_0$ we have $\{a_0, a_1, b_0\} \in E_1 \cap E_3$. Hence neither G_1 nor G_3 has an independent set of type $\omega+\omega^*$.

(iv) Let $X \subset S$ and $\mathrm{tp}_< X = \omega^*+\omega$. Then there are elements a_ν, b_ν of X such that, for $\mu < \nu < \omega$, we have $a_\nu < a_\mu < b_\mu < b_\nu$; $a_\mu \prec a_\nu$; $b_\mu \prec b_\nu$. If $a_0 \prec b_0$ then $\{a_1, a_0, b_0\} \in E_2 \cap E_3$, and if $a_0 \succ b_0$ then $\{a_0, b_0, b_1\} \in E_2 \cap E_3$. Hence neither G_2 nor G_3 has an independent set of type $\omega^*+\omega$. This proves the theorem.

6. *A special result*

It is easy to show that

$$\eta_0 \to \left(\eta_0, \begin{bmatrix} 4 \\ 2 \end{bmatrix}\right)^3. \tag{25}$$

This relation means that if $G = (R, E)$ is any 3-graph on the set R of rationals then either there is an independent set of type η_0 or there is a set of four vertices such that at least two of its three-element subsets are edges of G.

To prove (25), consider a 3-graph $G = (R, E)$ such that every set of power 4 contains at most one 3-edge of G. Let $R = \{r_0, \ldots, \hat{r}_\omega\}_{\neq}$. We define rational numbers $s_0, \ldots, \hat{s}_\omega$. Put $s_0 = r_0$. Let $1 \leqslant n < \omega$ and suppose that $\{s_0, \ldots, \hat{s}_n\}_{\neq} \subset R$ and that, for $i < j < n$, we have $s_i < s_j$ if and only if $r_i < r_j$. Also, let $\{s_0, \ldots, \hat{s}_n\}$ be independent. Then we can write $\{s_0, \ldots, \hat{s}_n\} = \{t_0, \ldots, \hat{t}_n\}_<$ and

$$R = S_0 \cup \{t_0\} \cup S_1 \cup \{t_1\} \cup \ldots \cup \{t_{n-1}\} \cup S_n$$

where, for $\nu < n$, $x < t_\nu$ if $x \in S_\nu$, and $y > t_\nu$ if $y \in S_{\nu+1}$. Then each S_ν is infinite. There is a number $p \leqslant n$ such that $\{i : i < n;\ r_i < r_n\} = \{i : i < n;\ s_i < x\}$ for all $x \in S_p$. Given $i < j < n$, there is at most one x in S_p such that $\{s_i, s_j, x\} \in E$. Therefore we can choose $s_n \in S_p - \bigcup(i < j < n)\{x : \{s_i, s_j, x\} \in E\}$. This defines $s_0, \ldots, \hat{s}_\omega \in R$, and the set $\{s_0, \ldots, \hat{s}_\omega\}$ is independent and of type η_0.

We cannot prove*

$$\eta_0 \not\to \left(\eta_0, \begin{bmatrix} 4 \\ 3 \end{bmatrix}\right)^3. \tag{26}$$

As a step towards proving (26) we can define a 3-graph $G = (R, E)$ which is such

* See footnote at the end of §1.

that every set $\{a, b, c, d\}_< \subset R$ contains at most two edges, and at the same time G contains no independent set which is dense in an interval of R. To do this we choose a well-order $\prec$ of R such that $\text{tp}_\prec R = \omega$. We define G by taking as E the set of all sets $\{x, y, z\}_< \subset R$ for which (i) $x \succ y \prec z$, (ii) there is no $y' \prec y$ with $x < y' < z$. Let $\{a, b, c, d\}_< \subset R$. If $\{a, b, c\} \in E$ then, by (i), $\{b, c, d\} \notin E$, and if $\{a, b, d\} \in E$ then, by (ii), $\{a, c, d\} \notin E$. Hence $\{a, b, c, d\}$ contains at most two edges of G. Now let $A \subset R$, and suppose that A is dense in the interval I of R. Let $y_0 = \min_\prec I \cap A$. Since $\{y' : y' \prec y_0\}$ is finite and A is dense in I we can choose numbers x_0, z_0 in $I \cap A$ so close to y_0 by magnitude that $\{x_0, y_0, z_0\} \in E$. Thus the set A is not independent. However, we cannot exclude the possibility that our graph contains an independent set of type η_0 as would be required in a proof of (26).

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Printed by C. F. Hodgson & Son, Ltd., 23 Pakenham Street, London, WC1X 0LB