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DECOMPOSITIONS OF COMPLETE GRAPHS INTO FACTORS WITH DIAMETER TWO

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In the present paper the question is studied from three points of view whether to any natural number $k \geqslant 2$ there exists a complete graph decomposable into k factors with diameters two. The affirmative answer to this question is given and some estimations for the minimal possible number of vertices of such a complete graph are deduced. As a corollary it follows that given k diameters $d_1, d_2, \ldots, d_k$ (where $k \geqslant 3$ and $d_i \geqslant 2$ for $i = 1, 2, 3, \ldots, k$), there always exists a finite complete graph decomposable into k factors with diameters $d_1, d_2, \ldots, d_k$. Thus Problem 1 from [1] is solved.

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In this paper we deal only with nonoriented graphs. By a *factor* of a graph G we mean any subgraph of G containing all the vertices of G. By a *diameter* of G we understand the supremum of the set of all distances between the pairs of vertices of G (e. g. a disconnected graph has the diameter ∞). The symbol $\langle n \rangle$ denotes the complete graph with n vertices.

Let k be a natural number. By a *decomposition* of a graph G into k factors we mean a finite system $\{\varphi_1, \varphi_2, \ldots, \varphi_k\}$ of factors of G such that every edge of G belongs to exactly one of the factors $\varphi_1, \varphi_2, \ldots, \varphi_k$. The symbol $F_k(d_1, d_2, \ldots, d_k)$ denotes the smallest natural number n such that the complete graph $\langle n \rangle$ can be decomposed into k factors with diameters $d_1, d_2, \ldots, d_k$; if such an n does not exists, we put $F_k(d_1, d_2, \ldots, d_k) = \infty$. Further, put $f_k(d) = F_k(d, d, \ldots, d)$. The main aim of the present paper is to find estimations for $f_k(2)$. From [1] it follows that $f_2(2) = 5$, $12 \leqslant f_3(2) \leqslant 13$.

Theorem 1. *For any integer $k \geqslant 3$ we have:*

$$4k - 1 \leqslant f_k(2) \leqslant \binom{6k - 7}{2k - 2}.$$

Proof. To prove the upper estimation it suffices to decompose the graph

$$G = \left\langle \binom{6k-7}{2k-2} \right\rangle$$

into k factors with diameters two. The vertices of G can be represented by $(2k-2)$-tuples formed from elements $1, 2, 3, \ldots, 6k-7$. The ith factor ($i = 1, 2, \ldots, k$) consists of all edges joining $(2k-2)$-tuples with just $i-1$ common elements. The remaining edges can be added to any factor. It is easy to prove that all the factors have diameter two.

Suppose that for some $k \geqslant 4$ we have $f_k(2) \leqslant 4k-2$. Then, according to Theorem 1 of [1], $\langle 4k-2 \rangle$ is decomposable into k factors $\varphi_1, \varphi_2, \ldots, \varphi_k$ with diameter two. Put $n = 4k-2$. None of the factors φ_i ($i = 1, 2, \ldots, k$) may have a vertex of degree $n-1$ (otherwise the other factors are not connected), therefore, by [4], φ_i has at least $2n-5$ edges. The number of all edges of $\langle n \rangle$ is

$$\binom{n}{2} \geqslant k(2n-5),$$

whence it follows that

$$n^2 + 10k \geqslant 4kn + n. \tag{1}$$

But

$$n^2 + 10k = 16k^2 - 6k + 4,$$

$$4kn + n = 16k^2 - 4k - 2,$$

thus for $k \geqslant 4$ we have $n^2 + 10k < 4kn + n$, which contradicts (1). For $k = 3$ our assertion follows from [1], Theorem 7.

Remark. The upper estimation given in Theorem 1 is too high. Therefore we later present some methods enabling to improve it, namely for a „small" k in the second part of this article, and for a „great" k in the third part.

Lemma 1. *Let* $k \geqslant 2$, $2 = d_1 \leqslant d_2 \leqslant d_3 \leqslant \ldots \leqslant d_k < \infty$. *We have:* $F_k(d_1, d_2, \ldots, d_k) \leqslant f_k(2) + d_1 + d_2 + \ldots + d_k - 2k$.

Proof. From Theorem 1 it follows that $f_k(2)$ is a natural number. If $d_1 = d_2 = \ldots = d_k = 2$, the assertion of the lemma is evident. Thus we can suppose that there exists an integer i ($1 \leqslant i \leqslant k-1$) such that $d_1 = d_2 = \ldots = d_i = 2 < d_{i+1} \leqslant \ldots \leqslant d_k$. Let us construct a decomposition of the graph

$$G = \langle f_k(2) + d_1 + d_2 + \ldots + d_k - 2k \rangle$$

into k factors with diameters $d_1, d_2, \ldots, d_k$.

The vertex set of G consists (as we may suppose) of vertices $u_1, u_2, u_3, \ldots, u_{f_k(2)}$ and of vertices $v_{j,1}, v_{j,2}, v_{j,3}, \ldots, v_{j,d_j-2}$ $(i+1 \leqslant j \leqslant k)$. Obviously, the total number of vertices is $f_k(2) + d_1 + d_2 + \ldots + d_k - 2k$. The complete subgraph of G generated by the vertices $u_1, u_2, u_3, \ldots, u_{f_k(2)}$ according to the definition of $f_k(2)$ can be decomposed into k factors $\varphi_1, \varphi_2, \ldots, \varphi_k$ with diameter two. Define a decomposition of G into factors φ'_m $(m = 1, 2, \ldots, k)$ thus: Into φ'_m there belong (i) all the edges of φ_m; (ii) all the edges $u_s v_{j,t}$ $(1 < s \leqslant f_k(2),\ i+1 \leqslant j \leqslant k,\ 1 \leqslant t \leqslant d_j - 2)$ such that the edge $u_s u_1$ belongs to φ_m and $j \neq m$; (iii) all the edges of the path $u_1 v_{m,1} v_{m,2} \ldots v_{m,d_m-2}$ (if $m \geqslant i+1$). All the remaining edges are placed into φ'_1.

It is easy to show that φ'_m has diameter d_m $(m = 1, 2, \ldots, k)$. The lemma follows.

Lemma 2. *Let* $k \geqslant 3,\ 2 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k < \infty$. *Then we have*:

$$F_k(d_1, d_2, \ldots, d_k) \leqslant \binom{6k-7}{2k-2} + d_1 + d_2 + \ldots + d_k - 2k.$$

Proof. Distinguish two cases:

I. $d_1 = 2$. Then the assertion follows from Lemma 1 and Theorem 1.

II. $d_1 > 2$. By [1], Theorem 4, we have:

$$F_k(d_1, d_2, \ldots, d_k) \leqslant d_1 + d_2 + \ldots + d_k - k.$$

Since for any $k \geqslant 2$ we have

$$k \leqslant \binom{6k-7}{2k-2},$$

the lemma follows.

Corollary. *Let* $k \geqslant 3,\ 2 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k \leqslant \infty$. *Then* $F_k(d_1, d_2, \ldots, d_k)$ *is a natural number.*

Proof. If $d_k < \infty$, our assertion follows from Lemma 2. If $d_2 = \infty$, the assertion follows from [1], Theorem 3. Therefore we may suppose that $d_2 < \infty$, $d_k = \infty$, i. e. there is an integer i $(2 \leqslant i \leqslant k-1)$ such that $2 \leqslant d_1 \leqslant d_2 \leqslant \ldots \leqslant d_i < \infty = d_{i+1} = d_{i+2} = \ldots = d_k$.

If $i \geqslant 3$, according to Lemma 2, $F_i(d_1, d_2, \ldots, d_i)$ is a natural number. Therefore the finite complete graph

$$G = \langle F_i(d_1, d_2, \ldots, d_i) \rangle$$

is decomposable into i factors with diameters $d_1, d_2, \ldots, d_i$. If we add $k - i$ *null factors* (i. e., factors without edges), we obtain a decomposition of G into k factors with diameters $d_1, d_2, \ldots, d_i, d_{i+1}, \ldots, d_k$.

If $i = 2$, then according to Theorem 8 of [1] $F_3(d_1, d_2, d_3 = \infty)$ is a natural number. Since

$$F_k(d_1, d_2, d_3 = \infty, \ldots, d_k = \infty) \leqslant F_3(d_1, d_2, d_3 = \infty),$$

then $F_k(d_1, d_2, \ldots, d_k)$ is also a natural number. The corollary follows.

Remark. As the supposition $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k$ is not essential, the preceding corollary completely solves Problem 1 from [1], p. 53.

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Let a natural number n and a set $A \subseteq \{1, 2, \ldots, n\}$ be given. A is called an *S_n-set* if each $x \in \{1, 2, \ldots, n\}$, $x \notin A$ can be written in at least one of the following forms

$$x = a + b,$$

$$x = a - b,$$

$$x = 2n + 1 - (a + b),$$

where $a, b \in A$.

Let k be a natural number. Denote by $g(k)$ the least natural number l such that the set $\{1, 2, \ldots, l\}$ can be partitioned into k disjoint S_l-sets. (If such a natural number l does not exist, put $g(k) = \infty$.)

Lemma 3. *$f_k(2) \leqslant 2g(k) + 1$ for any integer $k \geqslant 2$.*

Proof. Let natural numbers m and n be given. We shall call a finite graph (without loops or multiple edges) with m labelled vertices $v_1, v_2, \ldots, v_m$ *cyclic*, if it contains with each edge $v_i v_j$ $(i, j \in \{1, 2, \ldots, m\})$ the edge $v_{i+1} v_{j+1}$ (the indices taken modulo m) as well. By the *length* of an edge $v_i v_j$ we mean the number

$$\min \{|i - j|, m - |i - j|\}.$$

Evidently, a cyclic graph contains either every or no edge of length i for each $i \in \{1, 2, \ldots, [m/2]\}$.

Assign to a given S_n-set A a cyclic graph with $2n + 1$ vertices containing edges of length i if and only if $i \in A$ $(i = 1, 2, \ldots, n)$. It is clear that thus a one-to-one correspondence between cyclic graphs with $2n + 1$ labelled vertices with diameter two and S_n-sets is defined. Further, it is obvious that to different [disjoint] S_n-sets different [edge-disjoint, respectively] cyclic factors with diameter two of $\langle 2n + 1 \rangle$ are assigned. Therefore the assertion of the lemma follows immediately from the definitions of $f_k(2)$ and $g(k)$.

Let natural numbers n, i, integers c, d and a set $A \subseteq \{1, 2, \ldots, n\}$ be given. Denote by $\mathrm{red}_n c$ the (uniquely determined) integer r such that

$$r \equiv c \pmod{2n+1},$$
$$|r| \leqslant n.$$

Further, put

$$r^{(i)} = |\mathrm{red}_n r^i|,$$
$$c \circ d = |\mathrm{red}_n cd|,$$
$$c \circ A = \{c \circ d;\ d \in A\}.$$

Evidently, we always have

$$(*) \qquad 0 \leqslant c \circ d \leqslant n,$$
$$c \circ A \subseteq \{0, 1, 2, \ldots, n\}.$$

Lemma 4. *If n and r are such natural numbers that the greatest common divisor $(2n+1, r) = 1$ and A is an S_n-set, then $r \circ A$ is an S_n-set as well.*

Proof. Choose $x \in \{1, 2, \ldots, n\}$. It suffices to prove that either $x \in r \circ A$ or there exist $a, b \in A$ such that one of the equalities

$$x = r \circ a + r \circ b,$$
$$x = r \circ a - r \circ b,$$
$$x = (2n+1) - (r \circ a + r \circ b),$$

holds.

It is easy to see that there is a $y \in \{1, 2, \ldots, n\}$ such that $r \circ y = x$. In fact, as $(r, 2n+1) = 1$, the congruence

$$rz \equiv x \pmod{2n+1}$$

has a solution $z \in \{1, 2, \ldots, 2n\}$. If $1 \leqslant z \leqslant n$, we put $y = z$, and if $n + 1 \leqslant \leqslant z \leqslant 2n$, we put $y = 2n + 1 - z$.

Since A is an S_n-set, either $y \in A$ or there exist $a, b \in A$ such that one of the following cases occurs:

$$y = a - b,$$
$$y = a + b,$$
$$y = 2n + 1 - (a + b).$$

If $y \in A$, then evidently $x = r \circ y \in r \circ A$. Let us analyze the other cases (all the following congruences are related to the modul $2n + 1$).

(I) $y = a - b$. Obviously $\pm r \circ y \equiv ry = ra - rb$, where $ra \equiv \pm r \circ a$, $rb \equiv \pm r \circ b$.

By examining all 8 possibilities for choice of signs we find that one of the following 4 cases occurs (we use inequality (*)):

$$x = r\circ y \equiv \quad r\circ a + r\circ b, \text{ hence } x = r\circ a + r\circ b,$$

$$x = r\circ y \equiv \quad r\circ a - r\circ b, \text{ hence } x = r\circ a - r\circ b,$$

$$x = r\circ y \equiv -r\circ a + r\circ b, \text{ hence } x = r\circ b - r\circ a,$$

$$x = r\circ y \equiv -r\circ a - r\circ b \equiv (2n+1) - r\circ a - r\circ b,$$

$$\text{so } x = 2n+1-(r\circ a + r\circ b).$$

(II) $y = a+b$. Evidently

$$\pm k\circ y \equiv ky = ka + kb \equiv \pm k\circ a \pm k\circ b,$$

where we again have 8 possibilities for choice of the signs. Further procedure is the same as in case (I).

(III) $y = 2n+1-(a+b)$. We have: $\pm k\circ y \equiv ky = k(2n+1) - ka - kb \equiv -ka - kb \equiv \mp k\circ a \mp k\circ b$. Further we proceed as in case (I). The lemma follows.

Lemma 5. *Let r, n and k be such natural numbers that*

(1) $2n+1$ *is a prime number,*
(2) k *divides* n,
(3) r *is a primitive root of* $2n+1$, [1]
(4) $A = \{r^{(k)}, r^{(2k)}, r^{(3k)}, \ldots, r^{(n)} = 1\}$ *is an* S_n*-set.*

Then $g(k) \leqslant n$.

Proof. From (1) and (3) it follows that $(r, 2n+1) = 1$ and that the numbers $r, r^2, \ldots, r^n, \ldots, r^{2n}$ represent all non-zero residue classes modulo $2n+1$. From this fact it can be easily deduced that $\{r^{(1)}, r^{(2)}, \ldots, r^{(n)}\} = \{1, 2, \ldots, n\}$. From (2) and (4) it follows that the sets $A, r\circ A, r^2\circ A, \ldots, r^{k-1}\circ A$ are mutually disjoint. They are S_n-sets, as it follows from (4) and Lemma 4. Therefore the set $\{1, 2, \ldots, n\}$ can be decomposed into k disjoint S_n-sets, consequently $g(k) \leqslant n$.

Lemma 6. *We have*: $g(1) \leqslant 1$, $g(2) \leqslant 2$, $g(3) \leqslant 6$, $g(4) \leqslant 20$, $g(5) \leqslant 35$, $g(6) \leqslant 78$, $g(7) \leqslant 98$, $g(8) \leqslant 96$, $g(9) \leqslant 189$, $g(10) \leqslant 260$.

Proof. We use the method from Lemma 5: we look for such a multiple n of k that (1) is valid and the least primitive root r of $2n+1$ satisfies (4). With the help of tables of the least primitive roots of primes (see, e. g. [5]) we can construct the following S_n-sets A:

[1] A natural number r is called a *primitive root* of a prime number p if the numbers $r, r^2, r^3, \ldots, r^{p-1} \equiv 1$ represent all non-zero residue classes modulo p.

$k = 1,\ n = 1,\ r = 2,\ A = \{1\}$.

$k = 2,\ n = 2,\ r = 2,\ A = \{1\}$.

$k = 3,\ n = 6,\ r = 2,\ A = \{1, 5\}$.

$k = 4,\ n = 20,\ r = 3,\ A = \{1, 4, 10, 16, 18\}$.

$k = 5,\ n = 35,\ r = 7,\ A = \{1, 20, 23, 26, 30, 32, 34\}$.

$k = 6,\ n = 78,\ r = 5,\ A = \{1, 4, 14, 16, 27, 39, 46, 49, 56, 58, 64, 67, 75\}$.

$k = 7,\ n = 98,\ r = 2,\ A = \{1, 6, 14, 19, 20, 33, 36, 68, 69, 77, 83, 84, 87, 93\}$.

$k = 8,\ n = 96,\ r = 5,\ A = \{1, 7, 9, 12, 16, 43, 49, 55, 63, 81, 84, 85\}$.

$k = 9,\ n = 189,\ r = 2,\ A = \{1, 5, 25, 39, 51, 52, 57, 68, 76, 86, 91, 93, 94, 119, 124, 125, 133, 138, 162, 163, 184\}$.

$k = 10,\ n = 260,\ r = 3,\ A = \{1, 10, 18, 29, 32, 42, 52, 55, 62, 74, 98, 99, 100, 101, 106, 114, 176, 180, 197, 201, 219, 226, 231, 235, 237, 255\}$.

To check that they are S_n-sets is a matter of routine. The rest of the proof follows from Lemma 5.

Remark. It can be easily found that even $g(1) = 1$, $g(2) = 2$, $g(3) = 6$. By a systematic examination we can also establish that $g(4) = 20$, but, on the other hand, $g(5) = 30$. (The inequality $g(5) \leqslant 30$ follows from the fact that $A = \{1, 5, 6, 11, 14, 29\}$, $3 \circ A$, $3^2 \circ A$, $3^3 \circ A$ and $3^4 \circ A$ are disjoint S_{30}-sets.)

Theorem 2. *We have:* $f_2(2) \leqslant 5, f_3(2) \leqslant 13, f_4(2) \leqslant 41, f_5(2) \leqslant 61, f_6(2) \leqslant 157, f_7(2) \leqslant 193, f_8(2) \leqslant 193, f_9(2) \leqslant 379, f_{10}(2) \leqslant 521$.

Proof. For $k \neq 5$, $k \neq 7$ the upper estimation of $f_k(2)$ follows from Lemmas 3 and 6. For $k = 5$ it suffices to apply Lemma 3 and the preceding remark. For $k = 7$ we proceed thus: Evidently $f_7(2) \leqslant f_8(2)$, because from a decomposition of a complete graph into 8 factors with diameter two we obtain a decomposition into 7 factors with diameter two by unifying edges of any two of the 8 given factors leaving the other 6 factors without any change. Since $f_8(2) \leqslant 193$, we have $f_7(2) \leqslant 193$ as well.

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Lemma 7. *There exists a natural number N such that for all naturals $n > N$ we have: The number A_n of all factors of $\langle n \rangle$ with $t = [\sqrt{3n^3 \log n}]$ edges and with a diameter greater than two is less than*

$$\frac{1}{n} \binom{\binom{n}{2}}{t}.$$

Proof uses methods similar to those used in [2].

(I) Pick a vertex x of $\langle n \rangle$. Let i be an integer for which

$$0 \leqslant i \leqslant t$$

holds. Denote by a_i the number of factors of $\langle n \rangle$ with t edges, in which the degree of x is i. Evidently, we have:

$$a_i = \binom{n-1}{i} \binom{\binom{n-1}{2}}{t-i},$$

(II) Put $l = [\sqrt{3n \log n}]$. Prove that there is a number N_1 such that for $i = 0, 1, 2, \ldots, l$ and for every natural $n > N_1$ we have

$$\frac{a_i}{a_{2l}} < \frac{1}{n^3}.$$

It is easy to see that for any natural n the inequalities

$$nl \leqslant t,$$
$$2l \leqslant t$$

are valid. Now, we have:

$$\frac{a_i}{a_{2l}} = \frac{\binom{n-1}{i}\binom{\binom{n-1}{2}}{t-i}}{\binom{n-1}{2l}\binom{\binom{n-1}{2}}{t-2l}} =$$

$$= \frac{(i+1)(i+2)\ldots 2l}{(n-i-1)(n-i-2)\ldots(n-2l)} \times$$

$$\times \frac{\left(\binom{n-1}{2} - t + 2l\right)\left(\binom{n-1}{2} - t + 2l - 1\right)\ldots\left(\binom{n-1}{2} - t + i + 1\right)}{(t-2l+1)(t-2l+2)\ldots(t-i)} <$$

$$< \frac{(i+1)(i+2)\ldots 2l}{(n-i-1)(n-i-2)\ldots(n-2l)} \cdot \frac{\left(\frac{n^2}{2}\right)^{2l-i}}{(t-2l+1)(t-2l+2)\ldots(t-i)} =$$

$$= \frac{(i+1)(i+2)\ldots 2l}{2^{2l-i}} \cdot \left(\frac{n}{t}\right)^{2l-i} \cdot \frac{n^{2l-i}}{(n-i-1)(n-i-2)\ldots(n-2l)} \times$$

$$\times \frac{t^{2l-i}}{(t-2l+1)(t-2l+2)\dots(t-i)} \leqslant \frac{(i+1)(i+2)\dots 2l}{(2l)^{2l-i}} \times$$

$$\times \left(\frac{n}{n-2l}\right)^{2l-i} \cdot \left(\frac{t}{t-2l+1}\right)^{2l-i} \leqslant \frac{l+1}{2l} \cdot \frac{l+2}{2l} \dots \frac{2l}{2l} \cdot \left(\frac{n}{n-2l}\right)^{2l} \times$$

$$\times \left(\frac{t}{t-2l+1}\right)^{2l} \leqslant \left(\frac{3}{4}\right)^{l-1} \cdot \left(\sqrt[4]{\frac{5}{4}}\right)^{2l} \cdot \left(\sqrt[4]{\frac{5}{4}}\right)^{2l} = \frac{5}{4} \cdot \left(\frac{15}{16}\right)^{l-1} <$$

$$< \frac{5}{4}\left(\frac{15}{16}\right)^{\sqrt{n}} < \frac{1}{n^3}$$

for every natural $n > N_1$, if N_1 is a sufficiently large constant.

(III) Let us prove that the number $B_n(x)$ of the factors of $\langle n \rangle$ with t edges, in which the degree of x does not exceed l, is less than

$$\frac{1}{2} \frac{\binom{\binom{n}{2}}{t}}{n^2}$$

for every sufficiently large n.

Obviously, according to (II) for $n > N_1$ we have:

$$\frac{n^2 B_n(x)}{\binom{\binom{n}{2}}{t}} = n^2 \frac{a_0 + a_1 + \dots + a_l}{\binom{\binom{n}{2}}{t}} \leqslant$$

$$\leqslant n^2 \frac{a_0 + a_1 + \dots + a_l}{a_{2l}} = n^2 \left(\frac{a_0}{a_{2l}} + \frac{a_1}{a_{2l}} + \dots + \frac{a_l}{a_{2l}}\right) <$$

$$< n^2(l+1)\frac{1}{n^3} = \frac{[\sqrt{3n \log n}\,] + 1}{n}.$$

Evidently, the last expression tends to zero for $n \to \infty$. Therefore

$$\frac{[\sqrt{3n \log n}\,] + 1}{n} < \frac{1}{2}$$

for $n > N_2$, where N_2 is a sufficiently large constant so that

$$\frac{n^2 B_n(x)}{\binom{\binom{n}{2}}{t}} < \frac{1}{2},$$

i. e.

$$B_n(x) < \frac{1}{2} \frac{\binom{\binom{n}{2}}{t}}{n^2}$$

for $n > \max\{N_1, N_2\}$.

(IV) We prove now that the number B_n of the factors of $\langle n \rangle$ with t edges containing a vertex of degree $\leqslant l$, is less than

$$\frac{1}{2n}\binom{\binom{n}{2}}{t}$$

for $n > \max\{N_1, N_2\}$.

Evidently, we have

$$B_n \leqslant \sum_x B_n(x),$$

where x runs through the vertex set of $\langle n \rangle$. Therefore, using (III) we obtain

$$B_n \leqslant \sum_x B_n(x) < n \frac{1}{2} \frac{1}{n^2} \binom{\binom{n}{2}}{t} = \frac{1}{2n} \binom{\binom{n}{2}}{t}$$

for $n > \max\{N_1, N_2\}$.

(V) Fix now two different vertices x and y of $\langle n \rangle$ and two integers i and j satisfying the relations $l < i < n$, $l < j < n$.

Denote by $D_n(x, y, i, j)$ the number of factors of $\langle n \rangle$ with t edges in which x has degree i, y has degree j, and x is not joined with y by an edge. We have:

$$D_n(x, y, i, j) = \binom{n-2}{i} \binom{n-2}{j} \binom{\binom{n-2}{2}}{t-i-j}.$$

Further, denote by $E_n(x, y, i, j)$ the number of factors of $\langle n \rangle$ with t edges in which x has degree i, y has degree j, and the distance of x and y is greater than two. Evidently,

$$E_n(x, y, i, j) = \binom{n-2}{i}\binom{n-2-i}{j}\binom{\binom{n-2}{2}}{t-i-j}.$$

We shall find a natural number N_3 such that for every $n > N_3$ we have:

$$\frac{E_n(x, y, i, j)}{D_n(x, y, i, j)} < \frac{1}{n^3}.$$

Obviously, we have:

$$\frac{E_n(x, y, i, j)}{D_n(x, y, i, j)} = \frac{n-i-2}{n-2} \cdot \frac{n-i-3}{n-3} \cdots \frac{n-i-j-1}{n-j-1} <$$

$$< \left(\frac{n-i-2}{n-2}\right)^j \leqslant \left(\frac{n-3-l}{n-2}\right)^{l+1}.$$

It is easy to see that there exists a natural number N_3 such that for all $n > N_3$ we have

$$\frac{n-2}{l+1} > 1.$$

Evidently, it suffices to prove that for every $n > N_3$ we have:

$$\left(\frac{n-2}{n-3-l}\right)^{l+1} > n^3.$$

But for $n > N_3$ we have:

$$\left(1 + \frac{1}{\frac{n-2}{l+1} - 1}\right)^{\frac{n-2}{l+1}} > \mathrm{e}.$$

It follows that

$$\left(\frac{n-2}{n-3-l}\right)^{l+1} = \left(\left(1 + \frac{1}{\frac{n-2}{l+1} - 1}\right)^{\frac{n-2}{l+1}}\right)^{\frac{(l+1)^2}{n-2}} >$$

$$> \mathrm{e}^{\frac{(l+1)^2}{n-2}} > \mathrm{e}^{\frac{(\sqrt{3n \log n})^2}{n}} = n^3.$$

(VI) Let C_n be the number of factors of $\langle n\rangle$ with t edges in which all the vertices have degrees greater than l and with diameters greater than two. From (V) it follows that for every $n > N_3$ we have:

$$C_n \leqslant \sum_{(x,y)} \sum_{(i,j)} E_n(x, y, i, j) \leqslant$$

$$\leqslant \sum_{(x,y)} \sum_{(i,j)} \frac{D_n(x, y, i, j)}{n^3} = \frac{1}{n^3} \sum_{(x,y)} \sum_{(i,j)} D_n(x, y, i, j) <$$

$$< \sum_{(x,y)} \frac{\binom{\binom{n}{2}}{t}}{n^3} = \binom{n}{2} \cdot \frac{\binom{\binom{n}{2}}{t}}{n^3} < \frac{\binom{\binom{n}{2}}{t}}{2n},$$

where (x, y) runs through the set of all unordered pairs of different vertices of $\langle n\rangle$ (i, j) runs through the set of all ordered pairs of integers such that $l < i < n$, $l < j < n$.

(VII) Put $N = \max\{N_1, N_2, N_3\}$. Then, according to (IV) and (VI) for every natural number $n > N$ we have:

$$A_n \leqslant B_n + C_n < \frac{\binom{\binom{n}{2}}{t}}{2n} + \frac{\binom{\binom{n}{2}}{t}}{2n} = \frac{\binom{\binom{n}{2}}{t}}{n}.$$

The lemma follows.

Lemma 8. *A natural number M exists such that for every integer $n > M$ we have: $\langle n\rangle$ contains*

$$\left[\sqrt{\frac{n-2}{12\log n}}\right]$$

edge-disjoint factors with diameter two.

Proof. According to Lemma 7 there exists a positive integer N such that for every integer $n > N$ we have:

$$A_n < \frac{1}{n}\binom{\binom{n}{2}}{t}.$$

Put

$$u = \left[\frac{\binom{n}{2}}{t}\right].$$

Evidently there is a natural number N_4 such that for every $n > N_4$ we have $u < n$. Put $M = \max\{N, N_4\}$. Obviously for $n \geqslant 2$ we have:

$$u = \left[\frac{n(n-1)}{2[\sqrt{3n^3 \log n}]}\right] \geqslant \left[\frac{n(n-1)}{2\sqrt{3n^3 \log n}}\right] =$$

$$= \left[\sqrt{\frac{n^2 - 2n + 1}{12n \log n}}\right] \geqslant \left[\sqrt{\frac{n^2 - 2n}{12n \log n}}\right] = \left[\sqrt{\frac{n-2}{12 \log n}}\right].$$

Therefore it suffices to prove that for $n > M$ the graph $\langle n \rangle$ contains u edge-disjoint factors with diameter two.

If we assume the contrary, then each of the

$$p = \frac{\prod\limits_{i=0}^{u-1} \binom{\binom{n}{2} - it}{t}}{u!}$$

systems S consisting of u edge-disjoint factors of $\langle n \rangle$, each with t edges, contains at least one factor with diameter greater than two. Any such factor with t edges and with diameter greater than two occurs just in

$$q = \frac{\prod\limits_{i=1}^{u-1} \binom{\binom{n}{2} - it}{t}}{(u-1)\,!}$$

systems S. Therefore the number of factors of $\langle n \rangle$ with t edges and with a diameter greater than two is at least

$$\frac{p}{q} = \frac{1}{u}\binom{\binom{n}{2}}{t} > \frac{1}{n}\binom{\binom{n}{2}}{t},$$

which contradicts Lemma 7. Thus Lemma 8 follows.

Theorem 3. *There exists a positive integer K such that for any integer $k > K$ we have:*

$$f_k(2) \leqslant \left(\frac{49}{10}\right)^2 k^2 \log k\,.$$

Proof. Pick a natural number K_1 such that for every $k > K_1$ we have

$$\left[\left(\frac{49}{10}\right)^2 k^2 \log k\right] > M,$$

where M is the constant from Lemma 8.

Pick a natural number K_2 in such a way that for any $k > K_2$

$$k^2 \log k \geqslant 750,$$

and, consequently,

$$-3 \geqslant -\frac{1}{250} k^2 \log k.$$

Further, pick a natural number K_3 such that for every integer $k > K_3$ we have:

$$\left(\frac{49}{10}\right)^2 \log k \leqslant k^{\frac{1}{2000}}.$$

Put $K = \max\{K_1, K_2, K_3\}$. Pick an integer $k > K$. Put

$$n = \left[\left(\frac{49}{10}\right)^2 k^2 \log k\right].$$

Then we have:

$$\frac{n-2}{\log n} \geqslant \frac{\left(\left(\frac{49}{10}\right)^2 k^2 \log k - 1\right) - 2}{\log\left(\left(\frac{49}{10}\right)^2 k^2 \log k\right)} = \frac{\left(\frac{49}{10}\right)^2 k^2 \log k - 3}{2 \log k + \log\left(\left(\frac{49}{10}\right)^2 \log k\right)} \geqslant$$

$$\geqslant \frac{\left(\frac{49}{10}\right)^2 k^2 \log k - \frac{1}{250} k^2 \log k}{2 \log k + \log\left(k^{\frac{1}{2000}}\right)} = 12k^2.$$

It follows that

$$k \leqslant \sqrt{\frac{n-2}{12 \log n}},$$

where $n > M$. From Lemma 8 it follows that $\langle n \rangle$ can be decomposed into k edge-disjoint factors with diameter two (the remaining edges may be added to any factor). Consequently,

$$f_k(2) \leqslant n \leqslant \left(\frac{49}{10}\right)^2 k^2 \log k.$$

The theorem follows.

Remark. It can be proved that there exist positive constants C_1 and C_2 such that

$$C_1 k^2 < g(k) < C_2 k^2 \log k$$

for every sufficiently large k; the left inequality is obvious; the right one can be obtained using similar methods as in our Theorem 3 and in [3]; this remains true even if we do not allow representations of the form $2n + 1 - (a + b)$. Now, using Lemma 3 we can again obtain that $f_2(k) < Ck^2 \log k$ for certain constant C and all sufficiently large k.

Problem 1. Is $g(k)/k^2$ bounded?

Problem 2. Determine $\lim\limits_{k\to\infty} \dfrac{f_k(2)}{k}$.

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